Conservation of momentum on a rotating sphere

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Newton's statement of momentum conservation, as applied in an inertial (i.e., non-accelerating) frame of reference, can be written as follows:

$$\frac{D_a \mathbf{V}_a}{Dt} = -\nabla \phi_a - \alpha \nabla p - \alpha \nabla \cdot \mathbf{F} . \tag{1}$$

Here $\frac{D_a(\ )}{Dt}$ is the Lagrangian derivative in the inertial frame, and $\mathbf{V}_a$ is the velocity as seen in the inertial frame. The left-hand-side of (1) represents the acceleration of the air as seen in an inertial frame. The gravitational potential is ϕ_a, and $\mathbf{F}$ is the stress tensor associated with molecular effects.

The length of a sidereal day is 86,164 s, so the Earth rotates about its axis with an angular velocity of $\frac{2\pi}{(86{,}164 \text{ s})} \cong 7.292 \times 10^{-5} \text{ s}^{-1}$. This angular velocity can be represented by a vector, $\mathbf{\Omega}$, pointing towards the celestial North Pole, as shown in Fig. 1. In spherical coordinates (λ, φ, r), the unit vectors are $\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$, respectively, and the components of $\mathbf{\Omega}$ are $\Omega(0, \cos\varphi, \sin\varphi)$.

Let $\mathbf{r}$ be a position vector extending from the center of the Earth to a particle of air whose position is generally changing with time. The "absolute velocity" of the air, $\mathbf{V}_a \equiv \frac{D_a \mathbf{r}}{Dt}$, is related to its relative velocity, $\mathbf{V} \equiv \frac{D\mathbf{r}}{Dt}$, as seen in the rotating coordinate system, by

$$\frac{D_a \mathbf{r}}{Dt} = \frac{D\mathbf{r}}{Dt} + \mathbf{\Omega} \times \mathbf{r} , \tag{2}$$

or

$$\mathbf{V}_a = \mathbf{V} + \mathbf{\Omega} \times \mathbf{r}$$
$$\equiv \mathbf{V} + \mathbf{V}_e \ . \tag{3}$$

A transformation of the form (3) can be applied to any vector. In contrast, the time rate of change

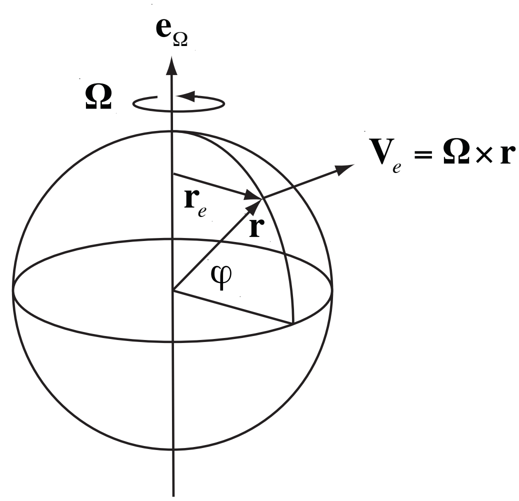

Fig. 1: Sketch defining vectors used in the text.

of a scalar quantity, such as temperature, is the same in the inertial and rotating frames. In the second line of (3), we define

$$\mathbf{V}_e \equiv \mathbf{\Omega} \times \mathbf{r}$$
$$= (\Omega r \cos\varphi)\mathbf{e}_\lambda \ . \tag{4}$$

Here $\mathbf{V}_e$ is the velocity (as seen in the inertial frame) that a particle at radius r and latitude φ experiences due to the Earth's rotation (refer to Fig. 1). Note that $\mathbf{V}_e$ always points towards the east. According to (3), the velocity as seen in the rotating frame is different from the velocity as seen in the inertial frame. Since $\mathbf{\Omega}$ is a constant, the first line of Eq. (4) implies that

$$\frac{D\mathbf{V}_e}{Dt} = \mathbf{\Omega} \times \mathbf{V} \ . \tag{5}$$

Now we apply the transformation used in (2) again, to relate the acceleration as seen in the inertial frame to the acceleration as seen in the rotating frame:

$$\frac{D_a \mathbf{V}_a}{Dt} = \frac{D\mathbf{V}_a}{Dt} + \boldsymbol{\Omega} \times \mathbf{V}_a \,. \tag{6}$$

