The Anelastic and Boussinesq Approximations

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Introduction

The atmosphere contains sound waves, and the linearized equations that describe the evolution of the atmosphere contain solutions corresponding to sound waves. These solutions are derived below.

Since sound waves have no meteorological significance, it is useful to have a system of equations that has no sound-wave solutions, but is still applicable to the study of turbulence and cumulus convection as well as large-scale motions. The familiar quasi-static approximation filters vertically propagating sound waves, while allowing the Lamb wave (which is a horizontally propagating sound wave), but seriously distorts the small-scale motions. The distortion arises because, for these motions, the perturbation pressure force is not hydrostatically balanced by the perturbation density field.

The anelastic approximation was invented by Ogura and Phillips (1962) in order to filter sound waves without assuming hydrostatic balance. The Boussinesq equations are a simplified subset of the anelastic equations, valid only for relatively shallow motions. Although the original anelastic and Boussinesq equations are very useful, they have important weaknesses. Improved alternatives are available now (Arakawa and Konor, 2009).

The exact equations

The basic equations in height coordinates, without rotation and friction, are

$$\frac{D\mathbf{V}_h}{Dt} = -\frac{1}{\rho}\nabla_z p \ , \tag{1}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \ , \tag{2}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_z + \nabla_z \cdot \left(\rho \mathbf{V}_h\right) + \frac{\partial}{\partial z}\left(\rho w\right) = 0 \ , \tag{3}$$

$$\dot{\theta} \equiv \frac{D\theta}{Dt} = \frac{Q}{\Pi} . \tag{4}$$

Here D / Dt is the Lagrangian time derivative, $\mathbf{V}_h$ is the horizontal velocity, ρ is density, p is pressure, w is the vertical velocity, z is height, g is the acceleration of gravity, θ is the potential temperature, Q is the heating rate per unit mass, and Π is the Exner function, which satisfies

$$c_p T = \Pi\theta , \tag{5}$$

where c_p is the heat capacity of air at constant pressure, and T is temperature. We can also write

$$\Pi = c_p \left(\frac{p}{p_0} \right)^{\kappa} , \tag{6}$$

where

$$\kappa \equiv \frac{R}{c_p} , \tag{7}$$

and R is the specific gas constant. Finally, we can include the prognostic equation for an arbitrary scalar, which is

$$\left(\frac{\partial \rho A}{\partial t} \right)_z + \nabla_z \cdot \left(\rho \mathbf{V}_h A \right) + \frac{\partial}{\partial z} \left(\rho w A \right) = \rho S_A , \tag{8}$$

where S_A is the source of A per unit mass.

We will need the ideal gas law, which is

$$p = \rho R T . \tag{9}$$

The following relationships can be derived using the ideal gas law:

$$p = p_0 \left(\frac{\rho R \theta}{p_0} \right)^{\frac{1}{1-\kappa}} , \tag{10}$$

$$\Pi = c_p \left(\frac{\rho R \theta}{p_0} \right)^{\frac{\kappa}{1-\kappa}} , \tag{11}$$

Acoustic-waves

Consider one-dimensional, small-amplitude motions, with no mean flow, no rotation, no stratification, no friction, and no heating. We adopt the following linearized system of equations:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial x} (\delta p) , \tag{12}$$

$$\lambda \frac{\partial}{\partial t} (\delta \rho) + \frac{\partial}{\partial x} (\rho_0 u) = 0 , \tag{13}$$

$$\frac{\partial}{\partial t} \left(\frac{\delta \theta}{\theta_0} \right) = 0 \tag{14}$$

$$\frac{\delta \theta}{\theta_0} = \frac{\delta T}{T_0} - \kappa \frac{\delta p}{p_0} , \tag{15}$$

$$\frac{\delta p}{p_0} = \frac{\delta \rho}{p_0} + \frac{\delta T}{T_0} , \tag{16}$$

where λ, which appears in (13), is normally equal to one but will be set equal to zero to obtain the anelastic system. We can eliminate unknowns so as to obtain a single wave equation in the single unknown δp:

$$\frac{\partial^2}{\partial t^2} (\delta p) - \frac{c_s}{\lambda} \frac{\partial^2}{\partial x^2} (\delta p) = 0 . \tag{17}$$

