Group Velocity

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Suppose that we have a superposition of two waves, with wave numbers k_1 and k_2, respectively. Then we can write

$$\begin{aligned} &\exp[ik_1(x - c_1\, t)] + \exp[ik_2(x - c_2\, t)] \\ &\cong \exp\{i(k + \Delta k)\, x - (kc + \Delta(kc))\, t]\} + \exp\{i[(k - \Delta k)\, x - (kc + \Delta(kc))\, t]\} \\ &= \exp[ik(x - ct)]\,(\exp\{\, i[\Delta kx - \Delta(kc)]\, t\} + \exp\{-i[\Delta kx - \Delta(kc)\, t]\}) \\ &= 2\cos[\Delta kx - \Delta(kc)\, t]\exp\{ik(x - ct)\}\ . \end{aligned} \tag{1}$$

Now define

$$k \equiv \frac{k_1 + k_2}{2},\ c \equiv \frac{c_1 + c_2}{2},\ \ \Delta k \equiv \frac{k_1 - k_2}{2},\ \ \Delta(kc) \equiv \frac{k_1\, c_1 + k_2\, c_2}{2}\ . \tag{2}$$

Note that $k_1 = k + \Delta k$ and $k_2 = k - \Delta k$ In the first line of (1), the exponents are the same within $\frac{1}{2}\,\Delta k$. If Δk is very small, the factor $\cos\Delta k[x\text{-}\frac{\Delta(kc)}{\Delta k}\ t]$ may appear schematically as the outer, slowly varying curves in the figure below. It "modulates" wave k, which is represented by the inner, rapidly varying curve in the figure. The wavelets (dashed lines) move with phase speed $c = \frac{kc}{k}$, but the wave packets (the solid curves, forming an envelope of the wavelets) move with speed $\frac{\Delta(kc)}{\Delta k}$. The differential expression $\frac{d(kc)}{dk} = c_g$ is called the "group velocity." Note that $c_g = c$ if c does not depend on k.

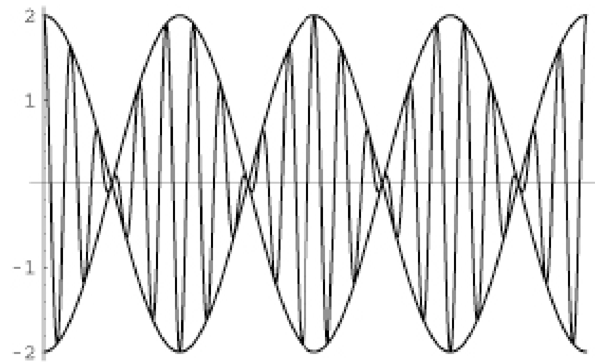

Figure 1: Sketch used to illustrate the concept of group velocity. The short waves are modulated by longer waves.