Vertical Coordinate Transformations

David Randall

Consider two vertical coordinates, denoted by z and ζ, respectively. Although the "z" symbol suggests height, no such implication is intended here; z and ζ can be any variables at all, so long as they vary monotonically with height.

Suppose that we have a rule telling how to compute ζ for a given value of z, and vice versa. For example, we might define $\zeta \equiv z - z_S(x,y,t)$, where $z_S(x,y,t)$ is the distribution of z along the Earth's surface.

Consider the variation of an arbitrary dependent variable, A, with the independent

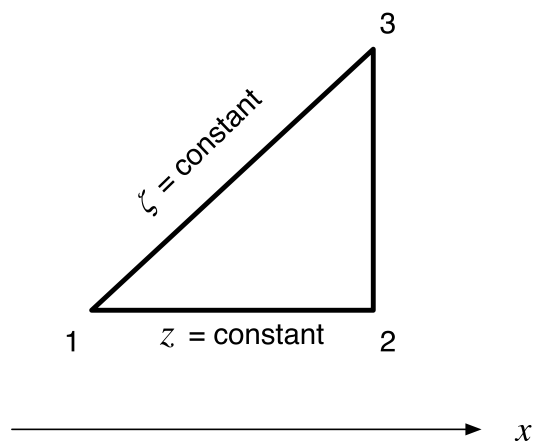

Figure 1: Sketch used to derive the rule relating derivatives on surfaces of constant z to those on surfaces of constant ζ.

variable x, as sketched in Fig. 1. Our goal is to relate $\left(\frac{\partial A}{\partial x}\right)_\zeta$ to $\left(\frac{\partial A}{\partial x}\right)_z$. With reference to Fig. 1, we can write

$$\begin{aligned}\frac{A_2 - A_1}{x_2 - x_1} &= \frac{A_3 - A_1}{x_3 - x_1} - \left(\frac{A_3 - A_2}{x_3 - x_1} \right) \\ &= \frac{A_3 - A_1}{x_3 - x_1} - \left(\frac{z_3 - z_2}{x_3 - x_1} \right)\left(\frac{A_3 - A_2}{z_3 - z_2} \right) \\ &= \frac{A_3 - A_1}{x_3 - x_1} - \left(\frac{z_3 - z_1}{x_3 - x_1} \right)\left(\frac{A_3 - A_2}{z_3 - z_2} \right) .\end{aligned} \tag{1}$$

Taking the limit as the increments become small, we obtain

$$\boxed{\left(\frac{\partial A}{\partial x} \right)_z = \left(\frac{\partial A}{\partial x} \right)_\zeta - \left(\frac{\partial z}{\partial x} \right)_\zeta \left(\frac{\partial A}{\partial z} \right)_x} . \tag{2}$$

Naturally, the derivation above works in exactly the same way if the independent variable is time, rather than a horizontal coordinate.

Starting from (2), we can show that the horizontal gradient satisfies

$$\boxed{\nabla_z A = \nabla_\zeta A - \nabla_\zeta z \left(\frac{\partial A}{\partial z} \right)_x} . \tag{3}$$

Analogous identities apply with other operators. For example, for an arbitrary horizontal vector $\mathbf{V}$, we can write

$$\boxed{\nabla_z \times \mathbf{V} = \nabla_\zeta \times \mathbf{V} - \nabla_\zeta z \times \left(\frac{\partial \mathbf{V}}{\partial z} \right)_{x,y}} , \tag{4}$$

and

$$\boxed{\nabla_z \cdot \mathbf{V} = \nabla_\zeta \cdot \mathbf{V} - \nabla_\zeta z \cdot \left(\frac{\partial \mathbf{V}}{\partial z} \right)_{x,y}} \tag{5}$$

In the example suggested earlier, with $\zeta \equiv z - z_S(x,y,t)$, Eq. (3) reduces to

$$\nabla_z A = \nabla_\zeta A - \left(\nabla z_S \right) \left(\frac{\partial A}{\partial z} \right)_x . \tag{6}$$

