Fourier Series

David Randall

Assume that $u(x,t)$ is real and integrable. If the domain is periodic, with period L, we can express $u(x,t)$ exactly by a Fourier series expansion:

$$u(x,t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t) e^{ikx} . \tag{1}$$

The complex coefficients $\hat{u}_k(t)$ can be evaluated using

$$\hat{u}_k(t) = \frac{1}{L}\int_{x-L/2}^{x+L/2} u(x,t) e^{-ikx}\, dx . \tag{2}$$

Recall that the proof of (1) and (2) involves use of the orthogonality condition

$$\frac{1}{L}\int_{x-L/2}^{x+L/2} e^{ikx} e^{ilx}\, dx = \delta_{kl} , \tag{3}$$

where

$$\delta_{kl} \equiv \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases} \tag{4}$$

is the Kronecker delta.

From (1), we see that the x-derivative of u satisfies

$$\frac{\partial u}{\partial x}(x,t) = \sum_{k=-\infty}^{\infty} ik\hat{u}_k(t) e^{ikx} . \tag{5}$$

Inspection of (5) shows that $\frac{\partial u}{\partial x}$ does not have a contribution from $\hat{u}_0$; the reason for this should be clear.

A numerical model uses equations similar to (1), (2), and (5), but with a finite set of wave numbers, and with x defined on a finite mesh:

$$u(x_j,t) \cong \sum_{k=-n}^{n} \hat{u}_k(t) e^{ikx_j} \ , \tag{6}$$

$$\hat{u}_k(t) \cong \frac{1}{M}\sum_{i=1}^{M} u(x_j,t) e^{-ikx_j} \ , \ -n \le k \le n \ , \tag{7}$$

$$\frac{\partial u}{\partial x}(x_j,t) \cong \sum_{k=-n}^{n} ik\hat{u}_k(t) e^{ikx_j} \ . \tag{8}$$

Note that we have used "approximately equal signs" in (6) - (8). In (7) we sum over a grid with M points. In the following discussion, we assume that the value of n is chosen by the user. The value of M , corresponding to a given value of n , is discussed below.

Substitution of (6) into (7) gives

$$\hat{u}_k(t) = \frac{1}{M}\sum_{i=1}^{M} \left\{ \left[\sum_{l=-n}^{n} \hat{u}_l(t) e^{ilx_j} \right] e^{ikx_j} \right\}, \ -n \le k \le n \ . \tag{9}$$

This is of course a rather circular substitution, but the result serves to clarify some basic ideas. If expanded, each term on the right-hand side of (9) involves the product of two wave numbers, l and k, each of which lies in the range $-n$ to n. The range for wave number l is explicitly spelled out in the inner sum on the right-hand side of (9); the range for wave number k is understood because, as indicated, we wish to evaluate the left-hand side of (9) for k in the range $-n$ to n. Because each term on the right-hand side of (9) involves the product of two Fourier modes with wave numbers in the range $-n$ to n, each term includes wave numbers up to $\pm 2n$. We therefore need $2n+1$ complex coefficients, i.e. $2n+1$ values of the $\hat{u}_k(t)$.

Because u is real, it must be true that $\hat{u}_{-k} = \hat{u}_k^*$, where the * denotes the conjugate. To see why this is so, consider the $+k$ and $-k$ contributions to the sum in (6):

$$T_k\left(x_j\right) \equiv \hat{u}_k\left(t\right)e^{ikx_j} + \hat{u}_{-k}\left(t\right)e^{-ikx_j}$$
$$\equiv R_k e^{i\theta} e^{ikx_j} + R_{-k} e^{i\mu} e^{-ikx_j} \ , \tag{10}$$

where $R_k e^{i\theta} \equiv \hat{u}_k\left(t\right)$ and $R_{-k}e^{i\mu} \equiv \hat{u}_{-k}\left(t\right)$, and R_k and R_{-k} are real and non-negative. By linear independence, our assumption that $u\left(x_j,t\right)$ for all x_j is real implies that the imaginary part of $T_{k\left(x_j\right)}$ must be zero, for all x_j. It follows that

$$R_k \sin\left(\theta + kx_j\right) + R_{-k}\sin\left(\mu - kx_j\right) = 0 \text{ for all } x_j . \tag{11}$$

The only way to satisfy this for all x_j is to set

$$\theta + kx_j = -\left(\mu - kx_j\right) = -\mu + kx_j \text{, which means that } \theta = -\mu , \tag{12}$$

and

$$R_k = R_{-k} . \tag{13}$$

Eqs. (12) and (13) imply that

$$\hat{u}_{-k} = \hat{u}_k^* , \tag{14}$$

as was to be demonstrated.

