An Introduction to the Energetics of Moving Air

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Conservation of kinetic energy and potential energy

As you probably already know, the kinetic energy equation can be derived from the equation of motion. It is easiest to start from the full three-dimensional equation of motion in the form

$$\frac{\partial \mathbf{V}}{\partial t} + (\nabla \times \mathbf{V} + 2\boldsymbol{\Omega}) \times \mathbf{V} + \nabla(K + \phi) = -\alpha\nabla p - \alpha\nabla \cdot \mathbf{F},$$

(1)

where $\mathbf{V}$ is the wind vector, $\boldsymbol{\Omega}$ is the angular velocity of the Earth's rotation,

$$K \equiv \frac{1}{2}\mathbf{V} \cdot \mathbf{V}$$

(2)

is the kinetic energy per unit mass, ϕ is the "geopotential" that includes both gravity and the centrifugal acceleration associated with the Earth's rotation, α is the specific volume, p is pressure, and $\mathbf{F}$ is the frictional stress tensor, which will be discussed below. In (1), we are using height coordinates. Dotting (1) with $\mathbf{V}$, we find that

$$\frac{\partial K}{\partial t} + \mathbf{V} \cdot \nabla(K + \phi) = -\alpha\mathbf{V} \cdot \nabla p - \alpha\mathbf{V} \cdot \nabla \cdot \mathbf{F}.$$

(3)

When we use height coordinates, ϕ is independent of time and horizontal position. This allows us to write "the potential energy equation" as

$$\frac{D\phi}{Dt} = \mathbf{V} \cdot \nabla\phi = wg.$$

(4)

Combining (3) and (4), we obtain

$$\frac{D(K+\phi)}{Dt} = -\alpha\mathbf{V}\cdot\nabla p - \alpha\mathbf{V}\cdot(\nabla\cdot\mathbf{F}).$$

(5)

We refer to $K+\phi$ as the mechanical energy per unit mass. Eq. (5) is sometimes called the mechanical energy equation.

In (5), the rate at which work is done by the pressure force, per unit mass, is represented by $-\alpha\mathbf{V}\cdot\nabla p$. This expression can be rewritten as follows:

$$\begin{aligned} -\alpha\mathbf{V}\cdot\nabla p &= -\alpha\nabla\cdot(p\mathbf{V}) + \alpha(p\nabla\cdot\mathbf{V}) \\ &= -\alpha\nabla\cdot(p\mathbf{V}) + p\frac{D\alpha}{Dt}. \end{aligned}$$

(6)

On the second line of (6), we have used the continuity equation to eliminate $\alpha\nabla\cdot\mathbf{V}$. The $\nabla\cdot(p\mathbf{V})$ term of (6) has the form of a flux divergence. It represents a spatial redistribution of energy by the pressure force. The $p\dfrac{D\alpha}{Dt}$ term represents the work done by volume expansion (analogous to the work done by inflating a balloon). We refer to $p\dfrac{D\alpha}{Dt}$ as the "expansion-work" term.

Similarly, the friction term of (5) can be expanded to reveal two physically distinct parts, as follows:

$$-\alpha\mathbf{V}\cdot(\nabla\cdot\mathbf{F}) = -\alpha\nabla\cdot(\mathbf{F}\cdot\mathbf{V}) - \delta,$$

(7)

where

$$\delta \equiv -\alpha(\mathbf{F}\cdot\nabla)\cdot\mathbf{V}$$

(8)

is the rate of kinetic energy *dissipation* per unit volume. The quantity $\nabla\cdot(\mathbf{F}\cdot\mathbf{V})$ in (7) has the form of a flux divergence. It represents a spatial redistribution of kinetic energy as friction (represented by $\mathbf{F}$) causes air parcels to do work on each other. Because this is just a spatial redistribution of energy, it does not change the total amount of kinetic energy in the atmosphere, except where friction does work on the lower boundary. In contrast, kinetic energy dissipation, δ, is a *true sink of kinetic energy*. It is shown below that

$$\delta \geq 0.$$

(9)

As discussed later, the dissipation of kinetic energy appears as a source of thermodynamic energy, i.e. as "frictional heating." It is a weak but persistent source of internal energy for the atmosphere.

At this point, I am conflicted. I want to explain enough about the effects of friction so that the preceding discussion is understandable, but I don't think that it is worth a major digression. Here, therefore, is a *small* digression: For simplicity, consider a Cartesian coordinate system that is applied in some small volume of the atmosphere. The coordinates will be named (x,y,z), and the corresponding velocity components will be named (u,v,w). The stress tensor can be written schematically as a matrix:

$$\mathbf{F} = \begin{bmatrix} 0 & F_{v,x} & F_{w,x} \\ F_{u,y} & 0 & F_{w,y} \\ F_{u,z} & F_{v,z} & 0 \end{bmatrix}.$$

(10)

As an example, $F_{u,y}$ is the flux of u-momentum in the y-direction. The diagonal elements of the matrix are set to zero because they would represent "normal stresses" (e.g., the pressure), and we want to consider only shearing stresses.

It can be shown that the stress tensor has to be symmetric about its diagonal, i.e., $F_{v,x} = F_{u,y}$, $F_{w,x} = F_{u,z}$, and $F_{w,y} = F_{v,z}$. The reason is that, if the tensor was not symmetric, the stresses would exert a finite torque on an infinitesimal air particle. Invoking this symmetry, we can rewrite (10) as

$$\mathbf{F} = \begin{bmatrix} 0 & F_{u,y} & F_{u,z} \\ F_{u,y} & 0 & F_{v,z} \\ F_{u,z} & F_{v,z} & 0 \end{bmatrix}.$$

(11)

The divergence can be written as a "row" vector, i.e.,

$$\nabla \cdot = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{bmatrix},$$

(12)

and so we find that

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial F_{u,y}}{\partial y} + \frac{\partial F_{u,z}}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_{u,y}}{\partial x} + \frac{\partial F_{v,z}}{\partial z} \right) \mathbf{j} + \left(\frac{\partial F_{u,z}}{\partial x} + \frac{\partial F_{v,z}}{\partial y} \right) \mathbf{k},$$

(13)

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors in the x, y, and z directions, respectively. Here you can see how it happens that the divergence of the tensor is a vector. Similarly,

$$\mathbf{F} \cdot \mathbf{V} = \left(F_{u,y}v + F_{u,z}w\right)\mathbf{i} + \left(F_{u,y}u + F_{v,z}w\right)\mathbf{j} + \left(F_{u,z}u + F_{v,z}v\right)\mathbf{k}$$

(14)

is the (vector) flux of kinetic energy due to work done by friction on the air "next door." For example, $F_{u,z}u$ is the energy exchange in the z-direction (hence, multiplied by $\mathbf{k}$) due to the work done as u-momentum is transferred in the z-direction by friction. The energy flux divergence is then

$$\nabla \cdot (\mathbf{F} \cdot \mathbf{V}) = \frac{\partial}{\partial x}\left(F_{u,y}v + F_{u,z}w\right) + \frac{\partial}{\partial y}\left(F_{u,y}u + F_{v,z}w\right) + \frac{\partial}{\partial z}\left(F_{u,z}u + F_{v,z}v\right).$$

(15)

The dissipation rate can be constructed as follows: We can write

$$\mathbf{F} \cdot \nabla = \begin{bmatrix} 0 & F_{u,y} & F_{u,z} \\ F_{u,y} & 0 & F_{v,z} \\ F_{u,z} & F_{v,z} & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix}$$

$$= \left[\left(F_{u,y}\frac{\partial}{\partial y} + F_{u,z}\frac{\partial}{\partial z}\right)\mathbf{i} \quad \left(F_{u,y}\frac{\partial}{\partial x} + F_{v,z}\frac{\partial}{\partial z}\right)\mathbf{j} \quad \left(F_{u,z}\frac{\partial}{\partial x} + F_{v,z}\frac{\partial}{\partial y}\right)\mathbf{k} \right].$$

(16)

Therefore

$$(\mathbf{F} \cdot \nabla) \cdot \mathbf{V} = \left(F_{u,y}\frac{\partial}{\partial y} + F_{u,z}\frac{\partial}{\partial z}\right)u + \left(F_{u,y}\frac{\partial}{\partial x} + F_{v,z}\frac{\partial}{\partial z}\right)v + \left(F_{u,z}\frac{\partial}{\partial x} + F_{v,z}\frac{\partial}{\partial y}\right)w$$

$$= F_{u,y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + F_{u,z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + F_{v,z}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right).$$

(17)

At this point, we have to introduce specific expressions for the stresses. Air is an example of a "Newtonian fluid," for which the molecular shear stress is related to the spatial derivatives of the motion (in particular, the "strain") by

$$F_{v,x} = F_{u,y} = -\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

(18)

$$F_{w,x} = F_{u,z} = -\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),$$

(19)

$$F_{w,y} = F_{v,z} = -\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),$$

(20)

where μ is the (positive, constant) molecular viscosity. Eq. (18)-(20) are called "stress-strain relationships." As an example, (19) shows that an upward increase of u will tend to favor a negative (downward) momentum flux, denoted by $F_{u,z}$. The flow of momentum is from "fast" to "slow," thus tending to homogenize the momentum over time. Such a flux is called "down-gradient." Substituting (18)-(20) into (17), we find that

$$(\mathbf{F}\cdot\nabla)\cdot\mathbf{V} = -\mu\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2\right],$$

(21)

which establishes (9). This ends our digression.

