Hermite Polynomials

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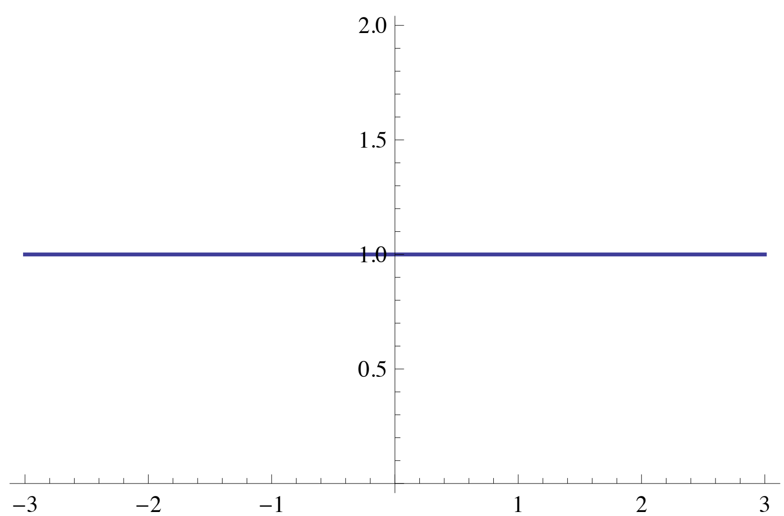
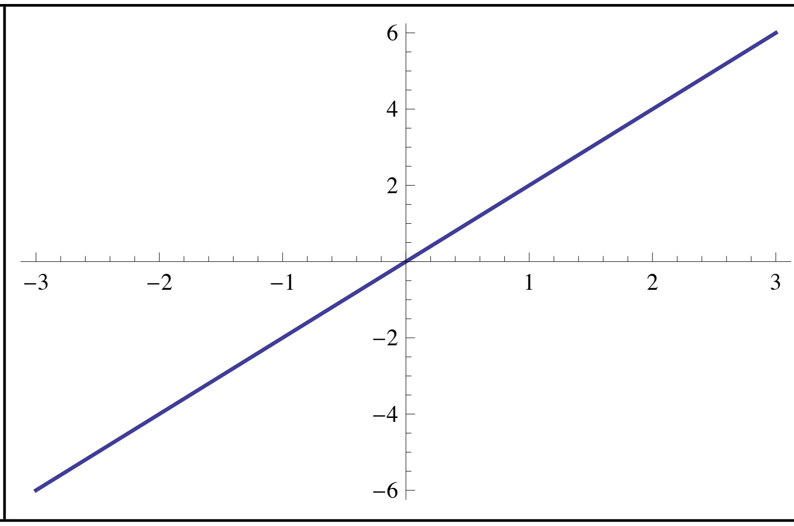
The Hermite polynomials are defined by:

$$H_n(y) \equiv (-1)^n e^{y^2} \frac{d^n}{dy^n}\left(e^{-y^2}\right), \text{ for } n \geq 0 . \tag{1}$$

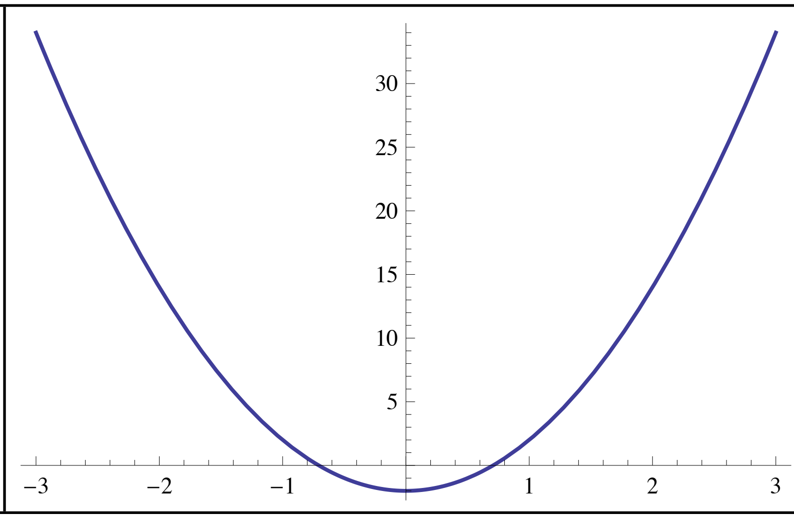
The first six Hermite polynomials are given in the Table below:

$H_0(y)=1$	

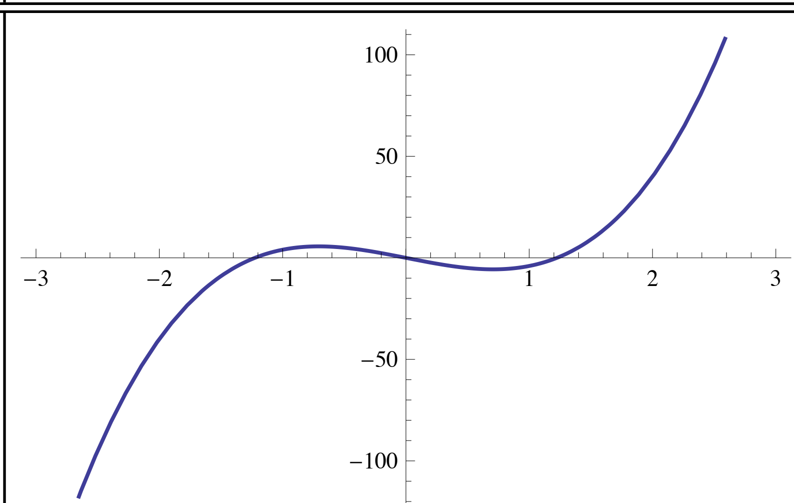
$$H_1(y) = 2y$$



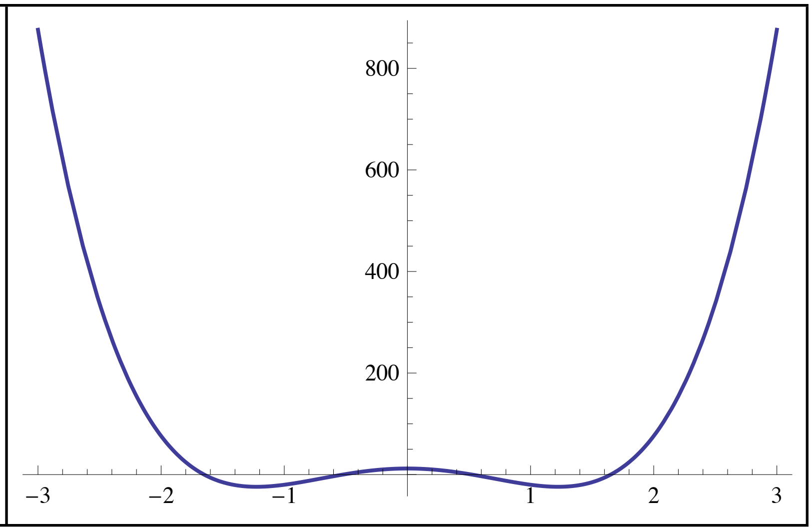
$$H_2(y) = 4y^2 - 2$$



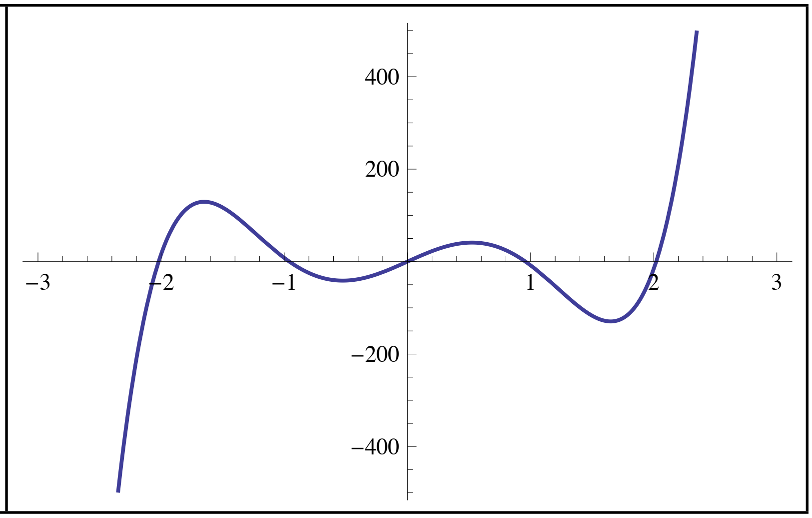
$$H_3(y) = 8y^3 - 12y$$



$$H_4(y) = 16y^4 - 48y^2 + 12$$



$$H_5(y) = 32y^5 - 160y^3 + 120y$$



Note that the even-numbered Hermite polynomials are even functions, and the odd-numbered Hermite polynomials are odd functions.

The Hermite polynomials are orthogonal with respect to e^{-y^2}:

$$\int_{-\infty}^{\infty} H_n(y) H_m(y) e^{-y^2}\, dy = 0 \text{ for } n \neq m, \tag{2}$$

$$\int_{-\infty}^{\infty} H_n(y) H_m(y) e^{-y^2}\, dy = 2^n n! \sqrt{\pi} \text{ for } n = m. \tag{3}$$

Two useful recursion relations are

$$\frac{d}{dy}H_n(y) = 2nH_{n-1}(y) \text{ for } n \geq 1 . \tag{4}$$

and

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y) . \tag{5}$$

Let

$$\psi_n(y) \equiv \frac{e^{-y^2/2}H_n(y)}{\sqrt{2^n n! \pi^{1/2}}} . \tag{6}$$

The $\psi_n(y)$ satisfy

$$\left\{ \frac{d^2}{dy^2} + \left[(2n+1) - y^2 \right] \right\} \psi_n = 0 , \tag{7}$$

as can be verified by substitution. Note that for $n \geq 0$, the expression $2n+1$, which appears in (7), generates all positive odd integers.

References and Bibliography

Courant, R., and D. Hilbert, 1989: *Methods of Mathematical Physics, Volume 1*. Wiley Interscience, 560 pp.