Here we have used

$$c_s^2 \equiv \gamma R T , \tag{18}$$

where

$$\gamma \equiv \frac{1}{1-\kappa} \cong 1.4 \; . \tag{19}$$

The solutions of (17) are

$$\delta p = P \exp\left[ik\left(x \pm \frac{c_S}{\lambda} t \right) \right], \tag{20}$$

and $\frac{c_S}{\lambda}$ is seen to be the signal velocity (the speed of sound).

For $\lambda = 0$, we get

$$\frac{\partial^2}{\partial x^2}(\delta p) = 0 \; , \tag{21}$$

which has the solutions $\delta p = A + Bx$. In this case, the effective "signal velocity" is infinite. There are no wave solutions.

This analysis shows that if we can justify neglect of the density-tendency term in the continuity equation, we can filter out all sound waves -- including both horizontally and vertically propagating waves. The next section presents a scale analysis designed to determine under what conditions such an approximation can be justified. The scale analysis is also used to introduce some additional simplifying approximations.

Scale analysis

Consider a hydrostatically balanced reference state for which the thermodynamic state variables may be functions of height z, but are independent of time and the horizontal coordinates. We denote this reference state by subscript 0, and departures from it by $\delta(\;)$, i.e.

$$(\;) \equiv (\;)_0 + \delta(\;) . \tag{22}$$

The reference state need not necessarily be identical to the "*initial" state,* or to a *"basic" state* upon which perturbations will be be imposed. Nevertheless, we assume that the actual range of variation of the thermodynamic variables is no greater, in order of magnitude, than the departure from the reference state, and that for each of the thermodynamic variables, this departure from the reference state is fractionally small.

For our scale analysis, we need thermodynamic scale heights, defined as follows:

$$H_\rho \equiv \left| \frac{1}{\rho_0} \frac{d\rho_0}{dz} \right|^{-1} \sim 10 \text{ km} , \tag{23}$$

$$H_p \equiv \left| \frac{1}{p_0} \frac{dp_0}{dz} \right|^{-1} \equiv \frac{RT_0}{g} \sim 8 \text{ km} , \tag{24}$$

$$H_\theta \equiv \left| \frac{1}{\theta_0} \frac{d\theta_0}{dz} \right|^{-1} \sim 70 \text{ km} , \tag{25}$$

$$H_T \equiv \left| \frac{1}{T_0} \frac{dT_0}{dz} \right|^{-1} \sim 40 \text{ km} . \tag{26}$$

The numerical values given in (23) - (26) are for typical tropospheric soundings. For the idealized special case of an isothermal atmosphere, it can be shown that the scales heights for density and pressure are equal and given by $\frac{RT_0}{g}$, while the scale height for temperature is (of course) infinite.

We first analyze the continuity equation, which can be written as

$$\frac{\partial}{\partial t}(\delta p) + \rho_0 \left(1 + \frac{\delta\rho}{p_0}\right)\left(\nabla_z \cdot \mathbf{V}_h + \frac{\partial w}{\partial z}\right) + \mathbf{V}_h \cdot \nabla_z (\delta p) + w\frac{\partial}{\partial z}(\delta p) + w\frac{\partial \rho_0}{\partial z} = 0 . \tag{27}$$

We have used our assumption that ρ_0 is horizontally homogeneous. Below we assume that $|\delta\rho| / \rho_0 << 1$. We proceed by comparing $\rho_0 \frac{\partial w}{\partial z}$ with each of the remaining terms. First, note that

$$\frac{\left|\frac{\partial}{\partial t}(\delta p)\right|}{\left|\rho_0 \frac{\partial w}{\partial z}\right|} \sim \frac{|\delta p| / \tau}{|w| \rho_0 / D}$$
$$= \frac{|\delta p|}{\rho_0} \frac{D}{|w| \tau}$$
$$= \frac{g\left(|\delta p| / \rho_0\right)}{\left(\frac{g}{D}\right) \tau^2 \frac{w}{\tau}} .$$