Substitution of (6) and (7) into (5) gives

$$\frac{D(K+\phi)}{Dt} = -\alpha\nabla\cdot(p\mathbf{V} + \mathbf{F}\cdot\mathbf{V}) + p\frac{D\alpha}{Dt} - \delta.$$

(22)

Using continuity, (22) can be rewritten in flux form:

$$\frac{\partial}{\partial t}\left[\rho(K+\phi)\right] + \nabla\cdot\left[\rho\mathbf{V}(K+\phi) + p\mathbf{V} + \mathbf{F}\cdot\mathbf{V}\right] = \rho p\frac{D\alpha}{Dt} - \rho\delta.$$

(23)

All of the contributions to $\frac{\partial}{\partial t}\left[\rho(K+\phi)\right]$ that appear on the left-hand side of (23), inside the $\nabla\cdot$, represent transport processes, which merely redistribute energy in space. In contrast, the expansion-work term, $p\frac{D\alpha}{Dt}$, need not integrate to zero. At a given place and time, $p\frac{D\alpha}{Dt}$ can be either positive or negative. Recall, however, that the dissipation term is always a sink. It follows that, *in an average over the whole atmosphere, and over time, the $p\dfrac{D\alpha}{Dt}$ term must be positive*, i.e., it has to act as a source of mechanical energy:

$$\int_V p \overline{\frac{D\alpha}{Dt}} \rho dV = \int_V \overline{\delta\rho} dV \geq 0 \ .$$

(24)

Here the integral is taken over the entire mass of the atmosphere, and an overbar represents a time average. Eq. (24) is a very fundamental result. It means that on the average the pressure force must do positive expansion work to compensate for the dissipation of kinetic energy. In order that $\int_V p \frac{D\alpha}{Dt} \rho dV$ be positive, expansion has to take place, in an average sense, at a higher pressure than compression. For example, we can have expansion in the lower troposphere, and compression in the upper troposphere.

Given that, on the average, the expansion work term of (23) must act as a source of mechanical energy, we should ask where this energy comes from. The answer is that it comes from the thermodynamic energy of the atmosphere. This will be demonstrated later. Expansion work represents an energy conversion process, which can have either sign locally but is positive when averaged over the whole atmosphere and over time.

Similarly, given that the dissipation term of (23) represents a sink of mechanical energy, we should ask where the energy goes. The answer is that it appears as a source of thermodynamic energy. Dissipation is, therefore, another energy conversion process -- a conversion that runs in only one direction.

The mechanical energy generation term can be written in another form. To see this, note that

$$p \frac{D\alpha}{Dt} = \frac{D}{Dt}(p\alpha) - \alpha \frac{Dp}{Dt}$$
$$= \frac{D}{Dt}(RT) - \omega\alpha \ .$$

(25)

This shows that in a time average over the whole atmosphere the expansion-work term is closely related to the product $\omega\alpha$. We can interpret $\omega\alpha$ as a rate of conversion between mechanical and thermodynamic energy. Substituting (25) into (22), we obtain

$$\frac{D(K+\phi-RT)}{Dt} = -\alpha\nabla\cdot(p\mathbf{V}+\mathbf{F}\cdot\mathbf{V}) - \omega\alpha - \delta \ .$$

(26)

This is somewhat easier to interpret when we convert to flux form:

$$\frac{\partial}{\partial t}[\rho(K+\phi)] + \nabla\cdot[\rho\mathbf{V}(K+\phi)+\mathbf{F}\cdot\mathbf{V}] = -\rho(\omega\alpha) - \rho\delta + \frac{\partial p}{\partial t} \ .$$

(27)

Note that the pressure-work term of (23), involving $\nabla \cdot (p\mathbf{V})$, has disappeared via a cancellation, but "in its place" we pick up a new term involving the local time-rate-of-change of the pressure, $\dfrac{\partial p}{\partial t}$. In an average over the whole atmosphere, and over time, the $-\omega\alpha$ term of (27) must be positive, i.e., it must act as a source of mechanical energy:

$$-\int_V \overline{\omega\alpha\rho}dV = \int_V \overline{\delta\rho}dV \ge 0 .$$

(28)

Comparing (24) and (28), we see that $-\int_V \overline{\omega\alpha\rho}dV = \int_V \overline{p\dfrac{D\alpha}{Dt}\rho}dV$.

Conservation of thermodynamic energy

The internal energy of a perfect gas is given by

$$e = c_v T ,$$

(29)

where c_v, the specific heat of air at constant volume, is a constant. For dry air, $c_v = 5R/2 \cong 713$ J kg^{-1} K^{-1}. More generally, the internal energy also includes the latent heat associated with the potential condensation of water vapor, and we find that for moist air

$$e \cong c_v T + Lq_v .$$

(30)

This equation is approximate because we have neglected the specific heat of the water vapor, as well as the specific heat of any liquid (or ice) that might be present. In atmospheric science we frequently define the internal energy as the internal energy of dry air, and treat the latent heat as an "external" source or sink of internal energy. This is not strictly correct, but does no harm in most applications. We will follow this convention in this course, i.e., we will define the internal energy per unit mass by (29).

When thermodynamic energy is added to a system, the energy input equals the sum of the work done and the change in the internal energy:

$$c_v \frac{DT}{Dt} + p\frac{D\alpha}{Dt} = -\alpha\nabla \cdot (\mathbf{R} + \mathbf{F}_S) + LC + \delta .$$

(31)

Here e is given by (29); $\mathbf{F}_S$ is the vector flux of internal energy due to molecular diffusion; $\mathbf{R}$ is the vector flux of energy due to radiation[1], L is the latent heat of water vapor, and C is the rate

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1 This notation conflicts with that used for the gas constant, but there should be little chance of confusion.

of condensation per unit mass. Note that the dissipation rate appears here as a source of internal energy. Equation (31) is a statement of the conservation of thermodynamic energy[2], applied to a moving particle.

An alternative statement of the conservation of thermodynamic energy, obtained using (25) in (31), is

$$c_p \frac{DT}{Dt} = \omega\alpha - \alpha\nabla\cdot\left(\mathbf{R}+\mathbf{F}_S\right) + LC + \delta,$$

(32)

where

$$c_p = R + c_v \cong 1000 \; \text{J kg}^{-1}\,\text{K}^{-1}, \text{ and}$$

$$\frac{Dp}{Dt} \equiv \omega.$$

(33)

Eq. (32) shows that, for an adiabatic, isobaric process, the enthalpy, η, is a conserved variable. For an ideal gas, the enthalpy is given by

$$\eta = c_p T.$$

(34)

More generally, the enthalpy can be written as

$$\eta = e + p\alpha.$$

(35)

A third useful form of the thermodynamic equation is

$$\frac{Dp}{Dt} - \left(\frac{c_p}{c_v} RT\right)\frac{D\rho}{Dt} = -\alpha\nabla\cdot\left(\mathbf{R}+\mathbf{F}_S\right) + LC + \delta.$$

(36)

This can be derived from (31) or (32) by using the equation of state and the continuity equation. The quantity $\frac{c_p}{c_v} RT$ turns out to be the square of the speed of sound.

