Bulk Boundary-Layer Model

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Ball (1960) was the first to propose a model in which the interior of the planetary boundary layer (PBL) is well-mixed in the conservative variables, while the PBL top is marked by discontinuties in these same variables. Geisler and Kraus (1969) were the first to extend the idea by treating momentum (not a conservative variable) in the same framework, so that the PBL could be said to move as a "slab." There is now a huge literature on mixed layer models, and in almost every paper the emphasis is on the question of what determines the time rate of change of the PBL depth. Actually, many of the papers are devoted to the *ocean* mixed layer, which is a kind of upside-down PBL. More generally, any model in which the PBL depth is an explicit parameter and the vertical structure of the PBL is described with just a few degrees of freedom can be called a "bulk" PBL model.

We begin our study of bulk models by deriving the equations that govern the vertically-averaged properties of the PBL. Let A be an arbitrary intensive scalar, satisfying the "flux-form" conservation equation

$$\frac{\partial}{\partial t}(\rho A) + \nabla \cdot (\rho \mathbf{V} A) + \frac{\partial}{\partial z}(\rho w A) = -\frac{\partial F_A}{\partial z} + S_A \, , \tag{1}$$

where $F_A \equiv \rho \overline{w'A'}$ is the upward turbulent flux of A, bars are omitted on the mean quantities, and S_A is a source or sink of A, per unit volume. The corresponding continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) + \frac{\partial}{\partial z}(\rho w) = 0 \, , \tag{2}$$

which can also be written as

$$\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho + w \frac{\partial \rho}{\partial z} = -\rho \left(\nabla \cdot \mathbf{V} + \frac{\partial w}{\partial z} \right) . \tag{3}$$

By use of (2), we can rewrite (1) in the "advective form:"

$$\rho\left(\frac{\partial A}{\partial t} + \mathbf{V}\cdot\nabla A + w\frac{\partial A}{\partial z}\right) = -\frac{\partial F_A}{\partial z} + S_A \,. \tag{4}$$

Integrating (1) from just below to just above the PBL top, and using Leibniz' rule, we get

$$\cancel{\frac{\partial}{\partial t}\left(\int_{z_B-\varepsilon}^{z_B+\varepsilon}\rho A\,dz\right)} - \Delta(\rho A)\frac{\partial z_B}{\partial t} + \cancel{\nabla\cdot\left(\int_{z_B-\varepsilon}^{z_B+\varepsilon}\rho\mathbf{V}A\,dz\right)} - \Delta(\rho\mathbf{V}A)\cdot\Delta z_B + \Delta(\rho w A) = -\cancel{(F_A)_{B+}} + \int_{z_B-\varepsilon}^{z_B+\varepsilon} S_A\,dz, \tag{5}$$

where the indicated terms drop out as the domain of integration shrinks to zero and/or because all of the turbulence variables go to zero above the PBL top. Here we have used the notation $\Delta(\;)\equiv(\;)_{z=z_B+\varepsilon}-(\;)_{z=z_B-\varepsilon}\equiv(\;)_{B+}-(\;)_B$, and henceforth subscripts B+ and B denote levels just above and just below the PBL top, respectively. For $A\equiv 1$, (5) reduces to mass conservation in the form

$$\rho_{B+}\left(\frac{\partial z_B}{\partial t} + \mathbf{V}_{B+}\cdot\nabla z_B - w_{B+}\right) = \rho_B\left(\frac{\partial z_B}{\partial t} + \mathbf{V}_B\cdot\nabla z_B - w_B\right) \equiv E - M_B \,, \tag{6}$$

where $E - M_B$ is the total mass flux across the PBL top. In essence, (6) simply says that the mass flux is continuous across the PBL top, i.e., no mass is created or destroyed between levels B and B+. We interpret M_B as the mass flux due to a loss of PBL mass into cumulus clouds, and E