(28)

where D is the depth of the motions, and τ *is the time scale of interest*. Since we are interested in boundary-layer eddies, we will choose in the range 100 s to 1000 s. If we were interested in sound waves, we would choose τ on the order of 10^{-3} s. Since

$$\left|\frac{w}{\tau}\right| \sim g \frac{|\delta p|}{\rho_0} ,$$

(29)

we can neglect $\frac{\partial}{\partial t}(\delta \rho)$ if

$$\tau^2 >> D / g \leq (30 \text{ seconds})^2 .$$

(30)

For large boundary-layer eddies and cumulus clouds, the term can safely be neglected. The continuity equation then becomes *diagnostic*, i.e., its time derivative term drops out.

Next, we analyze the $\partial \rho_0 / \partial z$ term of the continuity equation:

$$\frac{\left|w \frac{\partial \rho_0}{\partial z}\right|}{\left|\rho_0 \frac{\partial w}{\partial z}\right|} \sim \frac{D}{H_\rho} .$$

(31)

For $D / H_\rho \sim 1$, the two terms are the same size. For motions that are shallow in the sense that $D / H_\rho << 1$, we can neglect $\rho_0 \frac{\partial w}{\partial z}$. Here we allow the possibility that $D / H_\rho \sim 1$.

Now consider the horizontal advection of $\delta \rho$:

$$\frac{\left|\mathbf{V}_h \cdot \nabla_z \delta\rho\right|}{\left|\rho_0 \frac{\partial w}{\partial z}\right|} \sim \frac{V\left|\delta\rho\right|/L}{\rho_0\left|w\right|/D} \\ = \frac{\left|\delta\rho\right|}{\rho_0}\frac{V}{\left|w\right|}\frac{D}{L}.$$

(32)

Here L is a horizontal length scale. From (32), we conclude that $\mathbf{V}_h \cdot \nabla_z(\delta\rho)$ can be neglected if

$$\frac{V}{\left|w\right|}\frac{D}{L} \le 1.$$

(33)

This condition will be met if the aspect ratio D/L is sufficiently small.

Finally, we note that since

$$\frac{\left|w\frac{\partial}{\partial z}(\delta\rho)\right|}{\left|\rho_0 \frac{\partial w}{\partial z}\right|} \sim \frac{\delta\rho}{\rho_0} << 1,$$

(34)

the vertical advection of $\delta\rho$ is negligible.

Generally speaking, the remaining terms have to be kept. In summary, we have

$$\nabla_z \cdot \left(\rho_0 \mathbf{V}_h\right) + \frac{\partial}{\partial z}\left(\rho_0 w\right) = 0 \text{ for } D/H_\rho \sim 1,$$

(35)

and

$$\nabla_z \cdot \mathbf{V}_h + \frac{\partial w}{\partial z} = 0 \text{ for } D/H_\rho << 1.$$

(36)

Now consider the horizontal pressure gradient force. Since p_0 and ρ_0 are assumed to be horizontally homogeneous, we can write

$$-\frac{1}{\rho}\nabla_z p = -\frac{1}{\rho}\nabla_z(\delta p) \cong -\frac{1}{\rho_0}\nabla_z(\delta p) = -\nabla_z\left(\frac{\delta p}{\rho_0}\right).$$

(37)

The vertical pressure gradient force requires somewhat more analysis. Recall that the reference state is assumed to be in hydrostatic balance, i.e.,

$$\frac{dp_0}{dz} = -\rho_0 g \ . \tag{38}$$

We can then write

$$\begin{aligned}
\frac{-1}{\rho}\frac{\partial p}{\delta z} - g &= \frac{-1}{(\rho_0+\delta\rho)}\frac{\partial}{\partial z}(p_0+\delta p) - g \\
&= \frac{-1}{(\rho_0+\delta\rho)}\frac{\partial}{\partial z}(\delta p) + \left(\frac{\rho_0}{\rho_0+\delta\rho} - 1\right)g \\
&\cong \frac{-1}{\rho_0}\frac{\partial}{\partial z}(\delta p) - \frac{\delta\rho}{\rho_0}g \\
&= -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) - \frac{\delta p}{p_0}\left(\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z}\right) - \frac{\delta\rho}{\rho_0}g
\end{aligned} \tag{39}$$

