

# Virtual Temperature and Virtual Temperature Fluxes

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## 1 Definition of virtual temperature

For dry air, the equation of state is

$$p_d = \rho_d R_d T. \quad (1)$$

Similarly, water vapor obeys its own equation of state and has its own gas constant:

$$e = \rho_v R_v T. \quad (2)$$

Here  $R_d$  and  $R_v$  are the mixing ratios of water vapor and dry air, respectively. The gas constant for dry air is approximately  $287 \text{ J kg}^{-1} \text{ K}^{-1}$ , and the gas constant for water vapor is approximately  $462 \text{ J kg}^{-1} \text{ K}^{-1}$ .

The total pressure is

$$p = p_d + e \quad (3)$$

and the total density is

$$\rho = \rho_d + \rho_v + \rho_l \quad (4)$$

From (1) - (4), we see that

$$p = (\rho_d R_d + \rho_v R_v) T \quad (5)$$

We *define* the virtual temperature as satisfying the ideal gas law with the total pressure and the total density, and the gas constant for dry air:

$$\boxed{p \equiv \rho R_d T_v.} \quad (6)$$

Substituting on both sides of (6), we obtain

$$\begin{aligned} (\rho_d + \rho_v + \rho_l) R_d T_v &= (\rho_d R_d + \rho_v R_v) T \\ &= \left( \rho_d + \rho_v \frac{R_v}{R_d} \right) R_d T, \end{aligned} \quad (7)$$

where

$$\frac{R_d}{R_v} \cong 0.622 \quad (8)$$

This leads to

$$T_v = T \left( \frac{1 + q \frac{R_v}{R_d}}{1 + q + l} \right), \quad (9)$$

where

$$q \equiv \frac{\rho_v}{\rho_d} \quad (10)$$

is the mixing ratio of water vapor, and

$$l \equiv \frac{\rho_l}{\rho_d} \quad (11)$$

is the mixing ratio of liquid water. The quantity  $T_v$  is called the virtual temperature, or sometimes the density temperature.

Eq. (9) can be approximated by

$$T_v \cong T (1 + \delta q - l), \quad (12)$$

where

$$\delta \equiv \frac{R_v - R_d}{R_d} \cong 0.608. \quad (13)$$

## 2 Buoyancy fluctuations and fluxes on isobaric surfaces

The following discussion is based on ideas developed by Lilly (1968). The virtual dry static energy is

$$s_v \equiv c_p T_v + gz. \quad (14)$$

We use the moist static energy

$$h \equiv c_p T + gz + Lq, \quad (15)$$

which is approximately conserved under both moist and dry adiabatic processes, even when precipitation is occurring. The total water mixing ratio,  $q + l$ , is also approximately conserved under both moist and dry adiabatic processes, although it is of course affected by precipitation.

Consider fluctuations *at constant pressure*, denoted by primes. We can write

$$h' = c_p T' + Lq', \quad (16)$$

Here we neglect height fluctuations on the isobaric surfaces. From (12) and (14), we see that

$$\begin{aligned} s_v' &\cong c_p T_v' \\ &\cong c_p T' + c_p \bar{T} (\delta q' - l') \\ &= c_p T' + \varepsilon (\delta Lq' - Ll') , \end{aligned} \quad (17)$$

where for convenience we define the nondimensional ratio

$$\varepsilon \equiv \frac{c_p \bar{T}}{L} \cong 0.1. \quad (18)$$

Eq. (17) can be manipulated as follows:

$$\begin{aligned} s_v' &\cong (c_p T' + Lq') - (1 - \delta\varepsilon)Lq' - \varepsilon Ll' \\ &= h' - (1 - \delta\varepsilon)L(q' + l') + [1 - (1 + \delta)\varepsilon]Ll' . \end{aligned} \quad (19)$$

The coefficients  $(1 - \delta\varepsilon)$  and  $[1 - (1 + \delta)\varepsilon]$  are both positive and nondimensional.