Lorenz (1955) pointed out that the mass-weighted vertical integral of the enthalpy is equal to the sum of the mass-weighted vertical integrals of the internal and potential energies. To demonstrate this, we begin from hydrostatics in the form

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2 Also called the "First Law of Thermodynamics," although that terminology seems rather medieval.

$$\frac{\partial p}{\partial z} = -\rho g \ .$$

(37)

The vertically integrated potential energy, P, satisfies

$$
\begin{aligned}
P &\equiv \int_0^\infty gz\rho \, dz \\
&= -\int_0^\infty \left(\frac{\partial p}{\partial z} z\right) dz \\
&= -\int_0^\infty \left[\frac{\partial}{\partial z}(pz) - p\right] dz \\
&= -\left[(pz)\Big|_{z=0}^{z=\infty} - \int_0^\infty \rho RT \, dz\right] \\
&= \int_0^\infty \rho RT \, dz \ .
\end{aligned}
$$

(38)

Eq. (38) says that the total potential energy of the column is proportional to the average temperature of the column. The explanation is that warmer air occupies a larger volume, so that a warmer column is "taller." It follows from (38) that

$$
\begin{aligned}
P + I &= \int_0^\infty \rho RT \, dz + \int_0^\infty c_v T \rho \, dz \\
&= \int_0^\infty (c_v + R)T\rho \, dz \\
&= \int_0^\infty c_p T\rho \, dz \ .
\end{aligned}
$$

(39)

Here I is the vertically integrated internal energy. Note that $c_v T + \phi = c_p T$ is not true; this would imply that $\phi = RT$, which is obviously nonsense. We will use Eq. (39) later, when we discuss available potential energy.

The conservation of thermodynamic energy can also be expressed in a fourth way, in terms of the potential temperature. We can show that

$$c_p \frac{D\theta}{Dt} = \left(\frac{\theta}{T}\right)\left[-\alpha\nabla\cdot(\mathbf{R}+\mathbf{F}_S) + LC + \delta\right] .$$

(40)

In the absence of heating and dissipation, $\dfrac{D\theta}{Dt} = 0$, i.e., θ is conserved following a particle, in the absence of heating. This is one of the reasons that θ is a particularly useful quantity.

Conservation of total energy

Adding (22) and (31) gives

$$\frac{D(K+\phi+e)}{Dt} = -\alpha\nabla\cdot(p\mathbf{V}+\mathbf{F}\cdot\mathbf{V}) - \alpha\nabla\cdot(\mathbf{R}+\mathbf{F}_S) + LC .$$

(41)

Note that the $p\dfrac{D\alpha}{Dt}$ terms have cancelled, as have the dissipation terms. The cancellations occur because these terms represent conversions between thermodynamic and mechanical energy. Alternatively, we can add (26) and (32) to obtain

$$\frac{D(K+\phi+e)}{Dt} = -\alpha\nabla\cdot(p\mathbf{V}+\mathbf{F}\cdot\mathbf{V}) - \alpha\nabla\cdot(\mathbf{R}+\mathbf{F}_S) + LC .$$

(42)

Here we have used $c_p T - p\alpha = e$, and the $\omega\alpha$ and dissipation terms have cancelled because they represent energy conversions. As should be expected, (41) and (42) are identical. You can think of (22) and (31) as a "matched set," and (26) and (32) as a second, equivalent, matched set.

Although total energy conservation is expressed by (41) [and (42)], the energy conversions associated with latent heat release are not shown explicitly. To remedy this, we use the water vapor conservation equation in the form

$$\frac{Dq_v}{Dt} = -\alpha\nabla\cdot(\mathbf{F}_{q_v}) - C ,$$

(43)

where $\mathbf{F}_{q_v}$ is the flux of water vapor due to molecular diffusion. Next, we multiply (43) by the latent heat of condensation, L, neglect variations of L in time and space, and add the result to (41). This gives

$$\frac{D}{Dt}(K+\phi+e+Lq_v) = -\alpha\nabla\cdot(p\mathbf{V}+\mathbf{F}\cdot\mathbf{V}+\mathbf{R}+\mathbf{F}_h) .$$

(44)

Here $\mathbf{F}_h \equiv \mathbf{F}_S + L\mathbf{F}_{q_v}$ is the sum of the molecular fluxes of sensible and latent heat; for reasons explained later, this sum will be called the molecular flux of moist static energy. From (44) we see that the total energy per unit mass, which we will denote by e_T, is given by the sum of the kinetic, potential, internal, and latent energies:

$$e_T = K+\phi+e+Lq_v .$$

(45)

Every term on the right-hand side of (44) is the divergence of a flux, i.e., each term represents a spatial redistribution of energy. This shows that the total energy of the atmosphere is conserved apart from exchanges across its upper and lower boundaries.

Static energies

By using the equation of state, (33), and continuity we can rewrite the total energy equation, (44), as

$$\frac{\partial}{\partial t}\left[\rho\left(K+\phi+c_p T+Lq_v\right)\right]+\nabla\cdot\left[\rho\mathbf{V}\left(K+\phi+c_p T+Lq_v\right)+\mathbf{R}+\mathbf{F}_h+\mathbf{F}\cdot\mathbf{V}\right]=\frac{\partial p}{\partial t}.$$

(46)

Here the enthalpy appears in place of the internal energy in the time derivative and flux divergence terms. A key difference between (44) and (46) is that there is no pressure-work term in the latter. A price that we pay for this simplification is the appearance of the $\frac{\partial p}{\partial t}$ term on the right-hand side of (46). Note, however, that this term drops out in a time average.

The $\frac{\partial p}{\partial t}$ term of (46) looks funny. As an aid in its interpretation, write

$$\begin{aligned}\frac{\partial p}{\partial t} &=\frac{\partial}{\partial z}\left(z\frac{\partial p}{\partial t}\right)-z\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right)\\ &=\frac{\partial}{\partial z}\left(z\frac{\partial p}{\partial t}\right)-z\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial z}\right)\\ &=\frac{\partial}{\partial z}\left(z\frac{\partial p}{\partial t}\right)-\frac{\partial}{\partial t}\left(z\frac{\partial p}{\partial z}\right).\end{aligned}$$

(47)

This can be substituted into (46) to obtain

$$\frac{\partial}{\partial t}\left[\rho\left(K+\phi+c_p T+Lq_v\right)+z\frac{\partial p}{\partial z}\right]+\nabla\cdot\left[\rho\mathbf{V}\left(K+\phi+c_p T+Lq_v\right)+\mathbf{R}+\mathbf{F}_h+\mathbf{F}\cdot\mathbf{V}\right]=\frac{\partial}{\partial z}\left(z\frac{\partial p}{\partial t}\right).$$

(48)

In the hydrostatic limit the $\rho\phi$ and $z\frac{\partial p}{\partial z}$ terms inside the time derivative cancel. The second term on the right-hand side of (48) represents a vertical flux of energy associated with the time-rate-of-change of the pressure.

The contribution of the kinetic energy to the total energy is typically quite negligible. For example, an air parcel zipping along at a rather extreme 100 ms^{-1} has a kinetic energy per unit mass of 5 x 10^3 J kg^{-1}. (Keep in mind that the kinetic energy is proportional to the square of the

wind speed, so that a parcel moving at a more typical 10 ms⁻¹ has a kinetic energy 100 times smaller.) If a parcel traveling at 100 ms⁻¹ resides on the 200 mb surface, its potential energy per unit mass (relative to sea level) is about 1.2×10^5 J kg⁻¹, or about 24 times greater than its kinetic energy. If the temperature of the fast parcel is a mere 200 K, which is if anything a little too cold for the 200 mb surface, its internal energy per unit mass is about 1.5×10^5 J kg⁻¹, about 30 times greater than its kinetic energy.

For these reasons, we can usually neglect K in (48). In addition, the friction and pressure-tendency terms of (48) can often be neglected. With these simplifying approximations, (48) reduces to

$$\frac{\partial}{\partial t}(\rho h) + \nabla \cdot (\rho \mathbf{V} h + \mathbf{R} + \mathbf{F}_h) = 0 \, ,$$

(49)

where

$$h \equiv c_p T + \phi + L q_v$$

(50)

is the moist static energy, whose latitude-height distribution was discussed in Chapter 3. According to (49), the moist static energy is approximately conserved under both moist adiabatic and dry adiabatic processes. Since precipitation does not affect the water vapor mixing ratio, temperature, or geopotential height, the moist static energy is conserved even for pseudoadiabatic processes, in which condensed water is assumed to precipitate out immediately. For many practical purposes, conservation of total energy is (approximately) equivalent to conservation of moist static energy.