In the last line of (39), the basic state density appears inside the vertical derivative. Eq. (39) can be simplified as follows. From the definition of potential temperature,

$$\theta \equiv T\left(\frac{p_0}{p}\right)^{\kappa} , \tag{40}$$

and the ideal gas law, we can show that

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} = \frac{1}{\gamma}\frac{1}{p_0}\frac{\partial p_0}{\partial z} - \frac{1}{\theta_0}\frac{\partial \theta_0}{\partial z} , \tag{41}$$

and

$$\frac{\delta\rho}{\rho_0} = \frac{1}{\gamma}\frac{\delta p}{p_0} - \frac{\delta\theta}{\theta_0} , \tag{42}$$

where

$$\gamma \equiv c_p / c_v , \tag{43}$$

which is equivalent to (19). Substituting from (41) and (42) into (39), we find that

$$\frac{-1}{\rho}\frac{\partial p}{\delta z}-g\cong-\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right)+\frac{\delta p}{\rho_0}\left(\frac{1}{\theta_0}\frac{\partial\theta_0}{\partial z}-\frac{1}{\gamma}\frac{1}{p_0}\frac{\partial p_0}{\partial z}\right)+\left(\frac{\delta\theta}{\theta_0}-\frac{1}{\gamma}\frac{\delta p}{p_0}\right)g$$
$$=-\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right)+\frac{\delta p}{\rho_0}\left(\frac{1}{\theta_0}\frac{\partial\theta_0}{\partial z}\right)+g\frac{\delta\theta}{\theta_0},\tag{44}$$

where we have used (38) to obtain the final equality.

We now argue that, under some conditions, the $\frac{\delta p}{\rho_0}\left(\frac{1}{\theta_0}\frac{\partial\theta_0}{\partial z}\right)$ term of (44) can be neglected. Defining the Brunt-Vaisalla frequency, N, by

$$N^2\equiv\frac{g}{\theta_0}\frac{\partial\theta_0}{\partial z}=\frac{g}{H_\theta},\tag{45}$$

we can write

$$\frac{\left|\frac{\delta p}{\rho_0}\left(\frac{1}{\theta_0}\frac{\partial\theta_0}{\partial z}\right)\right|}{\left|\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right)\right|}\sim\frac{\left|\frac{\delta p}{\rho_0}\right|\frac{N^2}{g}}{\left|\frac{\delta p}{\rho_0}\right|\frac{1}{D}}$$
$$=\frac{N^2D}{g}$$
$$=\frac{N^2}{\gamma RT}\left(\frac{RT}{g}\right)\gamma D$$
$$=\frac{N^2}{\left(c_S/H_p\right)^2}\left(\frac{\gamma D}{H_p}\right),\tag{46}$$

where c_S is the isentropic sound speed, introduced earlier. From (46) we see that, provided that $\gamma D/H_\rho$ is *no greater than order 1*, we can neglect the $\frac{\delta p}{\rho_0}\left(\frac{1}{\theta_0}\frac{\partial\theta_0}{\partial z}\right)$ term of (44) when the frequency of sound waves with vertical wavelength H_ρ greatly exceeds the frequency of pure gravity waves. A typical value of H_ρ is 5 km or greater. Our conclusion is that

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z} - g \cong -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) + g\frac{\delta\theta}{\theta_0}}. \tag{47}$$

This is the origin of the familiar "buoyancy" term of the equation of vertical motion.