Substituting for $\mathbf{V}_a$ in (6), from (4), and using (5), we find that

$$\begin{aligned} \frac{D_a \mathbf{V}_a}{Dt} &= \frac{D}{Dt}\left(\mathbf{V} + \mathbf{V}_e\right) + \boldsymbol{\Omega} \times \left(\mathbf{V} + \mathbf{V}_e\right) \\ &= \frac{D\mathbf{V}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{V}_e \\ &= \frac{D\mathbf{V}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{V}_e \,. \end{aligned} \tag{7}$$

Eq. (7) relates the absolute acceleration, $\frac{D_a \mathbf{V}_a}{Dt}$, to the apparent acceleration as seen in the rotating frame, i.e., $\frac{D\mathbf{V}}{Dt}$. Using (7) in (1), we find that the equation of motion relative to the rotating frame is

$$\boxed{\frac{D\mathbf{V}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{V}_e = -\nabla \phi_a - \alpha \nabla p - \alpha \nabla \cdot \mathbf{F}} \,. \tag{8}$$

The term $2\boldsymbol{\Omega} \times \mathbf{V}$ is the Coriolis acceleration, which points in a direction perpendicular to $\mathbf{V}$. The term $\boldsymbol{\Omega} \times \mathbf{V}_e$ is the centrifugal acceleration, which can be written as

$$\begin{aligned} \boldsymbol{\Omega} \times \mathbf{V}_e &= -\left(\Omega^2 r \cos\varphi\right) \mathbf{e}_\lambda \times \mathbf{e}_\Omega \\ &= -\Omega^2 \mathbf{r}_e \,, \end{aligned} \tag{9}$$

where $\mathbf{r}_e$ is a vector that is parallel to the plane of the Equator, as shown in Fig. 1, and $\mathbf{e}_\Omega$ is a unit vector pointing toward the celestial north pole. Eq. (9) shows that the centrifugal acceleration points outward, in the direction of $\mathbf{r}_e$, which is perpendicular to the axis of the Earth's rotation. With reference to (9), the centrifugal acceleration can be expressed in spherical coordinates as

$$\boldsymbol{\Omega} \times \mathbf{V}_e = \Omega^2 r \cos\varphi \left(0, -\sin\varphi, \cos\varphi\right) . \tag{10}$$

It has an upward component that is largest at the Equator, and an equatorward component that is largest at the pole, but no zonal component.

Kinetic energy

The kinetic energy equation associated with (1) is

$$\frac{D_a K_a}{Dt} = \mathbf{V}_a \cdot \left(-\nabla \phi_a - \alpha \nabla p - \alpha \nabla \cdot \mathbf{F} \right), \tag{11}$$

where $K_a \equiv \frac{1}{2}\left(\mathbf{V}_a \cdot \mathbf{V}_a \right)$. The right-hand side of (11) represents work done by true forces. From (6), we see that

$$\frac{D_a K_a}{Dt} = \frac{DK_a}{Dt}. \tag{12}$$

This means that the time change of K_a is the same in the inertial and rotating frames, consistent with our earlier assertion that the time rate of change of a scalar is the same in the inertial and rotating frames. Using (12), we can rewrite (11) as

$$\frac{DK_a}{Dt} = \mathbf{V}_a \cdot \left(-\nabla \phi_a - \alpha \nabla p - \alpha \nabla \cdot \mathbf{F} \right). \tag{13}$$

The kinetic energies as seen in the inertial and rotating frames are different. We will derive the kinetic energy equation for the rotating frame in two different ways. First, the hard way: From (3), we can write

$$\mathbf{V}_a \cdot \mathbf{V}_a = \left(\mathbf{V} + \mathbf{V}_e \right) \cdot \left(\mathbf{V} + \mathbf{V}_e \right). \tag{14}$$