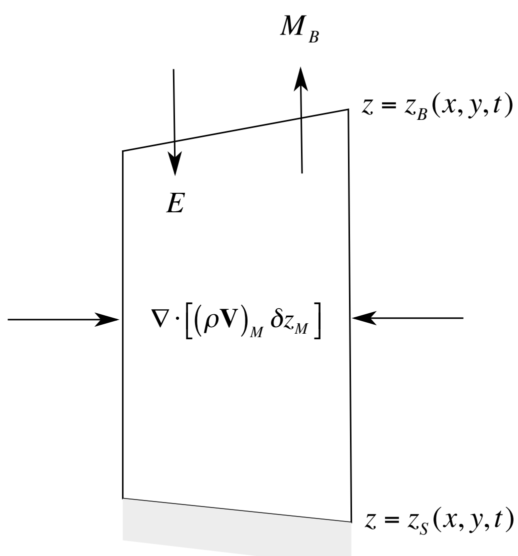

Figure 1: Sketch of a column of PBL air, with arrows indicating the lateral fluxes due to the horizontal winds, and the top-flux due to entrainment and the cumulus mass flux.

as the mass flux due to the turbulent *entrainment* of free atmospheric air into the PBL. See Fig. 1 for a sketch illustrating the physical system under consideration.

With the use of (6), we can rewrite (5) as

$$-\Delta A\left(E - M_B\right) = \left(F_A\right)_B + \int_{z_B-\varepsilon}^{z_B+\varepsilon} S_A \, dz \, , \tag{7}$$

Here we have assumed that $\left(F_A\right)_{B+} = 0$. For $S_A \equiv 0$, (7) simply says that the total flux of A must be continuous across the PBL top. Notice that for $\Delta A \neq 0$, a mass flux across the PBL top is generally associated with a turbulent flux of A at level B. This flux serves to change the A of entering particles from A_{B+} to A_B. For example, dry entrained air is moistened by an upward moisture flux which converges "discontinuously" at level B. Lilly (1968) was the first to derive (7) using the approach followed here.

Now integrate (1) through the PBL depth, from the surface to level B, to obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{z_S}^{z_B}(\rho A)\,dz - (\rho A)_B \frac{\partial z_B}{\partial t} + \\
&(\rho A)_S \frac{\partial z_S}{\partial t} + \nabla \cdot \int_{z_S}^{z_B}(\rho \mathbf{V} A)\,dz - (\rho \mathbf{V} A)_S \cdot \nabla z_S \\
&-(\rho \mathbf{V} A)_B \cdot \nabla z_B + (\rho w A)_B - (\rho w A)_S = \left(F_A\right)_S - \left(F_A\right)_B + \int_{z_S}^{z_B-\varepsilon} S_A \, dz
\end{aligned} \tag{8}$$

The condition that no mass crosses the Earth's surface can be written as

$$\rho_S\left(\frac{\partial z_S}{\partial t} + \mathbf{V}_S \cdot \nabla z_S - w_S\right) = 0 \, . \tag{9}$$

Use of (9) allows us to simplify (8) considerably, to

$$\frac{\partial}{\partial t}\left(\rho_M A_M \delta z_M\right) + \nabla \cdot \left[\rho_M \left(\mathbf{V}A\right)_M \delta z_M\right] - A_B\left(E - M_B\right) = \left(F_A\right)_S - \left(F_A\right)_M + \left(S_A\right)_M \delta_{z_M} \, . \tag{10}$$

Here we define the depth of the PBL as

$$\delta_{z_M} \equiv z_B - z_S \, . \tag{11}$$

and

$$\left(S_A\right)_M \delta_{z_M} \equiv \int_{z_S}^{z_B-\varepsilon} S_A \, dz \, . \tag{12}$$

By combining (7) with (10), we obtain

$$\frac{\partial}{\partial t}\left(\rho_M A_M \delta z_M\right) + \nabla \cdot \left[\rho_M \left(\mathbf{V}A\right)_M \delta z_M\right] - A_{B+}\left(E - M_B\right) = \left(F_A\right)_S - \left(S_A\right)_M \delta z_m + \int_{z_B-\varepsilon}^{z_B+\varepsilon} S_A \, dz \, . \tag{13}$$

For $A \equiv 1$, (10) reduces to a statement of mass conservation for the whole PBL:

$$\frac{\partial}{\partial t}\left(\rho_M \delta z_M\right) + \nabla \cdot \left[\left(\rho \mathbf{V}\right)_M \delta z_M\right] = E - M_B \, . \tag{14}$$

Again, refer to Fig. 1 for a sketch of the physical situation.