We now turn to the first law of thermodynamics, which can be written as

$$\frac{D\theta}{Dt} = \frac{\theta Q}{c_p T}, \tag{48}$$

where Q is the heating rate per unit mass. Using our assumptions that $|\delta\theta / \theta_0| << 1$ and $|\delta T / T_0| << 1$, we can immediately rewrite (48) as

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_h \cdot \nabla_z\right)\frac{\delta\theta}{\theta_0} + \frac{w}{\theta_0}\frac{\partial\theta_0}{\partial z} = \frac{Q}{c_p T_0}. \tag{49}$$

For shallow motions, a further simplification is possible whenever

$$\left|\frac{1}{\rho_0}\frac{\partial}{\partial z}(\delta p)\right| \le g\left|\frac{\delta\rho}{\rho_0}\right|. \tag{50}$$

This does not mean that the perturbations are in hydrostatic balance, but only that they are close to balance. This assumption is particularly appropriate whenever the buoyancy force plays a key role in the fluid motions, as in convection and gravity waves. Rewriting (50) as

$$\left|\frac{\delta p}{p_0}\right| \le \frac{D\rho_0 g}{p_0}\left|\frac{\delta\rho}{\rho_0}\right|, \tag{51}$$

and recognizing that

$$\frac{\rho_0 g}{p_0} = -\frac{1}{p_0}\frac{dp_0}{dz} \equiv -\frac{1}{H_p}, \tag{52}$$

we see that

$$\left|\frac{\delta p}{p_0}\right| \leq \frac{DH}{H_p} \sim \left|\frac{\delta p}{\rho_0}\right|.$$

(53)

For shallow convection, i.e., $D / H_p << 1$, we can neglect $\left|\delta p / p_0\right|$ in comparison with $\left|\delta\rho / \rho_0\right|$. Then from the state equation and the definition of θ we obtain

$$-\frac{\delta p}{\rho_0} \cong \frac{\delta T}{T_0} \cong \frac{\delta\theta}{\theta_0}.$$

(54)

From (47), the vertical pressure gradient force becomes

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g \cong -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) + g\frac{\delta T}{T_0},$$

(55)

and the first law of thermodynamics becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_H \cdot \nabla\right)\left(\frac{\delta T}{T_0}\right) + \frac{w}{T_0}\left(\frac{dT_0}{dz} + \frac{g}{c_p}\right) \cong \frac{Q}{c_p T_0},$$

(56)

where we have invoked the hydrostaticity of the reference state to write the vertical advection term in terms of T_0 rather than θ_0.

For simplicity, we have not considered the virtual temperature effect in the preceding analysis. It can be included simply by replacing θ by θ_v and T by T_v in the respective buoyancy terms of the anelastic and Boussinesq equations of motion.

Summary of the anelastic and Boussinesq systems

The *anelastic system of equations* is collected below.

Continuity:

$$\boxed{\nabla_z \cdot \left(\rho_0 \mathbf{V}_h\right) + \frac{\partial}{\partial z}\left(\rho_0 w\right) = 0}$$

(57)

Equation of motion:

$$\frac{\partial \mathbf{V}}{\partial t}+(2\mathbf{\Omega}+\nabla\times\mathbf{V})\times\mathbf{V}+\nabla K=-\nabla\left(\frac{\delta p}{\rho_0}\right)+g\mathbf{k}\frac{\delta\theta_v}{\theta_{v_0}}-\frac{\mathbf{F}}{\rho_0} \tag{58}$$

First Law of Thermodynamics:

$$\frac{D}{Dt}\left(\frac{\delta\theta}{\theta_0}\right)+\frac{w}{\theta_0}\frac{\partial\theta_0}{\partial z}=\frac{Q}{c_pT_0} \tag{59}$$

Here $\mathbf{V}$ is the *three-dimensional* velocity vector, $\mathbf{\Omega}$ is the angular velocity of the Earth's rotation, $K\equiv\frac{1}{2}|\mathbf{V}|^2$ is the kinetic energy per unit mass, and $\mathbf{F}$ is the frictional force, per unit mass. These equations are valid provided that the following conditions are met:

- All thermodynamic variables depart only slightly from their reference distributions.
- The time-scale of the motions is a few minutes or longer [see (30)].
- The aspect ratio of the motions is not too large [see (33)].
- The frequency of the motions is much less than the frequency of sound waves [see (46)]. This condition overlaps somewhat with the second condition above.