An expression for the virtual dry static energy flux,  $F_{sv}$ , can be obtained by multiplying (19) by  $w'$  and then averaging. (We neglect the contributions that arise from fluctuations of the leading factor of  $\rho$ .) The result is

$$\boxed{F_{sv} = F_h - (1 - \delta \varepsilon) L F_{q+l} + [1 - (1 + \delta) \varepsilon] L F_l.} \quad (20)$$

Eq. (20) is valid regardless of the cloud amount. It was used by Randall (1987). According to (20), for given values of  $F_h$  and  $F_{q+l}$ , the buoyancy flux increases as the liquid water flux increases.

Still following Lilly (1968), we consider two cases. First, if there is no cloud, then  $L F_l = 0$ , and (20) immediately reduces to

$$\boxed{F_{sv} = (F_{sv})_{clr} \equiv F_h - (1 - \delta \varepsilon) L F_{q+l} \text{ for clear air.}} \quad (21)$$

What we are going to do now is find a formula similar to (21) that holds in an “overcast,” where the fractional cloudiness is 100%. If the air is saturated everywhere, we can write

$$\gamma c_p T' \cong L q' \text{ in a uniform cloud,} \quad (22)$$

where

$$\gamma \equiv \frac{L}{c_p} \left( \frac{\partial q^*}{\partial T} \right)_p. \quad (23)$$

From (22), we get

$$\gamma h' = (1 + \gamma) L q' \text{ in a uniform cloud,} \quad (24)$$

or

$$L q' = \left( \frac{\gamma}{1 + \gamma} \right) h' \text{ in a uniform cloud,} \quad (25)$$

It follows that

$$L l' = L (q' + l') - \left( \frac{\gamma}{1 + \gamma} \right) h' \text{ in a uniform cloud,,} \quad (26)$$

which leads to

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$$\boxed{LF_l = (LF_l)_{cld} \equiv LF_{q+l} - \left(\frac{\gamma}{1+\gamma}\right) F_h \text{ in a uniform cloud.}} \quad (27)$$

Substituting (27) into (20), and collecting terms, we find that for the fully cloudy case

$$\boxed{F_{sv} = (F_{sv})_{cld} \equiv \beta F_h - \varepsilon LF_{q+l} \text{ in a uniform cloud,}} \quad (28)$$

where for convenience we define

$$\beta \equiv \frac{1 + (1 + \delta) \gamma \varepsilon}{1 + \gamma} \quad (29)$$

The notations  $(LF_l)_{cld}$  and  $(F_{sv})_{cld}$ , defined in (27) and (28) respectively, will be used below.

### 3 Buoyancy reversal

The fluxes just below the PBL top satisfy

$$(F_h)_B = -E\Delta h + \Delta R, \quad (30)$$

$$(F_h)_B = -E\Delta h + \Delta R, \quad (31)$$

$$(F_{q+l})_B = -E\Delta(q+l), \quad (32)$$

and

$$(F_{sv})_B = -E[\beta\Delta h - \varepsilon L\Delta(q+l)] + \beta\Delta R \text{ in a uniform cloud.} \quad (33)$$

We want to rewrite (33) in the form

$$\boxed{(F_{sv})_B = -E[\Delta s_v - (\Delta s_v)_{crit}] + \beta\Delta R \text{ in a uniform cloud}} \quad (34)$$

Here  $(\Delta s_v)_{crit}$  is defined as the value of  $\Delta s_v$  such that entrainment has no effect on  $(F_{sv})_B$ . For  $\Delta s_v > (\Delta s_v)_{crit}$  entrainment reduces  $(F_{sv})_B$ , which is what entrainment normally does in a clear boundary layer. But for  $\Delta s_v < (\Delta s_v)_{crit}$  entrainment increases  $(F_{sv})_B$ . This means that for  $\Delta s_v < (\Delta s_v)_{crit}$  entrainment drives convection! If the convection promotes additional entrainment, then the entrainment rate could amplify unstably.

We could use (34) for a cloud-free boundary layer, if we set  $(\Delta s_v)_{crit} = 0$  and  $\Delta R = 0$ . We can say that  $(\Delta s_v)_{crit} = 0$  for clear boundary layers.