Then

$$K_a = K + \mathbf{V}_e \cdot \mathbf{V} + \frac{1}{2} \mathbf{V}_e \cdot \mathbf{V}_e, \tag{15}$$

where $K \equiv \frac{1}{2}\left(\mathbf{V} \cdot \mathbf{V} \right)$. From (15), we see that

$$
\begin{aligned}
\frac{DK_a}{Dt} &= \frac{DK}{Dt} + \mathbf{V}_e \cdot \frac{D\mathbf{V}}{Dt} + \cancel{\mathbf{V} \cdot \frac{D\mathbf{V}_e}{Dt}} + \mathbf{V}_e \cdot \frac{D\mathbf{V}_e}{Dt} \\
&= \frac{DK}{Dt} + \mathbf{V}_e \cdot \frac{D\mathbf{V}}{Dt} + \mathbf{V}_e \cdot \frac{D\mathbf{V}_e}{Dt} \\
&= \frac{DK}{Dt} + \mathbf{V}_e \cdot \frac{D\mathbf{V}_a}{Dt} .
\end{aligned}
\tag{16}
$$

To obtain the second line of (16), we have used (5). To obtain the third line, we have used (3). Combining (16) with (6) and (1), we find that

$$
\begin{aligned}
\frac{DK_a}{Dt} &= \frac{DK}{Dt} + \mathbf{V}_e \cdot \left(\frac{D_a\mathbf{V}_a}{Dt} - \mathbf{\Omega} \times \mathbf{V}_a \right) \\
&= \frac{DK}{Dt} + \left(\mathbf{V}_a - \mathbf{V} \right) \cdot \left(\frac{D_a\mathbf{V}_a}{Dt} - \mathbf{\Omega} \times \mathbf{V}_a \right) \\
&= \frac{DK}{Dt} + \frac{D_aK_a}{Dt} - \mathbf{V} \cdot \left(\frac{D_a\mathbf{V}_a}{Dt} - \mathbf{\Omega} \times \mathbf{V}_a \right) \\
&= \frac{DK}{Dt} + \frac{D_aK_a}{Dt} - \mathbf{V} \cdot \left(-\nabla\phi_a - \alpha\nabla p - \alpha\nabla \cdot \mathsf{F} \right) + \mathbf{V} \cdot \left(\mathbf{\Omega} \times \mathbf{V}_a \right) .
\end{aligned}
\tag{17}
$$

Cancelling $\frac{D_aK_a}{Dt}$ in (17), and using $\mathbf{V} \cdot \left(\mathbf{\Omega} \times \mathbf{V}_a \right) = \mathbf{V} \cdot \left(\mathbf{\Omega} \times \mathbf{V}_e \right)$, we can rearrange the result to obtain

$$
\boxed{\frac{DK}{Dt} + \mathbf{V} \cdot \left(\mathbf{\Omega} \times \mathbf{V}_e \right) = \mathbf{V} \cdot \left(-\nabla\phi_a - \alpha\nabla p - \alpha\nabla \cdot \mathsf{F} \right)} .
\tag{18}
$$

We can also obtain (18) simply by dotting $\mathbf{V}$ with (8). That's the easy way.

Comparing (18) and (13), we see that

$$
\frac{D\left(K_a - K \right)}{Dt} = \mathbf{V} \cdot \left(\mathbf{\Omega} \times \mathbf{V}_e \right) + \mathbf{V}_e \cdot \left(-\nabla\phi_a - \alpha\nabla p - \alpha\nabla \cdot \mathsf{F} \right) .
\tag{19}
$$

The first term on the right-hand side of (19) appears to be the work done by the centrifugal acceleration. The second appears to be the work done by the true forces acting on $\mathbf{V}_e$. Of course, neither of these terms represents real work done. The first is due to a fictitious force, and the second is due to a fictitious velocity.

Effective gravity

The centrifugal acceleration can be written as

$$\begin{aligned}-\left(\mathbf{\Omega}\times\mathbf{V}_e\right) &= \Omega^2\mathbf{r}_e \\ &= \nabla\left(\frac{1}{2}\left|\mathbf{\Omega}\times\mathbf{r}\right|^2\right), \\ &= \nabla\left(\frac{1}{2}\left|\mathbf{V}_e\right|^2\right),\end{aligned} \tag{20}$$

i.e., it is the gradient of a potential, called the "centrifugal potential," which is equal to the "kinetic energy" associated with $\mathbf{V}_e$. From (20) and (4), we see that the centrifugal potential is given by $-\frac{1}{2}(\Omega r\cos\varphi)^2$.