We did not have to use the hydrostatic approximation to derive (49); this is important, because it means that (49) can be used in the analysis of non-hydrostatic processes, e.g., cumulus convection.

Note that *conservation of moist static energy is an approximation to the total energy equation, rather than the thermodynamic energy equation*. This is why there is no dissipation term in (49); such a term would of course appear in any version of the thermodynamic energy equation (although we might justify neglecting it under some conditions).

Because the water vapor mixing ratio, q_v, is conserved under dry adiabatic processes, conservation of moist static energy implies that the dry static energy,

$$s \equiv c_p T + \phi \, ,$$

(51)

is approximately conserved under dry adiabatic processes.

Available potential energy

Earlier we showed that, under dry adiabatic and frictionless processes,

$$\frac{d}{dt}\int_V \left(c_v T + \phi + K\right)\rho\, dV = 0 \ .$$

(52)

Here the integral is over the mass of the whole atmosphere. Also recall that, for each vertical column, hydrostatic balance implies that the vertically integrated sum of the mass-weighted internal and potential energies is equal to the vertical integral of the mass-weighted enthalpy, i.e.,

$$\int_0^{p_s} \left(c_v T + \phi\right) dp = \int_0^{p_s} c_p T\, dp \ .$$

(53)

Let H and K be the total (mass integrated) enthalpy and kinetic energy of the entire atmosphere. It follows from (52) and (53) that $H + K$ is invariant, under dry-adiabatic frictionless processes, i.e.,

$$\frac{d}{dt}\left(H + K\right) = 0 \ .$$

(54)

Imagine that we have the power to spatially rearrange the mass of the atmosphere at will, adiabatically and reversibly. As the parcels move, their entropy and potential temperature do not change, but their enthalpy and temperature do change. Suppose that we are given a state of the atmosphere - a set of maps, if you like. Starting from this given state, we move parcels around adiabatically and without friction until we find the unique state of the system that minimizes H. This means that we have reduced H as much as possible from its value in the given state. Because $H + K$ does not change, K is maximized in this special state, which Lorenz (1955) called the "reference state," and which we will call the "*A-state*."

You should prove for yourself that the mass-integrated potential energy of the entire atmosphere is lower in the A-state than in the given state. This means that the center of gravity of the atmosphere descends as the atmosphere passes from the given state to the A-state.

An adiabatic process that reduces H will also reduce the total potential energy of the atmospheric column. The non-kinetic energy that "disappears" in this process is converted into kinetic energy. Therefore, we can say that *an adiabatic process that reduces H will tend to increase the kinetic energy of the atmosphere. Two very important processes of this type are convection and baroclinic instability.*
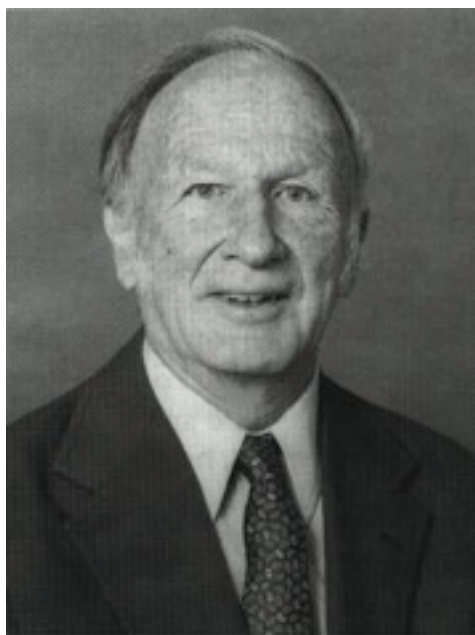
In passing from the given state to the A-state we have

$$H_{gs} \rightarrow H_{min} \, ,$$

$$K_{gs} \rightarrow K_{gs} + \left(H_{gs} - H_{min} \right) = K_{max} \, ,$$

(55)

where H_{min} is the value of H in the A-state, and subscript gs denotes the given state. The non-negative quantity

$$A = H_{gs} - H_{min} \geq 0$$

(56)

is called the "available potential energy," or APE. The APE was first defined by Lorenz (1955; see Fig. 1). Eq. (56) gives the fundamental definition of the APE.

Figure 1: Prof. Edward N. Lorenz, who proposed the concept of available potential energy, and has published a great deal of additional very fundamental research in the atmospheric sciences and the relatively new science of nonlinear dynamical systems.

Notice that the APE is a property of the entire atmosphere; it cannot be rigorously defined for a portion of the atmosphere, although the literature does contain studies in which the APE is computed, without rigorous justification, for a portion of the atmosphere, e.g., the Northern Hemisphere.

The A-state is invariant under adiabatic processes, because it depends only on the probability distribution of θ over the mass, rather than on any particular spatial arrangement of the air. Therefore

$$\frac{dA}{dt} = \frac{d}{dt}\left(H_{gs} - H_{min}\right)$$

$$= \frac{dH_{gs}}{dt} \, .$$

(57)

Then (3) implies that

$$\frac{d}{dt}(A + K) = 0 \, .$$

(58)

The sum of the available potential and kinetic energies is invariant under adiabatic frictionless processes. This means that such processes only convert between A and K.
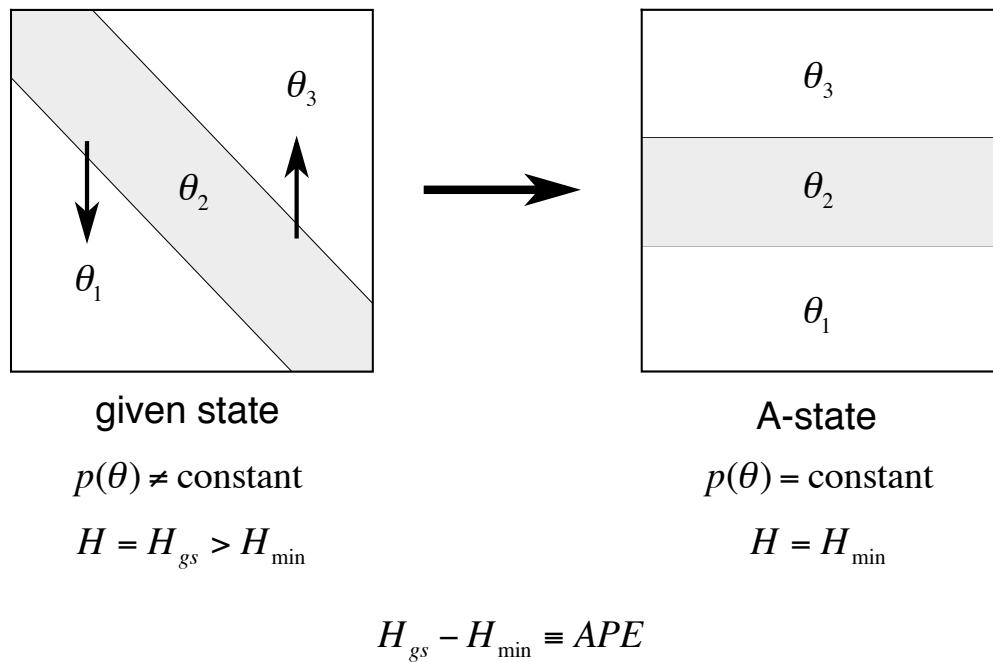
The APE of the A-state itself is obviously zero.

In order to compute A, we have to find the A-state and its (minimum) enthalpy. We can deduce the properties of the A-state as follows: There can be no horizontal pressure gradients in the A-state, because if there were K could increase. It follows that the potential temperature must be constant on isobaric surfaces, which is of course equivalent to the statement that p is constant on θ surfaces. This means that both the variance of θ on isobaric surfaces and the variance of p on isentropic surfaces are measures of the APE. There can be no static instability $(\partial\theta / \partial z < 0)$ in the A-state for a similar reason. From these considerations we conclude that, in the A-state, θ and T are uniform on each pressure-surface, or equivalently that p is uniform on each θ-surface, and also that θ does not decrease upward.

It would appear, and to a large extent it is true, that in passing from the given state to the A-state, no mass can cross an isentropic surface, because we allow only dry adiabatic processes for which $\dot{\theta} = 0$. This implies that $\bar{p}^{\theta}$, the average pressure on an isentropic surface, cannot change as we pass to the A-state. There is an important exception to this rule, however. If $\partial\theta / \partial z < 0$ in the given state, then the average pressure on an isentropic surface will be different in the A-state. This case of static instability is discussed below.

