Fluxes of Vorticity and Momentum

David Randall

Following Jung and Arakawa (2008), we define the x , y , and z components of the vorticity vector:

$$\xi \equiv \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \eta \equiv \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{1}$$

From (1), we see that

$$\begin{aligned} -w\eta + v\zeta &= -w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &= -w\frac{\partial u}{\partial z} + \frac{\partial}{\partial x}\left(\frac{v^2 + w^2}{2}\right) - v\frac{\partial u}{\partial y}, \end{aligned} \tag{2}$$

which can be rearranged to

$$\left(v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\left(-w\eta + v\zeta\right) + \frac{\partial}{\partial x}\left(\frac{v^2 + w^2}{2}\right). \tag{3}$$

The vector generalization of (3) is $\left(\mathbf{V}\cdot\nabla\right)\mathbf{V} = \boldsymbol{\omega}\times\mathbf{V} + \nabla\left(\frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right)$.

The anelastic continuity equation is

$$\frac{\partial}{\partial x}\left(\rho_0 u\right) + \frac{\partial}{\partial y}\left(\rho_0 v\right) + \frac{\partial}{\partial z}\left(\rho_0 w\right) = 0, \tag{4}$$

Where $\rho_0\left(z\right)$ is the reference-state density, which is assumed to be a function of height only. We can write

$$\frac{\partial}{\partial y}(\rho_0 uv) = \rho_0 v \frac{\partial u}{\partial y} + u \frac{\partial}{\partial y}(\rho_0 v),$$

(5)

and

$$\frac{\partial}{\partial z}(\rho_0 uw) = \rho_0 w \frac{\partial u}{\partial z} + u \frac{\partial}{\partial z}(\rho_0 w).$$

(6)

Adding (5) and (6), and using (4), we find that

$$\begin{aligned}
\frac{\partial}{\partial y}(\rho_0 uv) + \frac{\partial}{\partial z}(\rho_0 uw) &= \rho_0 v \frac{\partial u}{\partial y} + u \frac{\partial}{\partial y}(\rho_0 v) + \rho_0 w \frac{\partial u}{\partial z} + u \frac{\partial}{\partial z}(\rho_0 w) \\
&= \rho_0 \left(v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) + u \left[\frac{\partial}{\partial y}(\rho_0 v) + \frac{\partial}{\partial z}(\rho_0 w) \right] \\
&= \rho_0 \left(v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) - u \frac{\partial}{\partial x}(\rho_0 u) \\
&= \rho_0 \left(v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) - \frac{\partial}{\partial x}(\rho_0 uu) + \rho_0 u \frac{\partial u}{\partial x} \\
&= \rho_0 \left(v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) - \frac{\partial}{\partial x}(\rho_0 uu) + \rho_0 \frac{\partial}{\partial x}\left(\frac{u^2}{2} \right).
\end{aligned}$$

(7)

Comparing (7) with (3), we see that

$$\begin{aligned}
\frac{\partial}{\partial y}(\rho_0 uv) + \frac{\partial}{\partial z}(\rho_0 uw) &= \rho_0 \left[-(-w\eta + v\zeta) + \frac{\partial}{\partial x}\left(\frac{v^2 + w^2}{2} \right) \right] - \frac{\partial}{\partial x}(\rho_0 uu) + \rho_0 \frac{\partial}{\partial x}\left(\frac{u^2}{2} \right) \\
&= -\rho_0 (-w\eta + v\zeta) + \rho_0 \frac{\partial}{\partial x}\left(\frac{u^2 + v^2 + w^2}{2} \right) - \frac{\partial}{\partial x}(\rho_0 uu).
\end{aligned}$$

(8)

Using the fact that the reference-state density depends only on height, averaging in both the x and y directions, and assuming periodic boundary conditions in both directions, we find that (8) reduces to

$$\boxed{-\frac{1}{\rho_0} \frac{\partial}{\partial z}\left(\rho_0 \overline{uw} \right) = -\overline{w\eta} + \overline{v\zeta}}.$$

(9)

Here the overbars denote the horizontal average mentioned above.