If, *in addition* to the conditions required for application of the anelastic approximation, the depth of the motions is much less than H_p, and if the motions are strongly influenced by the buoyancy force (see (51)), we can further simplify to obtain the *Boussinesq equations*, which are collected below.

Continuity:

$$\nabla_z\cdot\mathbf{V}_h+\frac{\partial w}{\partial z}=0 \tag{60}$$

Equation of Motion:

$$\frac{\partial \mathbf{V}}{\partial t}+(2\mathbf{\Omega}+\nabla\times\mathbf{V})\times\mathbf{V}+\nabla K=-\nabla\left(\frac{\delta p}{\rho_0}\right)+g\mathbf{k}\frac{\delta T_v}{T_v}-\frac{\mathbf{F}}{\rho_0} \tag{61}$$

First Law of Thermodynamics:

$$\boxed{\frac{D}{Dt}\left(\frac{\delta T}{T_0}\right)+\frac{w}{T_0}\left(\frac{dT_0}{dz}+\frac{g}{c_p}\right)=\frac{Q}{c_pT_0}}$$

(62)

The anelastic pressure equation

One of the benefits of the anelastic system is the relatively simple *diagnostic* form of the anelastic continuity equation, which lacks a time derivative term. With the full system of equations (before simplification by the scale analysis), we must predict *two* of the three independent thermodynamic state variables, e.g., ρ and θ, as in (27) and (49). The third thermodynamic variable, p, is then determined by the equation of state.

In the anelastic system, on the other hand, only *one* thermodynamic variable, θ, is predicted. The other two, ρ and p, are determined diagnostically by the equation of state, and by our requirement that the three-dimensional mass flux be nondivergent, i.e., by the anelastic continuity equation.

For comparison, note that the quasi-static system, in which the equation of vertical motion is replaced by the hydrostatic equation, also has just one prognostic thermodynamic equation, and two diagnostic thermodynamic equations, one of which is the hydrostatic equation.

To derive the equation governing the pressure, we first use the continuity equation, (57), to write the equation of motion, (58), as

$$\frac{\partial}{\partial t}\left(\rho_0\mathbf{V}\right)+\mathbf{A}=-\rho_0\nabla\left(\frac{\delta p}{\rho_0}\right),$$

(63)

where, for convenience, we define

$$\mathbf{A}\equiv-\left\{\rho_0\left[\left(2\mathbf{\Omega}+\nabla\times\mathbf{V}\right)\times\mathbf{V}+\nabla K\right]+\mathbf{V}\left[\nabla\cdot\left(\rho_0\mathbf{V}\right)\right]\right\}-2\Omega\times\rho_0\mathbf{V}+g\mathbf{k}\left(\frac{\delta\theta_v}{\theta_{v_0}}\right)-\mathbf{F}.$$

(64)

Taking the divergence of (65), and using (58), we obtain

$$\boxed{\nabla\cdot\left[\rho_0\nabla\left(\frac{\delta p}{\rho_0}\right)\right]=-\nabla\cdot\mathbf{A}},$$

(65)

which is the *anelastic pressure equation*. The physical meaning of (65) is simply that the pressure field must be whatever it takes to keep the three-dimensional mass flux non-divergent. The pressure field does "air traffic control." Eq. (65) has the form of a Poisson equation, which must be solved over the whole three-dimensional domain, using appropriate boundary conditions.

The fact that the pressure is determined diagnostically in this way means that the pressure plays only a *passive* role in the dynamics of the motions we are considering. The distribution of the pressure at a given instant is completely determined by the distributions of the other variables; the past history of the pressure itself is irrelevant.