Lower boundary
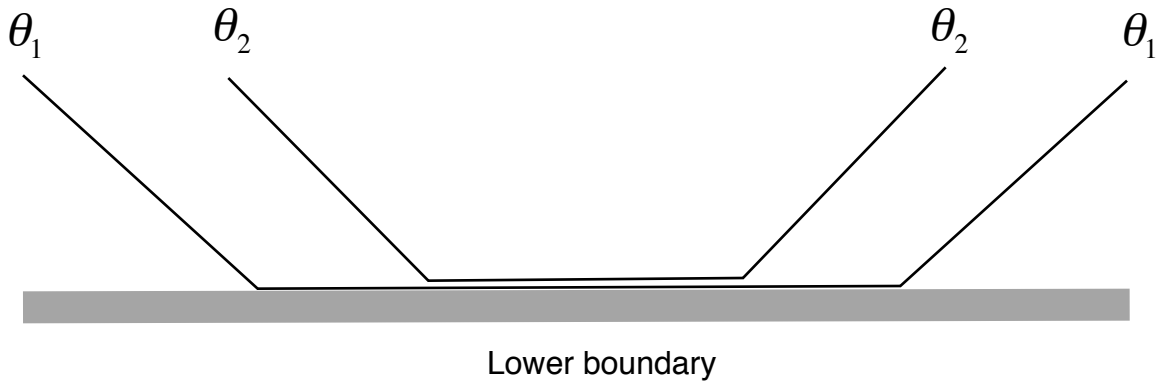
Figure 2: Sketch illustrating the concept of "massless layers."

A complication arises: What about -surfaces that intersect the ground in the given state? These are treated as though they continue along the ground, as shown in Fig. 2, Because the "layers" between the isentropic surfaces that are following the ground contain no mass, they have no effect on the physics. They are called "massless layers." Where a θ-surface intersects the Earth's surface, the pressure is $p = p_S$.

given state $\qquad$ A-state

$$p(\theta) \neq \text{constant} \qquad p(\theta) = \text{constant}$$

$$H = H_{gs} > H_{min} \qquad H = H_{min}$$

$$H_{gs} - H_{min} \equiv APE$$

Figure 3: Sketch illustrating the transition from a given state to the A-state. Here p increases downward and we assume that $\theta_3 > \theta_2 > \theta_1$, which means that the atmosphere is statically stable.

As shown by the heavy arrows in Fig. 3, warm air must rise (move to lower pressure) and cold air must sink (move to higher pressure) to pass from the given state to the A-state. This is

what happens as APE is released; to reach the A-state, the isobaric surfaces flatten out to coincide with isentropic surfaces.

The concept of massless layers allows us to write

$$\int_0^{p_s} (\ )\,dp = \int_0^{\infty} (\ )\frac{\partial p}{\partial \theta}\,d\theta,$$

(59)

where $(\ )$ can be anything. Note that the lower limit of integration on the right-hand side of (59) is zero. Eq. (59) will be used below and it is important for you to understand why it is true.

A useful expression for the APE can be derived as follows. The total enthalpy is given by

$$H = c_p p_0^{-\kappa} \int_M p^\kappa \theta \, dM$$

$$= \frac{c_p a^2}{g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^{p_s} p^\kappa \theta \cos\varphi \, dp \, d\lambda \, d\varphi \quad .$$

(60)

Integration by parts gives

$$H = \frac{c_p a^2}{(1+\kappa)g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^{\infty} p^{1+\kappa} \cos\varphi \, d\theta \, d\lambda \, d\varphi .$$

(61)

Note that in (61) vertical integration is with respect to θ rather than p, and that the lower limit of integration is $\theta = 0$. Let $\overline{p}^{\theta}$ be the average pressure on an isentropic surface (taking into account intersections with the ground). Recall that, provided that there are no regions of dry static instability, $\overline{p}^{\theta}$ is the same in the A-state as in the given state. Then use of (10) in (5) gives

$$A = \frac{c_p a^2}{(1+\kappa)g p_0^\kappa} \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^{\infty} \left[p^{1+\kappa}(\theta) - \left(\overline{p}^{\theta}\right)^{1+\kappa} \right] \cos\varphi \, d\theta \, d\lambda \, d\varphi .$$

(62)

Note that (62) is valid only if $\partial\theta / \partial z \geq 0$ everywhere in the given state, because we have assumed that $\overline{p}^{\theta}$ is the same in the A-state as in the given state. So long as this assumption is satisfied, (62) is exact. The most general expression for the available potential energy is the definition $A \equiv H_{gs} - H_{min}$.

Let p' be the departure of p from its average on an isentropic surface, so that $p = \overline{p}^{\theta} + p'$, where $\overline{(p')}^{\theta} = 0$. The binomial theorem tells us that

$$p^{1+\kappa} = \left(\bar{p}^\theta\right)^{1+\kappa}\left(1+\frac{p'}{\bar{p}^\theta}\right)^{1+\kappa}$$

$$= \left(\bar{p}^\theta\right)^{1+\kappa}\left[1+(1+\kappa)\frac{p'}{\bar{p}^\theta}+\frac{\kappa(1+\kappa)}{2!}\left(\frac{p'}{\bar{p}^\theta}\right)^2+\cdots\right].$$

(63)

Lorenz used (63) to write

$$\overline{p^{1+\kappa}(\theta)} \cong \left(\bar{p}^\theta\right)^{1+\kappa}\left[1+\frac{\kappa(1+\kappa)}{2!}\overline{\left(\frac{p'}{\bar{p}^\theta}\right)^2}\right],$$

(64)

and he showed that this is a pretty good approximation. Substitution of (63) into (62) gives

$$A \cong \frac{Ra^2}{2gp_0^\kappa}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^\infty \left(\bar{p}^\theta\right)^{1+\kappa}\left(\frac{p'}{\bar{p}^\theta}\right)^2\cos\varphi\, d\theta\, d\lambda\, d\varphi.$$

(65)

Because he wanted to express his results in terms of perturbations on isobaric surfaces, rather than pressure perturbations on isentropic surfaces, Lorenz also used

$$p' \cong \theta'\frac{\partial p}{\partial\theta}$$

$$\cong \theta'\frac{\partial\bar{p}^\theta}{\partial\theta}$$

$$= \theta'\left(\frac{\partial\bar{\theta}}{\partial p}\right)^{-1},$$

(66)

where, as before, p' represents the departure of p from its global average on an isentropic surface, and θ' represents the departure of θ from $\bar{\theta}$, its global average on a p-surface. Substitution of (66) into (65) gives

$$A \cong \frac{Ra^2}{2gp_0^\kappa}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^{p_s}\frac{\bar{\theta}^2}{\left(p^{1-\kappa}\right)\left(-\frac{\partial\bar{\theta}}{\partial p}\right)}\left(\frac{\theta'}{\bar{\theta}}\right)^2\cos\varphi\, dp\, d\lambda\, d\varphi.$$

(67)

Note that in (67) the independent variable used for vertical integration has been changed from θ to p. Eq. (67) involves a weighted average of the square of the departure of θ from its mean *on the pressure surface*. The average of the square of the departure from the mean is called the "variance about the mean," or just the variance. The variance is a measure of how variable a

quantity is; if the quantity is constant, and so everywhere equal to its mean, then its variance must be zero. If the quantity is not constant, its variance is positive. Because we are interested in variability, variances are quite important in the study of weather and climate.

Finally, Lorenz used the hydrostatic equation in the form

$$\frac{\partial \theta}{\partial p} = -\frac{\kappa \theta}{p}\left(\frac{\Gamma_d - \Gamma}{\Gamma_d}\right),$$

(68)

where $\Gamma \equiv -\dfrac{\partial T}{\partial z}$ is the lapse rate of temperature and $\Gamma_d \equiv \dfrac{g}{c_p}$ is the dry adiabatic lapse rate, as

well as

$$\frac{\theta'}{\bar{\bar{\theta}}} = \frac{T'}{\bar{\bar{T}}},$$

(69)

to rewrite (67) as

$$A = \frac{a^2}{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^{p_s}\frac{\bar{T}}{(\Gamma_d - \Gamma)}\left(\frac{T'}{\bar{\bar{T}}}\right)^2 \cos\varphi\, dp\, d\lambda\, d\varphi.$$

(70)

This result shows that the available potential energy is closely related to the variance of temperature on isobaric surfaces. It also increases as the lapse rate of temperature increases, i.e., as the atmosphere becomes less stable in the dry static sense.