Eq. (14) implies that $\hat{u}_k$ and $\hat{u}_{-k}$ together involve only two distinct real numbers. In addition, it follows from (14) that $\hat{u}_0$ is real. Therefore, the $2n+1$ complex values of $\hat{u}_k$ embody the equivalent of only $2n+1$ distinct real numbers. The Fourier representation up to wave number *n* is thus equivalent to representing the real function $u\left(x,t\right)$ on $2n+1$ grid points, in the sense that the information content is the same. We conclude that, in order to use a grid of *M* points to represent the amplitudes and phases of all waves up to $k = \pm n$, we need $M \geq 2n+1$; we can use more than $2n+1$ points, but not fewer.

As a very simple example, a highly truncated Fourier representation of *u* including just wave numbers zero and one is equivalent to a grid-point representation of *u* using 3 grid points. The real values of *u* assigned at the three grid points suffice to compute the coefficient of wave number zero (the mean value of u) and the phase and amplitude (or "sine and cosine coefficients") of wave number one.

Substituting (7) into (8) gives

$$\frac{\partial u}{\partial x}\left(x_l,t\right) \cong \sum_{k=-n}^{n}\left[\frac{ik}{M}\sum_{i=1}^{M}u\left(x_j,t\right)e^{-ikx_j}\right]e^{ikx_l} \ . \tag{15}$$

Reversing the order of summation leads to

$$\frac{\partial u}{\partial x}\left(x_l,t\right) \cong \sum_{i=1}^{M}\alpha_j^l u\left(x_j,t\right), \tag{16}$$

where

$$\alpha_j^l \equiv \frac{i}{M}\sum_{k=-n}^{n}ke^{ik\left(x_l-x_j\right)} \ . \tag{17}$$

The point of this little derivation is that (16) can be interpreted as a finite-difference approximation - a special one involving *all* grid points in the domain. From this point of view, spectral models can be regarded as a class of finite-difference models.

Eq. (9) can be rewritten as

$$\left(\hat{u}_k\right) = \frac{1}{M}\sum_{i=1}^{M}\left[\sum_{l=-n}^{n}\hat{u}_l e^{i(l-k)x_j}\right]. \tag{18}$$

The contribution of wave number l to $\hat{u}_k$ is

$$\left(\hat{u}_k\right)_l \equiv \frac{1}{M}\sum_{i=1}^{M}\left[\hat{u}_l e^{i(l-k)x_j}\right] = \frac{\hat{u}_l}{M}\sum_{i=1}^{M}e^{i(l-k)x_j} \ . \tag{19}$$

Then we can write

$$\hat{u}_k = \sum_{l=-n}^{n}\left(\hat{u}_k\right)_l \ . \tag{20}$$

For $l = k$, we recover

$$\left(\hat{u}_k\right)_k = \frac{u_k}{M}\sum_{i=1}^{M}1 = \hat{u}_k \ . \tag{21}$$

For $l \neq k$, we get

$$\left(\hat{u}_k\right)_l = \frac{\hat{u}_l}{M}\sum_{i=1}^{M}\left\{\cos\left[\left(l-k\right)x_j\right] + i\sin\left[\left(l-k\right)x_j\right]\right\}. \tag{22}$$

From (22) we can see that, provided that $l - k$ "fits" on the grid, we will have

$$\left(\hat{u}_k\right)_l = 0 \text{ for } l \neq k . \tag{23}$$

The results (21) and (23) are as would be expected.

The contribution of grid point j to the Fourier coefficient $\hat{u}_k$ is

$$\left(\hat{u}_k\right)_j \equiv \frac{1}{M}\sum_{l=-n}^{n}\hat{u}_l e^{i\left(l-k\right)x_j} . \tag{24}$$

Then

$$\hat{u}_k = \sum_{i=1}^{M}\left(\hat{u}_k\right)_j, \quad -n \le k \le n \tag{25}$$

We must get $2n+1$ independent complex values of $\hat{u}_k$. These are

$$\left(\hat{u}_k\right)_j = \frac{e^{-ikx_j}}{M}\sum_{l=-n}^{n}\hat{u}_l e^{ilx_j} . \tag{26}$$

Comparing with (7), we see that

$$\left(\hat{u}_k\right)_j = u\left(x_j\right)e^{ikx_j}, -n \le k \le n , \tag{27}$$

as expected.