

Empirical Orthogonal Functions

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Lorenz (1956) invented empirical orthogonal functions (EOFs) because he saw that they could be of use in statistical forecasting; EOFs were also invented, independently, by statisticians. Amazingly, Lorenz never published his EOF study in a journal. The goal of his study was to find a way to extract a compact or simplified but "optimal" representation of data with both space and time dependence, e.g. a time-sequence of sea-level pressure maps. His approach was to expand the data in terms of optimally defined functions of space, each of which is associated with a time-dependent "amplitude."

Consider M variables $p_m(t)$, which might represent the pressures at M stations as functions of time. Let these be observed at N times, $t_1, t_2, t_3, \ldots t_N$. Expand $p_m(t_i)$ as follows:

$$p_m(t_i) = \sum_{k=1}^{M} Y_{km} Q_k(t_i) . \tag{11.1}$$

Here the Y_{km} are unknown time-independent basis functions, which will be the EOFs, and the $Q_k(t_i)$ are unknown time-dependent coefficients or amplitudes. The total number of Y's is the same as the total number of stations, because the spatial information is contained in the Y's. If the sum in (11.1) is taken over all of the Y's, then we recover the input field, with no loss of information.

Suppose, however, that we truncate the series:

$$p_m^K(t_i) = \sum_{k=1}^{K} Y_{km} Q_k(t_i) + r_m^K(t_i) , \tag{11.2}$$

where $K < M$, and $r_m^K(t_i)$ is the error associated with the truncation. We would like to choose Y_{km} and $Q_k(t_i)$ in such a way that

$$R^K \equiv \sum_{m=1}^{M} (r_m^K)^2 \tag{11.3}$$

is minimized for a given K. Lorenz (1956) shows that R^K is minimized if we choose Y_{km} and $Q_k(t_i)$ so that

$$\sum_{m=1}^{M} Y_{km} Y_{jm} = \delta_{kj} \equiv \begin{cases} 1 \text{ if } k = j \\ 0 \text{ if } k \neq j \end{cases} \tag{11.4}$$

and

$$N\overline{Q_k^* Q_j^*} = a_k \delta_{kj} , \tag{11.5}$$

where $a_k \geq a_{k+1} \geq 0$. In (5), $\overline{(\ )}$ denotes a time average, and $(\ )^*$ denotes a departure from the time average. The meaning of (11.4) is that the EOFs are orthogonal in space. The meaning of (11.5) is that the amplitudes of the EOFs are orthogonal in time.

It is this orthogonality in both space and time that makes the EOFs an “optimal” representation of the data. The fact that R^K is minimized demonstrates this, but there is another way to see it. Suppose that we have chosen a first or “lowest-order” basis function to represent the spatial structure of our data, and that we now wish to make the best possible choice of a second basis function. Clearly the *worst* possible choice would be to make the second basis function the same as the first, because in that case the second function would contribute no additional information beyond what was already available in the first. This suggests that the second basis function should be “as different as possible” from the first; more precisely, the second function should be spatially uncorrelated with the first, and this is equivalent to the requirement of spatial orthogonality. Extending this reasoning, it is clear that a set of K basis functions should be chosen so that each is spatially orthogonal to each of the others. Similarly, the time-dependent amplitudes of the EOFs should be temporally orthogonal, to ensure that each new coefficient (with its EOF) contribute as much new information as possible.

A method to solve for Y_{km} and $Q_k(t)$ is as follows. First, we introduce some matrix notation. Let P, P^*, Q, and Q^* be matrices of N rows and M columns whose elements are $p_m(t_i)$, $p^*{}_m(t_i)$, $Q_k(t_i)$, and $Q^*{}_k(t_i)$, respectively. Let Y be a square matrix of order M whose elements are Y_{kj}. Then (11.1) can be rewritten as

$$P = QY , \tag{11.6}$$

and (11.4) and (11.5) become

$$YY^T = I\,, \tag{11.7}$$

$$Q^{*T}Q^* = D\,, \tag{11.8}$$

where $(\quad)^T$ denotes the transpose, I is the identity matrix, and D is a matrix whose nondiagonal elements vanish, and whose diagonal elements are a_k/N, as given by (11.5). It should be clear that (11.6)-(11.8) are merely restatements of (11.1), (11.4), and (11.5), respectively, using matrix notation.

From (11.6), we see that

$$Q = PY^T \tag{11.9}$$

Define

$$A \equiv P^{*T}P^* \tag{11.10}$$

so that the elements of A are proportional to the covariances of the $p_m(t_i)$. From (11.6), we have

$$P^{*T}P^* = (Q^*Y^*)^T(Q^*Y^*) \tag{11.11}$$

so, using (11.7) and (11.8),

$$\begin{aligned} Y(P^{*T}P^*)Y^T &= Y[(Q^*Y^*)^T(Q^*Y^*)]Y^T \\ &= Q^{*T}Q^* \\ &= D \end{aligned} \tag{11.12}$$

or, using (11.10),

$$YAY^T = D \tag{11.13}$$

From (11.7) and (11.13) we can solve for Y and D, since A is known from (11.10). This is a standard "eigenvalue-eigenvector" problem. Once Y is known, we can use (11.9) to find Q.

The p's do not have to be confined to one level, and they can even encompass more than one physical variable, e.g. both the temperature at 500 mb and the surface pressure. In such a case, the Y's are called "extended EOFs."

References and Bibliography

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