Observations show that the APE is only about 0.5% of $P + I$. The APE is comparable in magnitude to the total kinetic energy. Both are on the order of 10^6 -10^7 J m^{-2}.

How an upward temperature flux reduces the APE

Convection produces an upward flux of dry static energy, which means also an upward flux of θ and T. Suppose that the θ profile in a particular atmospheric column is altered by a vertical flux of θ, i.e.,

$$\frac{\partial \theta}{\partial t} = g\frac{\partial F_\theta}{\partial p}.$$

(71)

It follows that

$$\frac{\partial T}{\partial t} = g\frac{\partial}{\partial p}\left[\left(\frac{p}{p_0}\right)^{\kappa}F_{\theta}\right] - \left(\frac{g\kappa}{p_0^{\kappa}}\right)\frac{F_{\theta}}{p^{1-\kappa}} .$$

(72)

Integrating through the depth of the column, we find that

$$\frac{\partial}{\partial t}\left(\int_0^{p_S} T\,dp\right) = \left[T_S\frac{\partial p_S}{\partial t} + g\left(\frac{p_S}{p_0}\right)^{\kappa}(F_{\theta})_S\right] - \left(\frac{g\kappa}{p_0^{\kappa}}\right)\int_0^{p_S}\frac{F_{\theta}}{p^{1-\kappa}}\,dp .$$

(73)

The first term in square brackets on the right-hand side is zero if no mass is exchanged with neighboring columns. The second term in square brackets is diabatic, because it represents an exchange of energy between the atmosphere and the lower boundary. The remaining term, $-\left(\frac{g\kappa}{p_0^{\kappa}}\right)\int_0^{p_S}\frac{F_{\theta}}{p^{1-\kappa}}\,dp$, arises purely from θ redistribution within the column. The form of this redistribution term makes it clear that $F_{\theta} > 0$, i.e., an upward flux of θ, tends to reduce the total enthalpy of the column, especially when F_{θ} is large up high, where the pressure is small. An upward increase of F_{θ} cools the lower part of the column and warms the upper part; this is what we expect from convection, in which warm air rises and cold air sinks.

The conclusion of this little analysis is that *an adiabatic process that produces an upward flux of θ reduces the total enthalpy of the column, and so generates kinetic energy*. This is relevant to both convection and baroclinic instability.

Variance budgets

From (70) we see that the available potential energy is closely related to the variance of temperature or potential temperature on pressure surfaces.

We now examine a conversion process that couples the variance associated with the meridional gradient of the zonally averaged potential temperature with the eddy variance of potential temperature. This same process is closely related to the conversion between the zonal available potential energy, A_Z, and eddy available potential energy, A_E. The eddy potential temperature variance interacts with the meridional gradient of the zonally averaged potential temperature through

$$\frac{\partial}{\partial t}\left[\theta_*^2\right] \sim \frac{-\left[\theta_* v_*\right]}{a}\frac{\partial}{\partial \varphi}\left[\theta\right] .$$

(74)

The term shown in the right-hand side of (74) is called the "*meridional gradient-production term*." There are actually several additional terms; they are discussed below.

To gain an intuitive understanding of the gradient production terms, consider the simple example illustrated in Fig. 4. State A consists of two latitude belts of equal mass, each with
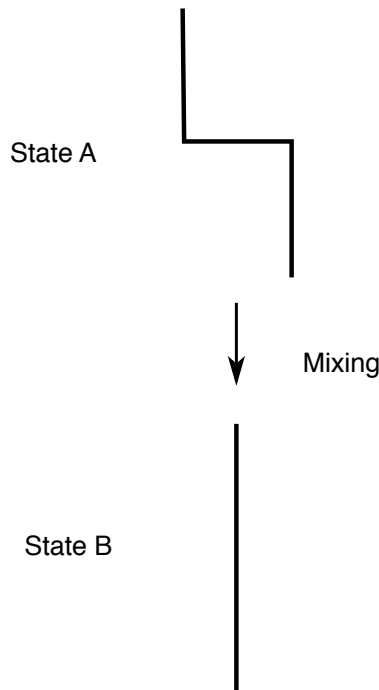
State A

Figure 4: A simple example to explain the idea of gradient production. State B is obtained by homogenizing State A. In both State A and State B, the average of θ is $(\theta_1 + \theta_2)/2$.

Variance seems to "disappear" in passing from State A to State B. In reality it is converted from the variance of the mean to an eddy variance.

Mixing

State B

uniform θ. State B is obtained by homogenizing State A. Consider the average of the square of θ, for each state. For State A,

$$\overline{\theta^2} = \frac{1}{2}\left(\theta_1^2 + \theta_2^2\right).$$

(75)

Here the overbar denotes an average over both latitude belts. The averaging accounts for the factor of 1/2. For State B,

$$\left(\overline{\theta}\right)^2 = \left[\frac{1}{2}\left(\theta_1 + \theta_2\right)\right]^2 = \frac{1}{4}\left(\theta_1^2 + 2\theta_1\theta_2 + \theta_2^2\right)$$

$$= \frac{1}{2}\left(\theta_1^2 + \theta_2^2\right) - \frac{1}{4}\left(\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2\right)$$

$$= \frac{1}{2}\left(\theta_1^2 + \theta_2^2\right) - \frac{1}{4}\left(\theta_1 - \theta_2\right)^2$$

$$\leq \frac{1}{2}\left(\theta_1^2 + \theta_2^2\right).$$

(76)

This shows that down-gradient transport (in this case, mixing) reduces the square of the mean state. It correspondingly increases the square of the fluctuations. In other words, it leads to a conversion of mean-state variance to eddy variance. This appears as a "dissipation" of the mean-state variance. Further discussion is given later.

To see where (74) comes from, start from the conservation equation for potential temperature, i.e.,

$$\frac{\partial \theta}{\partial t} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}(u\theta) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}(v\theta\cos\varphi) + \frac{\partial}{\partial p}(\omega\theta) = \dot{\theta},$$

(77)

and also mass continuity:

$$\frac{1}{a\cos\varphi}\frac{\partial u}{\partial \lambda} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}(v\cos\varphi) + \frac{\partial \omega}{\partial p} = 0.$$

(78)

Zonally average (77) and (78), to obtain

$$\frac{\partial}{\partial t}[\theta] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}([v\theta]\cos\varphi) + \frac{\partial}{\partial p}[\omega\theta] = [\dot{\theta}],$$

(79)

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}([v]\cos\varphi) + \frac{\partial}{\partial p}[\omega] = 0.$$

(80)

Subtracting (79) and (80) from (77) and (78), respectively, we find that

$$\frac{\partial \theta_*}{\partial t} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}(u\theta) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{(v\theta - [v\theta])\cos\varphi\right\} + \frac{\partial}{\partial p}(\omega\theta - [\omega\theta])$$

$$= \frac{\partial \theta_*}{\partial t} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}\left(u_*[\theta] + [u]\theta_* + u_*\theta_*\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{(v_*[\theta] + [v]\theta_* + v_*\theta_* - [v_*\theta_*])\cos\varphi\right\}$$

$$+ \frac{\partial}{\partial p}\left(\omega_*[\theta] + [\omega]\theta_* + \omega_*\theta_* - [\omega_*\theta_*]\right)$$

$$= \frac{\partial \theta_*}{\partial t} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}\left([u]\theta_*\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{([v]\theta_*)\cos\varphi\right\} + \frac{\partial}{\partial p}\left([\omega]\theta_*\right)$$

$$+ \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}\left(u_*\theta_*\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left(v_*\theta_*\cos\varphi\right) + \frac{\partial}{\partial p}\left(\omega_*\theta_*\right)$$

$$+ \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}\left(u_*[\theta]\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left(v_*[\theta]\cos\varphi\right) + \frac{\partial}{\partial p}\left(\omega_*[\theta]\right)$$

$$- \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left([v_*\theta_*]\cos\varphi\right) - \frac{\partial}{\partial p}\left([\omega_*\theta_*]\right)$$

$$= \dot{\theta}_* \ ,$$

(81)

$$\frac{1}{a\cos\varphi}\frac{\partial u_*}{\partial \lambda} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left(v_*\cos\varphi\right) + \frac{\partial \omega_*}{\partial p} = 0 \ .$$