Although the anelastic pressure equation simplifies things by taking us from two prognostic equations and one diagnostic equation (for the thermodynamic variables) to one prognostic equation and two diagnostic equations, the Poisson equation for the pressure field is inconvenient.

Jung and Arakawa (2008) show that if the three-dimensional (vector) vorticity equation is used instead of the momentum equation, then the elliptic equation corresponding to (65) governs the vertical velocity, rather than the pressure. This is advantageous because the boundary conditions on the vertical velocity are relatively straightforward.

A comparison with the quasi-static system

As mentioned in Section 1, the quasi-static approximation also filters vertically propagating sound waves, but it cannot be used to study PBL turbulence or cumulus convection because for these motions the perturbation pressure and the perturbation density are not quasi-statically balanced. If we tried to use the quasi-static system, serious errors would be introduced.

Some drawbacks

There are several problems with the anelastic and Boussinesq equations. The most fundamental weakness is that the equations have intrinsic errors on the order of a few percent for most motions, simply as a consequence of the various approximations made. We must always ask whether these errors are acceptable.

There is no guarantee that the solutions obtained with the anelastic system will actually be consistent with the assumptions made in their derivation. For example, it would be possible to obtain solutions in which the departures of the thermodynamic variables from the reference state were not fractionally small.

Lilly (1996) points out that the Boussinesq system conserves volume rather than mass. Giving up exact mass conservation should be enough to make anyone nervous.

A less obvious problem is that the classical anelastic system "leaks" energy, i.e., it does not have a conservation of energy theorem. To show this, we first dot the equation of motion (58) with the momentum vector $\rho_0 \mathbf{V}$, and use the continuity equation (57) to obtain the kinetic energy equation:

$$\frac{\delta(\rho_0 K)}{\delta t} + \nabla \cdot \left[\mathbf{V}(\rho_0 K + \delta p) \right] = g \rho_0 w \frac{\delta \theta}{\theta_0} , \tag{66}$$

For simplicity, we have neglected friction and the virtual temperature correction, which are irrelevant to the present discussion. By combining (58) and (60), we can derive

$$\frac{\partial}{\partial t}\left(P \frac{\delta \theta}{\theta_0} \right) + \nabla \cdot \left(\mathbf{V} P \frac{\delta \theta}{\theta_0} \right) = g \rho_0 w \frac{\delta \theta}{\theta_0} - P \frac{w}{\theta_0} \frac{\partial \theta_0}{\partial z} , \tag{67}$$

where we have neglected heating, for simplicity, and where

$$P \equiv \rho_0 g z \tag{68}$$

is the potential energy per unit volume. Subtracting (68) from (67), we obtain.

$$\frac{\partial}{\partial t}\left(\rho_0 K - P \frac{\delta \theta}{\theta_0} \right) + \nabla \cdot \left[\mathbf{V}\left(\rho_0 K - P \frac{\delta \theta}{\theta_0} - \delta p \right) \right] = P \frac{w}{\theta_0} \frac{\partial \theta_0}{\partial z} . \tag{69}$$

The term on the right-hand side of (69) is spurious; it is replaced by zero in a derivation that proceeds from the exact equations. It represents an infinite reservoir of energy associated with the stratification of the reference state. It can be forced to vanish by taking the reference state to be isentropic, but often this is unacceptable because it makes the departures from the reference state large.

Durran (1989), Lilly (1996), Bannon (1996), and Arakawa and Konor (2009) discuss improved anelastic systems that do conserve energy.

A final drawback to the anelastic system is that the anelastic pressure equation can only be solved through the imposition of boundary conditions that must sometimes be specified rather arbitrarily. Also, the numerical algorithms usually employed to solve the pressure equation are expensive and cumbersome.

Conclusions

The anelastic and Boussinesq equations have some useful properties, and they have been employed in many studies of PBL turbulence and cumulus convection. Their intrinsic errors, and particularly their failure to conserve total energy, make it important to proceed with caution in any application.

Improved anelastic systems have been developed, starting in the 1980s. In the future, some of these may be used in a new class of global atmospheric models.

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