A Non-Hydrostatic Model Using Isentropic Coordinates

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The basic equations in height coordinates

The basic equations in height coordinates, without rotation and friction, are

$$\frac{D\mathbf{V}_h}{Dt} = -\frac{1}{\rho}\nabla_z p \, , \tag{1}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \, , \tag{2}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_z + \nabla_z \cdot (\rho \mathbf{V}_h) + \frac{\partial}{\partial z}(\rho w) = 0 \, , \tag{3}$$

$$\dot{\theta} \equiv \frac{D\theta}{Dt} = \frac{Q}{\Pi} \, . \tag{4}$$

Here $\frac{D}{Dt}$ is the Lagrangian time derivative, $\mathbf{V}_h$ is the horizontal velocity, ρ is density, p is pressure, z is height, $w \equiv \frac{Dz}{Dt}$ is the vertical velocity, g is the acceleration of gravity, θ is the potential temperature, Q is the heating rate per unit mass, and Π is the Exner function, which satisfies

$$c_p T = \Pi\theta \, , \tag{5}$$

where c_p is the heat capacity of air at constant pressure, T is temperature, and

$$\Pi = c_p \left(\frac{p}{p_0} \right)^{\kappa} , \tag{6}$$

where

$$\kappa \equiv \frac{R}{c_p} , \tag{7}$$

and R is the specific gas constant. Finally, we include the prognostic equation for an arbitrary scalar A, which is

$$\left[\frac{\partial}{\partial t} (\rho A) \right]_z + \nabla_z \cdot (\rho \mathbf{V}_h A) + \frac{\partial}{\partial z} (\rho w A) = \rho S_A , \tag{8}$$

where S_A is the source of A per unit mass. Finally, we will need the ideal gas law, which is

$$p = \rho R T . \tag{9}$$

Transformation to θ *-coordinates*

As shown in the *QuickStudy* on coordinate transformations, the horizontal pressure gradient can be transformed to isentropic coordinates as follows:

$$\frac{1}{\rho} \nabla_z p = \frac{1}{\rho} \nabla_\theta p - \frac{1}{\rho} \frac{\partial p}{\partial z} \nabla_\theta z . \tag{10}$$

The first term on the right-hand side of (10) can be rewritten as follows, using (5), (6), (7) and (9):

$$\begin{aligned}\frac{1}{\rho}\nabla_\theta p &= RT\frac{\nabla_\theta p}{p}\\ &= \frac{RT}{\kappa}\frac{\nabla_\theta \Pi}{\Pi}\\ &= \frac{c_p T}{\Pi}\nabla_\theta \Pi\\ &= \theta\nabla_\theta \Pi\\ &= \nabla_\theta\left(\Pi\theta\right)\\ &= \nabla_\theta\left(c_p T\right)\\ &= \nabla_\theta s - g\nabla_\theta z\ ,\end{aligned} \tag{11}$$

where

$$s \equiv c_p T + gz \tag{12}$$

Is the Montgomery potential, also known as the dry static energy. Similarly, we write minus the vertical pressure-gradient force as

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial z} &= \frac{RT}{p}\frac{\partial p}{\partial z}\\ &= \frac{RT}{\kappa\Pi}\frac{\partial \Pi}{\partial z}\\ &= \frac{c_p T}{\Pi}\frac{\partial \Pi}{\partial z}\\ &= \theta\frac{\partial \Pi}{\partial z}\\ &= \frac{\partial}{\partial z}\left(\Pi\theta\right) - \Pi\frac{\partial \theta}{\partial z}\\ &= \frac{\partial}{\partial z}\left(c_p T\right) - \Pi\frac{\partial \theta}{\partial z}\\ &= \frac{\partial s}{\partial z} - g - \Pi\frac{\partial \theta}{\partial z}\\ &= \frac{\partial \theta}{\partial z}\left(\frac{\partial s}{\partial \theta} - \Pi\right) - g\ .\end{aligned} \tag{13}$$

Using (11) and (13) in (10), we find that

$$\frac{1}{\rho}\nabla_z p = \left(\nabla_\theta s - g\nabla_\theta z\right) - \left[\frac{\partial\theta}{\partial z}\left(\frac{\partial s}{\partial\theta} - \Pi\right) - g\right]\nabla_\theta z$$
$$= \nabla_\theta s - \frac{\partial\theta}{\partial z}\left(\frac{\partial s}{\partial\theta} - \Pi\right)\nabla_\theta z \, .$$

(14)

From (13), we see that the hydrostatic equation can be expressed as

$$\frac{\partial s}{\partial\theta} - \Pi = 0 \, .$$

(15)

Eq. (14) shows that in the hydrostatic limit the horizontal pressure-gradient force is a gradient, when expressed in θ-coordinates.

With the use of (13) and (14), we can now rewrite (1) and (2) as

$$\frac{D\mathbf{V}_h}{Dt} = -\left[\nabla_\theta s - \frac{\partial\theta}{\partial z}\left(\frac{\partial s}{\partial\theta} - \Pi\right)\nabla_\theta z\right],$$

(16)

$$\frac{Dw}{Dt} = -\frac{\partial\theta}{\partial z}\left(\frac{\partial s}{\partial\theta} - \Pi\right).$$

(17)

Using methods discussed in the *QuickStudy* on coordinate transformations, the continuity equation, (3), can be written as

$$\left(\frac{\partial\rho}{\partial t}\right)_\theta - \frac{\partial\theta}{\partial z}\frac{\partial\rho}{\partial\theta}\left(\frac{\partial z}{\partial t}\right)_\theta + \nabla_\theta \cdot (\rho\mathbf{V}_h) - \frac{\partial\theta}{\partial z}\left[\frac{\partial}{\partial\theta}\left(\rho\mathbf{V}_h\right)\right]\cdot\nabla_\theta z + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}\left(\rho w\right) = 0 \, ,$$