Since both the acceleration due to gravity and the centrifugal acceleration can be expressed as gradients of potentials, it is natural and convenient to combine them into an "apparent" gravity, $\mathbf{g}$, defined by

$$\mathbf{g} \equiv \mathbf{g}_a - \Omega^2\mathbf{r}_e\,, \tag{21}$$

where $\mathbf{g}_a \equiv \nabla\Phi_a$. Using (23) we see that the potential of $\mathbf{g}$ is

$$\phi = \phi_a - \frac{1}{2}\left|\mathbf{\Omega}\times\mathbf{r}\right|^2\,, \tag{22}$$

so that $\mathbf{g} \equiv \nabla\phi$. We refer to ϕ as the "geopotential." For most purposes $\mathbf{g} \cong \mathbf{g}_a = -g\mathbf{e}_r$, because the centrifugal acceleration is small compared to $\mathbf{g}_a$. Here $\mathbf{e}_r$ is a unit vector pointing upward, away from the center of the Earth. The acceleration due to the apparent gravity $\mathbf{g}$ is perpendicular to surfaces of constant ϕ , which differ slightly from spherical surfaces of constant radius. The Equatorial radius of the Earth is about 20 km larger than the polar radius, due to the centrifugal acceleration.

If the Earth's internal density distribution was a function of radius only, then ϕ_a , the true gravitational potential, would also depend only on distance from the center of the Earth, and would take the form

$$\phi_a = \frac{GM_e}{r}, \tag{23}$$

where $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} is the gravitational constant, and M_e is the total mass of the Earth. Eq. (23), which is called Newton's "Shell Theorem," would apply at any level at or above the Earth's surface. The geopotential ϕ is then given by

$$\phi = \frac{GM_e}{r} + \frac{1}{2}\left(\Omega r \cos\varphi\right)^2 . \tag{24}$$

Using (22) we can now write (8), the equation of motion in the rotating frame, as

$$\frac{D\mathbf{V}}{Dt} + 2\mathbf{\Omega} \times \mathbf{V} = -\nabla\phi - \alpha\nabla p - \alpha\nabla \cdot \mathbf{F} . \tag{25}$$

Similarly, the kinetic energy equation, (18), becomes

$$\frac{DK}{Dt} = \mathbf{V} \cdot \left(-\nabla\phi - \alpha\nabla p - \alpha\nabla \cdot \mathbf{F}\right) . \tag{26}$$

Component equations in spherical coordinates

Up to this point, we have not used any coordinate systems, except in connection with a comment on the centrifugal acceleration. We now consider spherical coordinates, (λ, φ, r). The unit vectors in the (λ, φ, r) coordinates are $\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$, respectively. You should be able to see that the direction of $\mathbf{e}_\lambda$ depends on longitude, and that the directions of $\mathbf{e}_\varphi$, and $\mathbf{e}_r$ depend on both longitude and latitude. This means that the directions of $\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$ are functions of space, although of course their magnitudes are spatially constant. Simple geometrical reasoning leads to the following formulae:

$$\begin{aligned}\frac{D\mathbf{e}_\lambda}{Dt} &= \frac{D\lambda}{Dt}\sin\varphi\mathbf{e}_\varphi - \cos\varphi\frac{D\lambda}{Dt}\mathbf{e}_r \\ &= \left(\frac{u\tan\varphi}{r}\right)\mathbf{e}_\varphi - \frac{u}{r}\mathbf{e}_r ,\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{D\mathbf{e}_\varphi}{Dt} &= \frac{D\lambda}{Dt}\sin\varphi\mathbf{e}_\lambda - \frac{D\varphi}{Dt}\mathbf{e}_r \\ &= \left(\frac{u\tan\varphi}{r}\right)\mathbf{e}_\lambda - \frac{v}{r}\mathbf{e}_r,\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{D\mathbf{e}_r}{Dt} &= \cos\varphi\frac{D\lambda}{Dt}\mathbf{e}_\lambda + \frac{D\varphi}{Dt}\mathbf{e}_\varphi \\ &= \frac{u}{r}\mathbf{e}_\lambda + \frac{v}{r}\mathbf{e}_\varphi \\ &= \frac{\mathbf{V}_h}{r}.\end{aligned} \tag{29}$$

In (29), $\mathbf{V}_h$ is the horizontal wind vector.