As a second example, let z be pressure and ζ be height, and let A be the geopotential, denoted by ϕ. Then (3) becomes

$$\begin{aligned}\nabla_p \phi &= \cancel{\nabla_z \phi} - \nabla_z p \frac{\partial \phi}{\partial p} \\ &= -\nabla_z p \frac{\partial \phi}{\partial p} \\ &\cong \alpha \nabla_z p .\end{aligned} \tag{7}$$

The third line follows in the hydrostatic limit, where $\alpha = RT / p$ is the specific volume. As a third example, again using hydrostatics, we can write

$$\begin{aligned}\nabla_p \phi &= \nabla_\theta \phi - \frac{\partial \phi}{\partial p} \nabla_\theta p \\ &\cong \nabla_\theta \phi + \alpha \nabla_\theta p \\ &= \nabla_\theta \phi + \frac{RT}{p} \nabla_\theta p .\end{aligned} \tag{8}$$

By logarithmic differentiation of

$$T = \theta \left(\frac{p}{p_0} \right)^{R/c_p} , \tag{9}$$

we obtain

$$\frac{\nabla_\theta p}{p} = \frac{c_p}{R} \frac{\nabla_\theta T}{T} . \tag{10}$$

Use of (10) in (8) gives

$$\nabla_p \phi = \nabla_\theta \left(c_p T + \phi \right) . \tag{11}$$

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Check the sign in (4)

In Cartesian coordinates (x, y), the curl of a horizontal vector (u, v) is given by

$$\mathbf{k}\cdot\nabla_z \times \mathbf{V} = \left(\frac{\partial v}{\partial x}\right)_z - \left(\frac{\partial u}{\partial y}\right)_z \tag{12}$$

Using (2), we can rewrite (11) as

$$\begin{aligned}\mathbf{k}\cdot\nabla_z \times \mathbf{V} &= \left[\left(\frac{\partial v}{\partial x}\right)_\zeta - \left(\frac{\partial v}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_\zeta\right] - \left[\left(\frac{\partial u}{\partial y}\right)_\zeta - \left(\frac{\partial u}{\partial z}\right)_x \left(\frac{\partial z}{\partial y}\right)_\zeta\right] \\ &= \left[\left(\frac{\partial v}{\partial x}\right)_\zeta - \left(\frac{\partial u}{\partial y}\right)_\zeta\right] - \left[\left(\frac{\partial v}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_\zeta - \left(\frac{\partial u}{\partial z}\right)_x \left(\frac{\partial z}{\partial y}\right)_\zeta\right] \\ &= \left(\mathbf{k}\cdot\nabla_\zeta \times \mathbf{V}\right) + \mathbf{k}\cdot\left[\left(\frac{\partial \mathbf{V}}{\partial z}\right)_{x,y} \times \nabla_\zeta z\right].\end{aligned} \tag{13}$$

We know that

$$\mathbf{k}\cdot\left[\left(\frac{\partial \mathbf{V}}{\partial z}\right)_{x,y} \times \nabla_\zeta z\right] = \left(\frac{\partial u}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial y}\right)_\zeta - \left(\frac{\partial v}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial x}\right)_\zeta , \tag{14}$$

so we conclude that

$$\mathbf{k}\cdot\nabla_z \times \mathbf{V} = \left(\mathbf{k}\cdot\nabla_\zeta \times \mathbf{V}\right) + \mathbf{k}\cdot\left[\left(\frac{\partial \mathbf{V}}{\partial z}\right)_{x,y} \times \nabla_\zeta z\right], \tag{15}$$

or

$$\nabla_z \times \mathbf{V} = \nabla_\zeta \times \mathbf{V} - \left[\nabla_\zeta z \times \left(\frac{\partial \mathbf{V}}{\partial z}\right)_{x,y}\right] \tag{16}$$

This is consistent with (4).