To find an expression for  $(\Delta s_v)_{crit}$  that can be used for boundary layers that are uniformly cloudy just below the inversion, we use (17) to write

$$\begin{aligned}
 \Delta s_v &\cong \Delta s + \delta \varepsilon L \Delta q - \varepsilon L \Delta l \\
 &= \Delta s + \delta \varepsilon L \Delta q - \varepsilon L \Delta (q + l) + \varepsilon L \Delta q \\
 &= \Delta s + (1 + \delta) \varepsilon L \Delta q - \varepsilon L \Delta (q + l) \\
 &= \Delta h - [1 - (1 + \delta) \varepsilon] L \Delta q - \varepsilon L \Delta (q + l) .
 \end{aligned} \tag{35}$$

Eq. (35) is valid whether or not a cloud is present. For the special case of a uniform cloud, the water vapor jump can be written as

$$\begin{aligned}
 L \Delta q &= L (q_{B+} - q_B) \\
 &= L (q_{B+} - q_{*B}) \\
 &= L (q_{*B+} - q_{*B}) - L (q_{*B+} - q_{B+}) \\
 &= \gamma \Delta s - L (q_{*B+} - q_{B+}) \text{ for a uniform cloud.}
 \end{aligned} \tag{36}$$

From (36), we see that

$$(1 + \gamma) L \Delta q = \gamma \Delta h - L (q_{*B+} - q_{B+}) \text{ for a uniform cloud,} \tag{37}$$

or

$$L \Delta q = \frac{\gamma \Delta h - L (q_{*B+} - q_{B+})}{(1 + \gamma)} \text{ for a uniform cloud.} \tag{38}$$

Substituting (38) back into (35), we find that

$$\begin{aligned}
 L\Delta q &= L(q_{B+} - q_B) \\
 &= L(q_{B+} - q_{*B}) \\
 &= L(q_{*B+} - q_{*B}) - L(q_{*B+} - q_{B+}) \\
 &= \gamma\Delta s - L(q_{*B+} - q_{B+}) \text{ for a uniform cloud.}
 \end{aligned} \tag{39}$$

Eq. (39) can be rearranged to

$$\beta\Delta h - \varepsilon L\Delta(q+l) = \Delta s_v - \left[ \frac{1 - (1 + \delta)\varepsilon}{1 + \gamma} \right] L(q_{*B+} - q_{B+}) \text{ for a uniform cloud.} \tag{40}$$

for a uniform cloud.

Comparing (33), (34), and (39), we see that

$$\boxed{(\Delta s_v)_{crit} = \left[ \frac{1 - (1 + \delta)\varepsilon}{1 + \gamma} \right] L(q_{*B+} - q_{B+}) \text{ for a uniform cloud.}} \tag{41}$$

This result, derived by Randall (1980), shows that  $(\Delta s_v)_{crit}$  is a measure of the relative humidity of the air above cloud top. When the air is drier,  $(\Delta s_v)_{crit}$  is larger. The drier the entrained air is, the more liquid can be evaporated into it, and the stronger the evaporative cooling can be.

#### 4 The buoyancy of mixed parcels

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#### 5 Buoyancy fluxes in partly cloudy layers

Eq. (20) can be used to determine the buoyancy flux in partly cloudy layers. The following discussion is taken from Randall (1987), but with some changes in notation. We use a mass flux approach for all fluxes, i.e.,

$$F_\psi = M_c(\psi_u - \psi_d) \tag{42}$$

where  $\psi$  is a generic intensive variable,

$$M_c \equiv \rho \sigma (1 - \sigma) (w_u - w_d), \quad (43)$$

$$\bar{\psi} = \sigma \psi_u + (1 - \sigma) \psi_d, \quad (44)$$

and  $\sigma$  is the fractional area covered by rising motion. From (44), we see that

$$\psi_u = \bar{\psi} + (1 - \sigma) (\psi_u - \psi_d) \quad (45)$$

and .

$$\psi_d = \bar{\psi} - \sigma (\psi_u - \psi_d). \quad (46)$$

Eqs. (45) and (46) play an important role in the analysis below.