The vector operators that are most commonly used in atmospheric science can be expressed in spherical coordinates as follows:

$$\nabla A = \left(\frac{1}{r\cos\varphi}\frac{\partial A}{\partial\lambda},\ \frac{1}{r}\frac{\partial A}{\partial\varphi},\ \frac{\partial A}{\partial r}\right), \tag{30}$$

$$\nabla\cdot\mathbf{V} = \frac{1}{r\cos\varphi}\frac{\partial V_\lambda}{\partial\lambda} + \frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}\left(V_\varphi\cos\varphi\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(V_r r^2\right), \tag{31}$$

$$\nabla\times\mathbf{V} = \left\{\frac{1}{r}\left[\frac{\partial V_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rV_\varphi\right)\right], \frac{1}{r}\frac{\partial}{\partial r}(rV_\lambda) - \frac{1}{r\cos\varphi}\frac{\partial V_r}{\partial\lambda}, \frac{1}{r\cos\varphi}\left[\frac{\partial V_\varphi}{\partial\lambda} - \frac{\partial}{\partial\varphi}(V_\lambda\cos\varphi)\right]\right\} \tag{32}$$

$$\nabla^2 A = \frac{1}{r^2\cos\varphi}\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{\cos\varphi}\frac{\partial A}{\partial\lambda}\right) + \frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial A}{\partial\varphi}\right) + \frac{\partial}{\partial r}\left(r^2\cos\varphi\frac{\partial A}{\partial r}\right)\right]. \tag{33}$$

Here A is an arbitrary scalar, and $\mathbf{V}$ is an arbitrary vector.

We can express the velocity vector in spherical coordinates as

$$\mathbf{V} \equiv u\mathbf{e}_\lambda + v\mathbf{e}_\varphi + w\mathbf{e}_r, \tag{34}$$

where

$$u \equiv r\cos\varphi \frac{D\lambda}{Dt} \text{ , } v \equiv r\frac{D\varphi}{Dt} \text{ , and } w \equiv \frac{Dr}{Dt} \text{ .} \tag{35}$$

Using (35), the Lagrangian time derivative can then be expanded as

$$\begin{aligned} \frac{D}{Dt} &\equiv \frac{Dt}{Dt}\frac{\partial}{\partial t} + \frac{D\lambda}{Dt}\frac{\partial}{\partial \lambda} + \frac{D\varphi}{Dt}\frac{\partial}{\partial \varphi} + \frac{Dr}{Dt}\frac{\partial}{\partial r} \\ &= \frac{\partial}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial r} . \end{aligned} \tag{36}$$

Eq. (34) shows that the directions in which the u , v , and w components actually point depend on where you are. Taking this into account, for an arbitrary vector $\mathbf{Q} \equiv a\mathbf{e}_\lambda + b\mathbf{e}_\varphi + c\mathbf{e}_r$, we can write

$$\begin{aligned} \frac{D\mathbf{Q}}{Dt} &\equiv \frac{D}{Dt}\left(a\mathbf{e}_\lambda + b\mathbf{e}_\varphi + c\mathbf{e}_r\right) \\ &= \mathbf{e}_\lambda \frac{Da}{Dt} + a\frac{D\mathbf{e}_\lambda}{Dt} + \mathbf{e}_\varphi \frac{Db}{Dt} + b\frac{D\mathbf{e}_\varphi}{Dt} + \mathbf{e}_r \frac{Dc}{Dt} + c\frac{D\mathbf{e}_r}{Dt} \\ &= \mathbf{e}_\lambda \frac{Da}{Dt} + a\left[\left(\frac{u\tan\varphi}{r}\right)\mathbf{e}_\varphi - \frac{u}{r}\mathbf{e}_r\right] + \mathbf{e}_\varphi \frac{Db}{Dt} + b\left[\left(\frac{u\tan\varphi}{r}\right)\mathbf{e}_\lambda - \frac{v}{r}\mathbf{e}_r\right] + \mathbf{e}_r \frac{Dc}{Dt} + c\left(\frac{u}{r}\mathbf{e}_\lambda + \frac{v}{r}\mathbf{e}_\varphi\right) \\ &= \mathbf{e}_\lambda\left(\frac{Da}{Dt} + \frac{bu\tan\varphi}{r} + \frac{uc}{r}\right) + \mathbf{e}_\varphi\left(\frac{Db}{Dt} + \frac{au\tan\varphi}{r} + \frac{vc}{r}\right) + \mathbf{e}_r\left[\frac{Dc}{Dt} - \left(\frac{ua + vb}{r}\right)\right] . \end{aligned} \tag{37}$$