To derive an analogous expression for $-\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 vw)$, we repeat the derivation by starting with ξ and ζ, instead of η and ζ. First, we write

$$\begin{aligned} -w\xi + u\zeta &= -w\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &= w\frac{\partial v}{\partial z} + u\frac{\partial v}{\partial x} - \frac{\partial}{\partial y}\left(\frac{u^2 + w^2}{2}\right), \end{aligned} \tag{10}$$

which can be rearranged to

$$w\frac{\partial v}{\partial z} + u\frac{\partial v}{\partial x} = (-w\xi + u\zeta) + \frac{\partial}{\partial y}\left(\frac{u^2 + w^2}{2}\right). \tag{11}$$

We write

$$\frac{\partial}{\partial z}(\rho_0 wv) = \rho_0 w\frac{\partial v}{\partial z} + v\frac{\partial}{\partial z}(\rho_0 w), \tag{12}$$

$$\frac{\partial}{\partial x}(\rho_0 uv) = \rho_0 u\frac{\partial v}{\partial x} + v\frac{\partial}{\partial x}(\rho_0 u). \tag{13}$$

Adding (12)-(13), and using (4), we obtain

$$\begin{aligned} \frac{\partial}{\partial z}(\rho_0 wv) + \frac{\partial}{\partial x}(\rho_0 uv) &= \rho_0\left(w\frac{\partial v}{\partial z} + u\frac{\partial v}{\partial x}\right) + v\left[\frac{\partial}{\partial x}(\rho_0 u) + \frac{\partial}{\partial z}(\rho_0 w)\right] \\ &= \rho_0\left(w\frac{\partial v}{\partial z} + u\frac{\partial v}{\partial x}\right) - v\frac{\partial}{\partial y}(\rho_0 v) \\ &= \rho_0\left(w\frac{\partial v}{\partial z} + u\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}(\rho_0 vv) + \rho_0 v\frac{\partial v}{\partial y} \\ &= \rho_0\left(w\frac{\partial v}{\partial z} + u\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}(\rho_0 vv) + \rho_0\frac{\partial}{\partial y}\left(\frac{v^2}{2}\right). \end{aligned} \tag{14}$$

Comparing with (11), we see that

$$\frac{\partial}{\partial z}(\rho_0 wv)+\frac{\partial}{\partial x}(\rho_0 uv)=\rho_0\left[(-w\xi+u\zeta)+\frac{\partial}{\partial y}\left(\frac{u^2+w^2}{2}\right)\right]-\frac{\partial}{\partial y}(\rho_0 vv)+\rho_0\frac{\partial}{\partial y}\left(\frac{v^2}{2}\right)$$
$$=\rho_0(-w\xi+u\zeta)+\rho_0\frac{\partial}{\partial y}\left(\frac{u^2+v^2+w^2}{2}\right)-\frac{\partial}{\partial y}(\rho_0 vv). \tag{15}$$

Averaging over both the x and y directions, assuming periodic boundary conditions in both directions, and using the fact that ρ_0 is independent of y, we obtain

$$\boxed{-\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\overline{wv}\right)=-\left(-\overline{w\xi}+\overline{u\zeta}\right)}. \tag{16}$$

Using vector notation, we can combine (9) and (16) to write

$$-\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\overline{w\mathbf{V}_h}\right)=\left(-\overline{w\eta}+\overline{v\zeta}\right)\mathbf{i}-\left(-\overline{w\xi}+\overline{u\zeta}\right)\mathbf{j}$$
$$=\overline{w(-\eta\mathbf{i}+\xi\mathbf{j})}+\overline{\zeta(v\mathbf{i}-u\mathbf{j})}, \tag{17}$$

or

$$\boxed{-\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\overline{w\mathbf{V}_h}\right)=\left(-\overline{w\mathbf{k}\times\boldsymbol{\eta}}\right)+\left(\overline{\zeta\mathbf{k}\times\mathbf{V}_h}\right)}, \tag{18}$$

where $\mathbf{V}_h\equiv u\mathbf{i}+v\mathbf{j}$ is the horizontal wind vector and $\boldsymbol{\eta}\equiv\xi\mathbf{i}+\eta\mathbf{j}$ is the horizontal vorticity vector.

References and Bibliography

Jung, J.-H., and A. Arakawa, 2008: A Three-Dimensional Anelastic Model Based on the Vorticity Equation. *Mon. Wea. Rev.*, **136**, 276-294.