In order to use (20), we have to determine the flux of liquid water. The liquid water mixing ratio in the updraft is given by

$$\begin{aligned} l_u &= \text{Max} \{ (q+l)_u - (q^*)_u, 0 \} \\ &= \text{Max} \left\{ \left[ (q+l)_u - \overline{(q+l)} \right] + \left[ \overline{(q+l)} - \bar{q}^* \right] + \left[ \bar{q}^* - (q^*)_u \right], 0 \right\}, \end{aligned} \quad (47)$$

where  $\bar{q}^*$  is the saturation mixing ratio at temperature  $\bar{T}$ . The second line of (47) shows that, for a given mean state, the liquid water in the updraft depends on three things:

- the total water difference between the updraft and the mean state,  $(q+l)_u - \overline{(q+l)}$ ,
- the relative humidity of the mean state itself, as measured by  $\overline{(q+l)} - \bar{q}^*$ , and
- the difference in saturation mixing ratio between the mean state and the updraft, as measured by  $\bar{q}^* - (q^*)_u$ .

The difference in saturation mixing ratios can be written as

$$\begin{aligned} L [\bar{q}^* - (q^*)_u] &= \left( \frac{\gamma}{1+\gamma} \right) [\bar{h}^* - (h^*)_u] \\ &= \left( \frac{\gamma}{1+\gamma} \right) [(\bar{h}^* - \bar{h}) - (h_u - \bar{h})] . \end{aligned} \quad (48)$$

Here we have replaced  $h_u^*$  by  $h_u$ , which is only correct when the updraft is saturated. Substituting (48) into (47), we obtain

$$Ll_u = \text{Max} \left\{ L\tilde{l} + L[(q+l)_u - \overline{(q+l)}] - \left( \frac{\gamma}{1+\gamma} \right) (h_u - \bar{h}), 0 \right\} , \quad (49)$$

where

$$L\tilde{l} \equiv L[\overline{(q+l)} - \bar{q}^*] - \left( \frac{\gamma}{1+\gamma} \right) (\bar{h}^* - \bar{h}) \quad (50)$$

is a property of the mean state. Finally, we use (45) to rewrite (49) as

$$Ll_u = \text{Max} \left\{ L\tilde{l} + (1 - \sigma)L[(q+l)_u - (q+l)_d] - \left( \frac{\gamma}{1+\gamma} \right) (1 - \sigma)(h_u - h_d), 0 \right\} \quad (51)$$

In a similar way, we can show that

$$Ll_d = \text{Max} \left\{ L\tilde{l} - \sigma L[(q+l)_u - (q+l)_d] + \left( \frac{\gamma}{1+\gamma} \right) \sigma (h_u - h_d), 0 \right\} . \quad (52)$$

To interpret the meaning of  $L\tilde{l}$ , consider some particular cases. Inspection of (50) shows that

$$L\bar{l} = L\tilde{l} \text{ when both updraft and downdraft are saturated.} \quad (53)$$

This means that when both the updraft and downdraft are saturated  $\tilde{l}$  is simply equal to the mean liquid water mixing ratio. A similar conclusion can be drawn directly from (50): When the mean state is saturated, we have  $\overline{(q+l)} - \bar{q}^* = \bar{l}$  and  $\bar{h}^* - \bar{h} = 0$ , so that again (50) reduces to  $L\bar{l} = L\tilde{l}$ . If only the updraft is saturated, it is possible (even likely) for  $\tilde{l}$

to be negative. In that case,  $\tilde{l}$  can be interpreted as the amount of liquid water that would have to be isobarically evaporated into the mean state in order to bring it to saturation. In general,  $\tilde{l}$  is a measure of the relative humidity of the mean state.

We now see that Eq. (51) expresses the liquid water content of the updraft in terms of a measure of the relative humidity of the mean state, namely  $\tilde{l}$ , and the updraft-downdraft differences in  $q+l$  and  $h$ , weighted by  $1-\sigma$ . If the updraft-downdraft differences are set to zero, then (51) reduces to  $l_u = \tilde{l} \equiv \bar{l}$ .

Similar comments apply to (52).

