Alternative Forms of the Hydrostatic Equation

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The hydrostatic equation can be written in many different ways. Here are some of the possibilities:

$$\frac{\partial p}{\partial z} = -\rho g \tag{1}$$

$$\frac{\partial \phi}{\partial p} = -\alpha \tag{2}$$

$$\frac{\partial \phi}{\partial \sigma} = -\pi\alpha \tag{3}$$

$$\frac{\partial \Pi}{\partial z} = -\frac{g}{c_p \theta} \tag{4}$$

$$\frac{\partial \phi}{\partial \Pi} = -\theta \tag{5}$$

$$\frac{\partial p}{\partial \theta} = -\rho_\theta g \tag{6}$$

$$\frac{\partial s}{\partial \theta} = \Pi \tag{7}$$

$$\frac{\partial \phi}{\partial \theta} = \alpha \rho_\theta g \tag{8}$$

$$\frac{\partial s}{\partial \eta} = T \tag{9}$$

The notation is as follows:

z is height

p is pressure

ρ is density

g is the acceleration of gravity

$\phi \equiv gz$ is the geopotential

$\alpha \equiv \frac{1}{\rho}$ is the specific volume

$\sigma \equiv \frac{p - p_T}{p_S - p_T}$ is the terrain-following sigma coordinate, where p_T and p_S are the pressures at the model top and the surface, respectively

$\pi \equiv p_S - p_T$

$\Pi \equiv c_p \left(\frac{p}{p_0} \right)^{\kappa}$ is the Exner function, where c_p is the specific heat of air at constant pressure, p_0 is a constant reference pressure, $\kappa \equiv \frac{R}{c_p}$, and R is the specific gas constant for air

$\theta \equiv \frac{c_p T}{\Pi}$ is the potential temperature, and T is temperature,

$s \equiv \Pi \theta + \phi = c_p T + \phi$ is the dry static energy,

$\rho_\theta \equiv \rho \frac{\partial z}{\partial \theta} \cong -\frac{1}{g} \frac{\partial p}{\partial \theta}$ is the isentropic pseudo-density, and

$\eta \equiv c_p \ln \left(\frac{\theta}{\theta_0} \right)$ is the entropy, where θ_0 is a constant reference value of the potential temperature.