(82)

To obtain the first equality of (81), we have used

$$v\theta = \left([v] + v_*\right)\left([\theta] + \theta_*\right) = [v][\theta] + v_*[\theta] + [v]\theta_* + v_*\theta_* \ ,$$

(83)

$$[v\theta] = [v][\theta] + [v_*\theta_*] \ ,$$

(84)

$$v\theta - [v\theta] = v_*[\theta] + [v]\theta_* + v_*\theta_* - [v_*\theta_*] \ ,$$

(85)

and so on. We can use the continuity equations (80) and (82) to rewrite (81) as follows:

$$\left(\frac{\partial}{\partial t} + \frac{[u]}{a\cos\varphi}\frac{\partial}{\partial \lambda} + \frac{[v]}{a}\frac{\partial}{\partial \varphi} + [\omega]\frac{\partial}{\partial p}\right)\theta_* + \left(\frac{u_*}{a\cos\varphi}\frac{\partial}{\partial \lambda} + \frac{v_*}{a}\frac{\partial}{\partial \varphi} + \omega_*\frac{\partial}{\partial p}\right)\theta_* + \frac{v_*}{a}\frac{\partial}{\partial \varphi}[\theta] + \omega_*\frac{\partial}{\partial p}[\theta]$$

$$= \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left([v_*\theta_*]\cos\varphi\right) + \frac{\partial}{\partial p}[w_*\theta_*] + \dot{\theta}_* \ .$$

(86)

Multiplying (86) by θ_*, and using (82) again, we obtain:

$$\left(\frac{\partial}{\partial t} + \frac{[u]}{a\cos\varphi}\frac{\partial}{\partial \lambda} + \frac{[v]}{a}\frac{\partial}{d\varphi} + [\omega]\frac{\partial}{\partial p} \right)\left(\frac{1}{2}\theta_*^2 \right)$$

$$+ \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}\left\{ u_*\left(\frac{1}{2}\theta_*^2\right) \right\} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{ v_*\left(\frac{1}{2}\theta_*^2\right)\cos\varphi \right\} + \frac{\partial}{\partial p}\left\{ \omega_*\left(\frac{1}{2}\theta_*^2\right) \right\}$$

$$= \theta_*\left\{ \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left([v_*\theta_*]\cos\varphi\right) + \frac{\partial}{\partial p}[\omega_*\theta_*] \right\} - \frac{v_*\theta_*}{a}\frac{\partial}{\partial \varphi}[\theta] - \omega_*\theta_*\frac{\partial}{\partial p}[\theta] + \dot{\theta}_*\theta_* \; .$$

(87)

Zonally averaging gives

$$\left(\frac{\partial}{\partial t} + \frac{[v]}{a}\frac{\partial}{d\varphi} + [\omega]\frac{\partial}{\partial p} \right)\left[\frac{1}{2}\theta_*^2 \right] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{ \left[v_*\left(\frac{1}{2}\theta_*^2\right) \right]\cos\varphi \right\} + \frac{\partial}{\partial p}\left\{ \left[\omega_*\left(\frac{1}{2}\theta_*^2\right) \right] \right\}$$

$$= -\frac{[v_*\theta_*]}{a}\frac{\partial}{\partial \varphi}[\theta] - [\omega_*\theta_*]\frac{\partial}{\partial p}[\theta] + [\dot{\theta}_*\theta_*] \; .$$

(88)

Finally, we can use (80) to rewrite (88) in flux form:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\theta_*^2 \right] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{ \left([v]\frac{1}{2}[\theta_*^2] \right)\cos\varphi \right\} + \frac{\partial}{\partial p}\left([\omega]\frac{1}{2}[\theta_*^2] \right)$$

$$\underbrace{+ \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{ \left[v_*\left(\frac{1}{2}\theta_*^2\right) \right]\cos\varphi \right\} + \frac{\partial}{\partial p}\left\{ \left[\omega_*\left(\frac{1}{2}\theta_*^2\right) \right] \right\}}_{\text{eddy transport}}$$

$$= \underbrace{-\frac{[v_*\theta_*]}{a}\frac{\partial}{\partial \varphi}[\theta] - [\omega_*\theta_*]\frac{\partial}{\partial p}[\theta]}_{\text{gradient production}} + [\dot{\theta}_*\theta_*] \; .$$

(89)

According to (89), $\left[\frac{1}{2}\theta_*^2 \right]$ can change due to advection by the mean meridional circulation, or due to transport by the eddies themselves, or due to "gradient production."

Eq. (89) governs the "eddy variance" at a particular latitude. There is also a contribution to the *global* variance of θ that comes from the meridional and vertical gradients of $[\theta]$. To derive an equation for this part of the global variance of θ, start by using (80) to rewrite (79) as

$$\left(\frac{\partial}{\partial t} + \frac{[v]}{a}\frac{\partial}{\partial \varphi} + [\omega]\frac{\partial}{\partial p} \right)[\theta] = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left([v_*\theta_*]\cos\varphi\right) - \frac{\partial}{\partial p}[\omega_*\theta_*] + [\dot{\theta}] \; .$$

(90)

Multiplication by $[\theta]$ gives

$$\left(\frac{\partial}{\partial t}+\frac{[v]}{a}\frac{\partial}{\partial\varphi}+[\omega]\frac{\partial}{\partial p}\right)\frac{1}{2}[\theta]^2 = -\frac{[\theta]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([v_*\theta_*]\cos\varphi\right)-[\theta]\frac{\partial}{\partial p}\left([\omega_*\theta_*]\right)+[\theta]\left[\dot\theta\right].$$

(91)

This can be rearranged to

$$\left(\frac{\partial}{\partial t}+\frac{[v]}{a}\frac{\partial}{\partial\varphi}+[\omega]\frac{\partial}{\partial p}\right)\frac{1}{2}[\theta]^2$$

$$= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([\theta][v_*\theta_*]\cos\varphi\right)-\frac{\partial}{\partial p}\left([\theta][\omega_*\theta_*]\right)+\frac{[v_*\theta_*]}{a}\frac{\partial}{\partial\varphi}[\theta]+[\omega_*\theta_*]\frac{\partial}{\partial p}[\theta]+[\theta]\left[\dot\theta\right].$$

(92)

Converting back to flux form, we find that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}[\theta]^2\right)+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v]\frac{1}{2}[\theta]^2\right)\cos\varphi\right\}+\frac{\partial}{\partial p}\left([\omega]\frac{1}{2}[\theta]^2\right)$$

$$= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([\theta][v_*\theta_*]\cos\varphi\right)-\frac{\partial}{\partial p}\left([\theta][\omega_*\theta_*]\right)+\frac{[v_*\theta_*]}{a}\frac{\partial}{\partial\varphi}[\theta]+[\omega_*\theta_*]\frac{\partial}{\partial p}[\theta]+[\theta]\left[\dot\theta\right].$$

(93)

When we add (93) and (89), the gradient production terms cancel. This shows that those terms represent a "conversion" between $[\theta]^2$ and $\left[\theta_*^2\right]$. We obtain:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}[\theta]^2+\frac{1}{2}\left[\theta_*^2\right]\right)+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{[v]\left(\frac{1}{2}[\theta]^2+\frac{1}{2}\left[\theta_*^2\right]\right)\cos\varphi\right\}+\frac{\partial}{\partial p}\left\{[\omega]\left(\frac{1}{2}[\theta]^2+\frac{1}{2}\left[\theta_*^2\right]\right)\right\}$$

$$+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left[v_*\left(\frac{1}{2}\theta_*^2\right)\right]\cos\varphi\right\}+\frac{\partial}{\partial p}\left\{\left[\omega_*\left(\frac{1}{2}\theta_*^2\right)\right]\right\}$$

$$= \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([\theta][v_*\theta_*]\cos\varphi\right)-\frac{\partial}{\partial p}\left([\theta][\omega_*\theta_*]\right)+[\theta]\left[\dot\theta\right]+\left[\dot\theta_*\theta_*\right].$$

(94)

Finally, integration of (94) over the entire atmosphere gives

$$\frac{\partial}{\partial t}\int_M\left(\frac{1}{2}[\theta]^2+\frac{1}{2}\left[\theta_*^2\right]\right)dM = \int_M\left([\theta]\left[\dot\theta\right]+\left[\dot\theta_*\theta_*\right]\right)dM.$$

(95)

This shows that, in the absence of heating, the sum of the eddy variance and the variance of the zonal mean is a constant. Note also that the potential temperature variance increases with time if we "heat where it's hot and cool where it's cold."