According to (51),  $l_u$  increases as  $\sigma$  decreases, for a given mean state and given updraft-downdraft differences in  $q+l$  and  $h$ . Why should that be true? From (45), we see that for a generic variable

$$\psi_u - \bar{\psi} = - \left( \frac{1-\sigma}{\sigma} \right) (\psi_d - \bar{\psi}). \quad (54)$$

This means that as  $\sigma$  decreases towards zero, the updraft properties become increasingly different from those of the mean, and of course the downdraft properties become closer to those of the mean. For a given mean state and given updraft-downdraft differences in  $h$  and  $q+l$ , the wettest possible updraft is obtained in the limit as  $\sigma \rightarrow 0$ .

Now define

$$L(F_l)_u \equiv M_c L l_u \geq 0, \quad (55)$$

and

$$L(F_l)_d \equiv -M_c L l_d \leq 0. \quad (56)$$

With these definitions, the total liquid water flux can be expressed as

$$\begin{aligned} LF_l &= M_c L (l_u - l_d) \\ &= L(F_l)_u + L(F_l)_d. \end{aligned} \quad (57)$$

Using (51), we can now write

$$\begin{aligned}
 L(F_l)_u &= M_c L l_u \\
 &= M_c \text{Max} \left\{ \tilde{L} + (1 - \sigma) L [(q+l)_u - (q+l)_d] - (1 - \sigma) \left( \frac{\gamma}{1 + \gamma} \right) (h_u - h_d), 0 \right\} \\
 &= \text{Max} \left\{ M_c \tilde{L} + (1 - \sigma) L(F_l)_{cld}, 0 \right\}.
 \end{aligned} \tag{58}$$

Here we have used the notation defined in (27). Similarly, we find that

$$\begin{aligned}
 L(F_l)_d &\equiv -M_c L l_d \\
 &= -M_c \text{Max} \left\{ \tilde{L} - \sigma L [(q+l)_u - (q+l)_d] + \left( \frac{\gamma}{1 + \gamma} \right) \sigma (h_u - h_d), 0 \right\} \\
 &= -\text{Max} \left\{ M_c \tilde{L} - \sigma (L F_l)_{cld}, 0 \right\}.
 \end{aligned} \tag{59}$$

Suppose now that the updraft contains liquid water but the downdraft does not. In that case, the total liquid water flux satisfies

$$\boxed{L(F_l) = M_c \tilde{L} + (1 - \sigma) (L F_l)_{cld} \text{ for partly cloudy layers.}} \tag{60}$$

Recall that  $(L F_l)_{cld}$  is the liquid water flux in a uniformly cloudy layer. It is therefore surprising to see that is weighted, in (56), by  $1 - \sigma$ , which is the fraction of the area that is *not* cloudy.

Substituting (60) into (20), we obtain a similarly strange result:

$$\begin{aligned}
 F_{sv} &= F_h - (1 - \delta \varepsilon) L F_{q+l} + [1 - (1 + \delta) \varepsilon] \left[ M_c \tilde{L} + (1 - \sigma) (L F_l)_{cld} \right] \\
 &= \sigma \left[ F_h - (1 - \delta \varepsilon) L F_{q+l} \right] \\
 &\quad + (1 - \sigma) \left\{ F_h - (1 - \delta \varepsilon) L F_{q+l} + [1 - (1 + \delta) \varepsilon] (L F_l)_{cld} \right\} \\
 &\quad + [1 - (1 + \delta) \varepsilon] M_c \tilde{L},
 \end{aligned} \tag{61}$$

or

$$\boxed{F_{sv} = \sigma (F_{sv})_{clr} + (1 - \sigma) (F_{sv})_{cld} + [1 - (1 + \delta) \varepsilon] M_c \tilde{L} \text{ in partly cloudy layers.}} \tag{62}$$

## Virtual Temperature and Virtual Temperature Fluxes

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According to (62), the total buoyancy flux in a partly cloudy layer involves a weighted sum of and , as might be expected (e.g., Sommeria and Deardorff 1977), but the weights are “backward,” in that  $(F_{sv})_{clr}$  is weighted by the cloud fraction, and  $(F_{sv})_{cld}$  is weighted by one minus the cloud fraction. When I first derived (62), I thought that I had made an algebra mistake. The term involving  $\tilde{l}$  is expected to be negative in most cases.

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