As a special case of (37), the acceleration in the rotating frame can be expanded as

$$\begin{aligned} \frac{D\mathbf{V}}{Dt} &\equiv \frac{D}{Dt}\left(u\mathbf{e}_\lambda + v\mathbf{e}_\varphi + w\mathbf{e}_r\right) \\ &= \mathbf{e}_\lambda \frac{Du}{Dt} + u\frac{D\mathbf{e}_\lambda}{Dt} + \mathbf{e}_\varphi \frac{Dv}{Dt} + v\frac{D\mathbf{e}_\varphi}{Dt} + \mathbf{e}_r \frac{Dw}{Dt} + w\frac{D\mathbf{e}_r}{Dt} \\ &= \mathbf{e}_\lambda \frac{Du}{Dt} + u\left[\left(\frac{u\tan\varphi}{r}\right)\mathbf{e}_\varphi - \frac{u}{r}\mathbf{e}_r\right] + \mathbf{e}_\varphi \frac{Dv}{Dt} + v\left[\left(\frac{u\tan\varphi}{r}\right)\mathbf{e}_\lambda - \frac{v}{r}\mathbf{e}_r\right] + \mathbf{e}_r \frac{Dw}{Dt} + w\left(\frac{u}{r}\mathbf{e}_\lambda + \frac{v}{r}\mathbf{e}_\varphi\right) \\ &= \mathbf{e}_\lambda\left(\frac{Du}{Dt} + \frac{uv\tan\varphi}{r} + \frac{uw}{r}\right) + \mathbf{e}_\varphi\left(\frac{Dv}{Dt} + \frac{u^2\tan\varphi}{r} + \frac{vw}{r}\right) + \mathbf{e}_r\left[\frac{Dw}{Dt} - \left(\frac{u^2 + v^2}{r}\right)\right] . \end{aligned} \tag{38}$$

Using (38), we can separate (8) into component form:

$$\frac{Du}{Dt}-\left(2\Omega+\frac{u}{r\cos\varphi}\right)(v\sin\varphi-w\cos\varphi)=-\frac{\alpha}{r\cos\varphi}\frac{\partial p}{\partial\lambda}-\alpha(\nabla\cdot\mathbf{F})_\lambda,$$
$$\frac{Dv}{Dt}+\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\sin\varphi+\frac{vw}{r}=-\frac{\alpha}{r}\frac{\partial p}{\partial\varphi}-\alpha(\nabla\cdot\mathbf{F})_\varphi,$$
$$\frac{Dw}{Dt}-\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\cos\varphi-\frac{v^2}{r}+g=-\alpha\frac{\partial p}{\partial r}-\alpha(\nabla\cdot\mathbf{F})_r.$$

(39)

These are the components of the equation of motion in spherical coordinates.

We can modify (39) to show “true gravity” and the centrifugal acceleration explicitly:

$$\frac{Du}{Dt}-\left(2\Omega+\frac{u}{r\cos\varphi}\right)(v\sin\varphi-w\cos\varphi)=-\frac{\alpha}{r\cos\varphi}\frac{\partial p}{\partial\lambda}-\alpha(\nabla\cdot\mathbf{F})_\lambda,$$
$$\frac{Dv}{Dt}+\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\sin\varphi+\frac{vw}{r}+\Omega^2 r\sin\varphi\cos\varphi=-\frac{\alpha}{r}\frac{\partial p}{\partial\varphi}-\alpha(\nabla\cdot\mathbf{F})_\varphi,$$
$$\frac{Dw}{Dt}-\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\cos\varphi-\frac{v^2}{r}-\Omega^2 r\cos^2\varphi+g_a=-\alpha\frac{\partial p}{\partial r}-\alpha(\nabla\cdot\mathbf{F})_r.$$

(40)

By using the continuity equation in spherical coordinates, we can rewrite (39) in flux form:

$$\frac{\partial}{\partial t}(\rho u)+\nabla\cdot(\rho\mathbf{V}u)-\rho\left(2\Omega+\frac{u}{r\cos\varphi}\right)(v\sin\varphi-w\cos\varphi)=-\frac{1}{r\cos\varphi}\frac{\partial p}{\partial\lambda}-(\nabla\cdot\mathbf{F})_\lambda,$$
$$\frac{\partial}{\partial t}(\rho v)+\nabla\cdot(\rho\mathbf{V}v)+\rho\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\sin\varphi+\rho\frac{vw}{r}=-\frac{1}{r}\frac{\partial p}{\partial\varphi}-(\nabla\cdot\mathbf{F})_\varphi,$$
$$\frac{\partial}{\partial t}(\rho w)+\nabla\cdot(\rho\mathbf{V}w)-\rho\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\cos\varphi-\rho\frac{v^2}{r}+\rho g=-\frac{\partial p}{\partial r}-(\nabla\cdot\mathbf{F})_r.$$

(41)

These equations are fairly exact. Various approximations are introduced below.