Now define a transformed vertical coordinate that follows the PBL top, given by

$$z' \equiv z - z_B\left(x, y, t\right) . \tag{15}$$

Using the methods described in the *QuickStudy* on vertical coordinate transformations, we can write

$$\left(\frac{\partial \rho}{\partial t}\right)_z = \left(\frac{\partial \rho}{\partial t}\right)_{z'} - \frac{\partial \rho}{\partial z}\frac{\partial z_B}{\partial t} , \tag{16}$$

$$\nabla \cdot \left(\rho \mathbf{V}\right) = \nabla_{z'} \cdot \left(\rho \mathbf{V}\right) - \frac{\partial}{\partial z}\left(\rho \mathbf{V}\right) \nabla z_B \, . \tag{17}$$

Substitution of (15) and (16) into (2) gives

$$\left(\frac{\partial \rho}{\partial t}\right)_{z'} + \nabla_{z'} \cdot \left(\rho \mathbf{V}\right) + \frac{\partial\left(\rho w'\right)}{\partial z} = 0 , \tag{18}$$

where we define

$$w'\left(z\right) \equiv -\left[\frac{\partial z_B}{\partial t} + \mathbf{V}\left(z\right) \cdot \nabla z_B - w\left(z\right)\right] . \tag{19}$$

Note that w' is a function of height, and that

$$\rho_B w'(z_B) = \rho_{B+} w'(z_{B+}) = -(E - M_B) .$$

(20)

Similarly, Eq. (1) can be rewritten as

$$\left[\frac{\partial(\rho A)}{\partial t}\right]_{z'} + \nabla_{z'} \cdot (\rho \mathbf{V} A) + \frac{\partial(\rho w' A)}{\partial z'} = -\frac{\partial F_A}{\partial z} + S_A .$$

(21)

By combining (17) and (20), we obtain

$$\left(\frac{\partial A}{\partial t}\right)_{z'} + \mathbf{V} \cdot \nabla_{z'} A + w' \frac{\partial A}{\partial z} = -\alpha \frac{\partial F_A}{\partial z} + S_A .$$

(22)

Eq. (21) describes the time-rate-of-change of A as seen on any surface of constant z'. One such surface is the top of the boundary layer, so we can apply (21) to determine the time rate of change of a quantity on the surface $z = z_B(x, y, t)$. Evaluating (21) at levels *B*+ and *B*, subtracting, and using (6), we find that

$$\frac{\partial(\Delta A)}{\partial t} + \Delta(\mathbf{V} \cdot \nabla_{z'} A) - (E - M_B) \Delta\left(\alpha \frac{\partial A}{\partial z}\right) = a_B \left(\frac{\partial F_A}{\partial z}\right)_B + \Delta(\alpha S_A) .$$

(23)

This governs the time rates of change of the “jumps.” In writing (22), we have assumed that the turbulent flux divergence vanishes at level *B*+.

Consider the special case of the horizontal momentum equation, i.e., $A \rightarrow \mathbf{V}$. Then we have $\alpha S_A \equiv -\alpha \nabla_z p - f\mathbf{k} \times \mathbf{V}$, where p is pressure and f is the Coriolis parameter. The gradient operator satisfies

$$\begin{aligned} -\alpha \nabla_z p &= -\alpha\left(\nabla_{z'} p - \frac{\partial p}{\partial z} \nabla z_B\right) \\ &= \alpha\left(\nabla_{z'} p + \rho g \nabla z_B\right) \\ &= -\alpha \nabla_{z'} p - g \nabla z_B . \end{aligned}$$

(24)

Here we have used hydrostatics. Applying (23) at levels *B*+ and *B*, and subtracting, we find that

$$-\Delta(\alpha \nabla_z p) = -\Delta \alpha \nabla p_B .$$

(25)

Then, corresponding to (22), we obtain

$$\frac{\partial(\Delta\mathbf{V})}{\partial t} + \Delta\left(\mathbf{V}\cdot\nabla_{z'}A\right) - \left(E - M_B\right)\nabla\left(\alpha\frac{\partial\mathbf{V}}{\partial z}\right) = \alpha_B\left(\frac{\partial\mathbf{F}_\mathbf{v}}{\partial z}\right)_B - \Delta\alpha\nabla p_B - f\mathbf{k}\times\Delta\mathbf{V}\,. \tag{26}$$

For a steady state in which advection and friction are negligible, this reduces to

$$0 = -\Delta\alpha\nabla p_B - f\mathbf{k}\times\Delta\mathbf{V}\,, \tag{27}$$

which we recognize as a form of the thermal wind equation, sometimes called Margules' equation.

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