The governing equations for the eddy kinetic energy, zonal kinetic energy, and total kinetic energy

We now present a discussion of the eddy kinetic energy, zonal kinetic energy, and total kinetic energy equations. The derivations of these equations follow methods similar to those used to derive the conservation equation for the potential energy variance, and so will be omitted here for brevity. A *QuickStudy* showing the details is available on the instructor's web site.

We define the eddy kinetic energy per unit mass by

$$KE \equiv \frac{1}{2}\left[(u*)^2 + (v*)^2 \right].$$

(96)

It satisfies the following equation:

$$\frac{\partial}{\partial t}KE + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{ \left([v]KE + \frac{1}{2}[v*u*u*] + \frac{1}{2}[v*v*v*] \right)\cos\varphi \right\}$$

$$+\frac{\partial}{\partial p}\left([\omega]KE + \frac{1}{2}[\omega*u*u*] + \frac{1}{2}[\omega*v*v*] \right) + \left\{ \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v*\phi*]\cos\varphi) + \frac{\partial}{\partial p}[\omega*\phi*] \right\}$$

$$= \frac{-[v*u*]}{a}\frac{\partial}{\partial\varphi}[u] - \frac{[v*v*]}{a}\frac{\partial}{\partial\varphi}[v] - [\omega*u*]\frac{\partial}{\partial p}[u] - [\omega*v*]\frac{\partial}{\partial p}[v]$$

$$-[\omega*\alpha*] + ([u*u*][v] - [u*v*][u])\frac{\tan\varphi}{a} + \left[u*g\frac{\partial}{\partial p}\tau_\lambda* \right] + \left[v*g\frac{\partial}{\partial p}\tau_\varphi* \right].$$

(97)

The terms on the first line of the right-hand side represent gradient production, i.e. the conversion between the kinetic energy of the mean flow and that of the eddies. This conversion is in the sense of increasing the eddy kinetic energy when the eddy momentum flux is "down the gradient," i.e. when it is from higher mean momentum to lower mean momentum. The $\omega*\alpha*$ term represents eddy kinetic energy generation from eddy available potential energy, while the terms involving $\phi*$ represent the effects of "pressure work."

The appearance of the metric terms in (97) may be somewhat surprising. They arise because we have defined "eddies" in terms of departures from the zonal mean, so that a particular latitude-longitude coordinate system is implicit in the very definition of KE. Obviously there cannot be any metric terms in the equation for the total kinetic energy per unit mass, which we denote by K.

Define the zonal kinetic energy by

$$KZ \equiv \frac{1}{2}\left([u]^2 + [v]^2 \right),$$

(98)

and note that

$$[K] = KZ + KE \, .$$

(99)

All three quantities in Eq. (99) are independent of longitude. The zonal kinetic energy satisfies

$$\frac{\partial}{\partial t} KZ + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([v] KZ \cos\varphi\right) + \frac{\partial}{\partial p}\left([\omega] KZ\right) = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([v][\phi]\cos\varphi\right) - \frac{\partial}{\partial p}\left([\phi][\omega]\right) - [\omega][\alpha]$$

$$-\frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left[v^*u^*\right]\cos\varphi\right\} - [u]\frac{\partial}{\partial p}\left(\left[\omega^*u^*\right]\right) - \frac{[v]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left[v^*v^*\right]\cos\varphi\right\} - [v]\frac{\partial}{\partial p}\left(\left[\omega^*v^*\right]\right)$$

$$+\left([u]\left[v^*u^*\right] - [v]\left[u^*u^*\right]\right)\frac{\tan\varphi}{a} + [u]g\frac{\partial}{\partial p}[\tau_\lambda] + [v]g\frac{\partial}{\partial p}[\tau_\varphi] \, .$$

(100)

The terms on the second and third lines of the right-hand side of (100) can be interpreted as representing the work done by the mean flow against the "forces" exerted on the mean flow by the eddies, through eddy momentum transport.

Comparison with the eddy kinetic energy equation shows that, as expected, the metric terms do not affect the zonally averaged total kinetic energy. Adding the equations for KZ and KE gives the equation for the zonally averaged total kinetic energy, $[K]$:

$$\frac{\partial}{\partial t}[K] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v][K] + \frac{1}{2}[v^*u^*u^*] + \frac{1}{2}[v^*v^*v^*]\right)\cos\varphi\right\}$$

$$+\frac{\partial}{\partial p}\left([\omega][K] + \frac{1}{2}[\omega^*u^*u^*] + \frac{1}{2}[\omega^*v^*v^*]\right)$$

$$= \frac{-[v^*u^*]}{a}\frac{\partial}{\partial\varphi}[u] - \frac{[v^*v^*]}{a}\frac{\partial}{\partial\varphi}[v] - [\omega^*u^*]\frac{\partial}{\partial p}[u] - [\omega^*v^*]\frac{\partial}{\partial p}[v]$$

$$-\frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left[v^*u^*\right]\cos\varphi\right\} - [u]\frac{\partial}{\partial p}\left(\left[\omega^*u^*\right]\right) - \frac{[v]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left[v^*v^*\right]\cos\varphi\right\} - [v]\frac{\partial}{\partial p}\left(\left[\omega^*v^*\right]\right)$$

$$-\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([v][\phi]\cos\varphi\right) + \frac{\partial}{\partial p}\left([\phi][\omega]\right)\right\} - [\omega][\alpha]$$

$$-\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(v^*\phi^*\cos\varphi\right) + \frac{\partial}{\partial p}\left(\omega^*\phi^*\right)\right\} - [\omega^*\alpha^*]$$

$$+\left[u^*g\frac{\partial}{\partial p}\tau_\lambda^*\right] + \left[v^*g\frac{\partial}{\partial p}\tau_\varphi^*\right] + [u]g\frac{\partial}{\partial p}[\tau_\lambda] + [v]g\frac{\partial}{\partial p}[\tau_\varphi] \, .$$

(101)

The first three lines on the right-hand side of (101) come from the "gradient production" terms of the eddy kinetic energy equation and the terms of the zonal kinetic energy equation that represent

the work done by the mean flow against the forces exerted on the mean flow by the eddies. Inspection shows that these terms can be combined, as shown in Eq. (102):

$$
\frac{\partial}{\partial t}[K] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v][K] + \frac{1}{2}[v*u*u*] + \frac{1}{2}[v*v*v*]\right)\cos\varphi\right\}
$$

$$
+ \frac{\partial}{\partial p}\left([\omega][K] + \frac{1}{2}[\omega*u*u*] + \frac{1}{2}[\omega*v*v*]\right)
$$

$$
= -\frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([u]\left[v^*u^*\right] + [v]\left[v^*v^*\right]\right)\cos\varphi\right\} - \frac{\partial}{\partial p}\left([u]\left[\omega^*u^*\right] + [v]\left[\omega^*v^*\right]\right)
$$

$$
- \left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([v][\phi]\cos\varphi\right) + \frac{\partial}{\partial p}\left([\phi][\omega]\right)\right\} - [\omega][\alpha]
$$

$$
- \left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(v*\phi*\cos\varphi\right) + \frac{\partial}{\partial p}\left(\omega*\phi*\right)\right\} - [\omega*\alpha*]
$$

$$
+ \left[u*g\frac{\partial}{\partial p}\tau_\lambda*\right] + \left[v*g\frac{\partial}{\partial p}\tau_\varphi*\right] + [u]g\frac{\partial}{\partial p}[\tau_\lambda] + [v]g\frac{\partial}{\partial p}[\tau_\varphi].
$$

(102)

Here the terms mentioned above have been combined on the first line of the right-hand side, and it is apparent that they take the form of a divergence. This implies, of course, that the two terms together integrate to zero over the whole atmosphere. The interpretation of this result is that the gradient production terms represent conversion between the kinetic energy of the mean flow and the kinetic energy of the eddies. We do not see simple cancellation between the corresponding terms of the *KE* and *KZ* equations, because the conversion process is not local; it occurs over a region that is extended in the meridional and vertical directions. Cancellation occurs only when we integrate over the whole atmosphere.