Solid-body rotation

It is instructive to consider the special case of zonal solid-body rotation, in which

$$u=\omega r\cos\varphi,\ v=0,\ w=0,\ \text{and}\ p=p(\varphi,r),$$

(42)

where $\omega=$ constant. This type of motion is called “solid-body rotation,” because the fluid rotates as if it were a solid, i.e., neighboring particles remain neighbors for all time. The flow is purely zonal. Eqs. (41) reduce to

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\rho u) = 0, \\
&\rho u^2 \frac{\tan\varphi}{r} = -\rho f u - \frac{1}{r}\frac{\partial p}{\partial \varphi}, \\
&-\rho\left(\frac{u^2}{r}\right) = \rho f u - \frac{\partial p}{\partial r} - \rho g.
\end{aligned}
\tag{43}
$$

Here we have assumed that the stress tensor vanishes (it does). The zonal wind equation is in trivial balance. The meridional momentum equation is in "gradient-wind balance," which is a generalization of geostrophic balance. The vertical momentum equation is in a modified hydrostatic balance, in which the centrifugal and Coriolis accelerations enter.

Approximations

Up to here the discussion has been fairly exact. We now introduce some very useful approximations:

- Replace r by a everywhere, where a is the radius of the Earth. An approximation of this form can be justified for an atmosphere which is thin compared to the radius of the planet, and so it is called the "thin atmosphere approximation." It is a good approximation for Earth, but would not apply, e.g., to Jupiter.
- Replace (33) by

$$
\nabla \cdot \mathbf{V} = \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial \lambda}V_\lambda + \frac{\partial}{\partial \varphi}\left(V_\varphi \cos\varphi\right)\right] + \frac{\partial V_r}{\partial r}.
\tag{44}
$$

  This is also justified by the thinness of the Earth's atmosphere.

- Drop the terms involving the horizontal component of $\mathbf{\Omega}$, i.e., $\Omega\cos\varphi$. This is often called "the traditional approximation." There is an ongoing discussion as to whether or not this is always justified.
- Neglect $\frac{uw}{r}$ and $\frac{vw}{r}$, the curvature terms involving w, in the equations for u and v, respectively, neglect $\frac{u^2+v^2}{r}$ in the equation of vertical motion.
- Simplify the vertical component of (40) to

$$
\frac{\partial p}{\partial z} = -\rho g .
\tag{45}
$$

Eq. (45) is called the hydrostatic equation. With an appropriate boundary condition, (45) allows us to compute $p(z)$ from $\rho(z)$. Even when the air is moving, (45) gives a good *approximation* to $p(z)$, simply because $\frac{Dw}{Dt}$ and the vertical component of the friction force are small compared to *g*. Eq. (45) as applied to moving air is called the hydrostatic approximation, and it is applicable to virtually all meteorological phenomena, including violent thunderstorms.

For large-scale circulations, the approximate $p(z)$ determined through the use of (45) can be used to compute the pressure gradient force in the equation of horizontal motion . To do so is to use the quasi-static approximation. The quasi-static approximation applies very well for large-scale motions, but it is not applicable to many small-scale motions, such as thunderstorms. When the quasi-static approximation is made, the effective kinetic energy is due entirely to the horizontal motion; the contribution of the vertical component, *w*, is neglected. For large-scale motions, $w << (u,v)$, so that this quasi-static kinetic energy is very close to the true kinetic energy. Further discussion is given in the *QuickStudy* on the quasi-static approximation.

With the approximations discussed above, (39) is replaced by

$$\frac{Du}{Dt} - \frac{uv\tan\varphi}{a} = fv - \frac{\alpha}{a\cos\varphi}\frac{\partial p}{\partial\lambda} - \alpha(\nabla\cdot\mathbf{F})_\lambda ,$$
$$\frac{Dv}{Dt} + \frac{u^2\tan\varphi}{a} = -fu - \frac{\alpha}{a}\frac{\partial p}{\partial\varphi} - \alpha(\nabla\cdot\mathbf{F})_\varphi ,$$
$$0 = -g - \alpha\frac{\partial p}{\partial z}. \tag{46}$$

Component equations in a local Cartesian coordinate system

Now consider a locally defined "(x,y)" or Cartesian coordinate system, as shown in Fig. 2. Here "locally defined" means that the origin of the coordinate system is attached to a specific point on the rotating Earth, e.g., Fort Collins. With this coordinate system,

$$\mathbf{\Omega} = \Omega(0,\ \cos\varphi,\ \sin\varphi), \tag{47}$$

where φ is the latitude of the origin of the coordinate system, and

$$-\mathbf{\Omega}\times\mathbf{V}=-2\Omega\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & \cos\varphi & \sin\varphi\\ u & v & w\end{vmatrix}.$$

(48)

The directions of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are independent of position, although they do of course depend on the freely chosen position of the origin of the coordinate system. The components of the equation of motion in this local Cartesian coordinate system are

$$\begin{aligned}\frac{Du}{Dt}&=-\bar{f}w+fv-\alpha\frac{\partial p}{\partial x}-\alpha\left(\nabla\cdot\mathbf{F}\right)_x,\\ \frac{Dv}{Dt}&=-fu-\alpha\frac{\partial p}{\partial y}-\alpha\left(\nabla\cdot\mathbf{F}\right)_y,\\ \frac{Dw}{Dt}&=\bar{f}u-\alpha\frac{\partial p}{\partial z}--\alpha\left(\nabla\cdot\mathbf{F}\right)_z-g.\end{aligned}$$

(49)

Here we define

$$\mathbf{V}\equiv u\mathbf{i}+v\mathbf{j}+w\mathbf{k},$$

(50)

$$u\equiv\frac{Dx}{Dt},\ v\equiv\frac{Dy}{Dt},\ w\equiv\frac{Dz}{Dt},$$

(51)

$$f\equiv 2\Omega\sin\varphi\text{, and }\bar{f}\equiv 2\Omega\cos\varphi,$$

(52)

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

(53)

At the origin, and at all points with the same longitude as the origin, the x -coordinate points east; but because this Cartesian coordinate system is defined with respect to a fixed location on the Earth's surface, the x -coordinate does not point east at other longitudes. For example, at points 90° to the east of the origin, the x-coordinate points up, and on the opposite side of the

Earth from the origin, the x -coordinate points west. Such a coordinate system could be used to

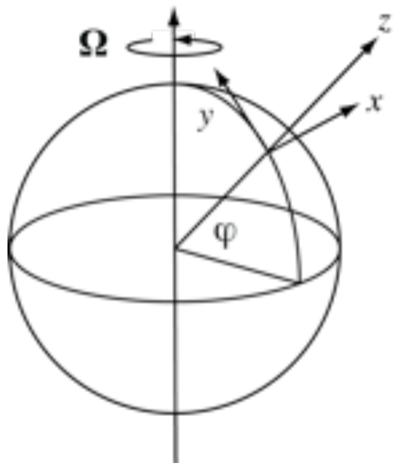

Fig. 2: Sketch depicting a local Cartesian coordinate system.

study the general circulation, but clearly it is not very well suited to such an application.

Comparing (49) with (39), we see that (49) does not contain the terms $\frac{uw}{r}$, $\frac{uv\tan\varphi}{r}$, etc. These so-called "metric" terms arise in (39) because in spherical coordinates the directions of the unit vectors $\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$ vary with λ and φ. This does not happen in the local Cartesian coordinate system.

Summary

We have presented the momentum equation that describes motion on a rotating sphere, as seen in the rotating frame of reference. We have also introduced several commonly used approximations.

References and Bibliography

Phillips, N. A., 1966: The equations of motion for a shallow rotating atmosphere and the "traditional approximation." *J. Atmos. Sci.*, **23**, 626-628.

White, A. A., and R. A. Bromley, 1995: Dynamically consistent, quasi-hydrostatic equations for global models with a complete representation of the Coriolis force. *Quart. J. Roy. Meteor. Soc.*, **121**, 399-418.