(18)

or

$$\frac{\partial z}{\partial\theta}\left(\frac{\partial\rho}{\partial t}\right)_\theta - \frac{\partial\rho}{\partial\theta}\left(\frac{\partial z}{\partial t}\right)_\theta + \nabla_\theta \cdot (\rho\mathbf{V}_h) - \left[\frac{\partial}{\partial\theta}\left(\rho\mathbf{V}_h\right)\right]\cdot\nabla_\theta z + \frac{\partial}{\partial\theta}\left(\rho w\right) = 0 \, .$$

(19)

Note that

$$\left[\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial\theta}\right)\right]_\theta = \frac{\partial z}{\partial\theta}\left(\frac{\partial\rho}{\partial t}\right)_\theta + \rho\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial t}\right)_\theta ,$$

(20)

and

$$\nabla_\theta \cdot \left(\rho \frac{\partial z}{\partial \theta}\mathbf{V}_h\right) = \frac{\partial z}{\partial \theta}\nabla_\theta \cdot (\rho \mathbf{V}_h) + \rho \mathbf{V}_h \cdot \nabla_\theta \left(\frac{\partial z}{\partial \theta}\right). \tag{21}$$

Substituting (20) and (21) into (19), we obtain

$$\left(\frac{\partial \rho_\theta}{\partial t}\right)_\theta + \nabla_\theta \cdot (\rho_\theta \mathbf{V}_h) - \frac{\partial}{\partial \theta}\left\{\rho\left[\left(\frac{\partial z}{\partial t}\right)_\theta + \mathbf{V}_h \cdot \nabla_\theta z - w\right]\right\} = 0\,, \tag{22}$$

where

$$\rho_\theta \equiv \rho \frac{\partial z}{\partial \theta} \tag{23}$$

is the pseudo-density. We recognize

$$\mu \equiv -\rho\left[\left(\frac{\partial z}{\partial t}\right)_\theta + \mathbf{V}_h \cdot \nabla_\theta z - w\right] \tag{24}$$

as the upward mass flux across an isentropic surface. This allows us to rewrite (22) as

$$\left(\frac{\partial \rho_\theta}{\partial t}\right)_\theta + \nabla_\theta \cdot (\rho_\theta \mathbf{V}_h) + \frac{\partial \mu}{\partial \theta} = 0\,. \tag{25}$$

In a similar way, the conservation equation for an intensive scalar, (8), becomes

$$\left[\frac{\partial}{\partial t}(\rho_\theta A)\right]_\theta + \nabla_\theta \cdot (\rho_\theta \mathbf{V}_h A) + \frac{\partial}{\partial \theta}(\mu A) = \rho_\theta S_A\,. \tag{26}$$

The advective form can be obtained by combining (26) with (25):

$$\left(\frac{\partial A}{\partial t}\right)_\theta + \mathbf{V}_h \cdot \nabla_\theta A + \frac{\mu}{\rho_\theta}\frac{\partial A}{\partial \theta} = S_A\,. \tag{27}$$

As a special case of (27), the advective form of the potential temperature equation is

$$\mu = \rho_\theta \dot{\theta} ,$$

(28)

where we have used $\dot{\theta} \equiv S_\theta$. By using (28), we can rewrite (26) as

$$\left[\frac{\partial}{\partial t} \left(\rho_\theta A \right) \right]_\theta + \nabla_\theta \cdot (\rho_\theta \mathbf{V}_h A) + \frac{\partial}{\partial \theta} \left(\rho_\theta \dot{\theta} A \right) = \rho_\theta S_A .$$

(29)

The total time derivative is given by

$$\frac{D}{Dt}(\ ) = \left(\frac{\partial}{\partial t} \right)_\theta (\ ) + \mathbf{V}_h \cdot \nabla_\theta (\ ) + \dot{\theta} \frac{\partial}{\partial \theta} (\ ).$$

(30)

Summary of the nonhydrostatic equations in θ *-coordinates*

The prognostic equations needed to describe non-hydrostatic motions in θ -coordinates can now be summarized as follows:

$$\boxed{\frac{D\mathbf{V}_h}{Dt} = -\left[\nabla_\theta s - \frac{\partial \theta}{\partial z} \left(\frac{\partial s}{\partial \theta} - \Pi \right) \nabla_\theta z \right]},$$

(31)

$$\boxed{\frac{Dw}{Dt} = -\frac{\partial \theta}{\partial z} \left(\frac{\partial s}{\partial \theta} - \Pi \right)},$$

(32)

$$\boxed{\left(\frac{\partial \rho_\theta}{\partial t} \right)_\theta + \nabla_\theta \cdot \left(\rho_\theta \mathbf{V}_h \right) + \frac{\partial}{\partial \theta} \left(\rho_\theta \dot{\theta} \right) = 0},$$

(33)

$$\boxed{\left(\frac{\partial z}{\partial t} \right)_\theta = -\mathbf{V}_h \cdot \nabla_\theta z + w - \frac{\partial z}{\partial \theta} \dot{\theta}},$$

(34)

$$\boxed{\left[\frac{\partial}{\partial t} \left(\rho_\theta A \right) \right]_\theta + \nabla_\theta \cdot (\rho_\theta \mathbf{V}_h A) + \frac{\partial}{\partial \theta} \left(\rho_\theta \dot{\theta} A \right) = \rho_\theta S_A}.$$

(35)

Not counting the scalar A , the prognostic variables of the θ -coordinate model are $\mathbf{V}_h$, w , ρ_θ , and z . The corresponding prognostic variables of the z -coordinate model are $\mathbf{V}_h$, w , ρ , and θ .

Finally, we have to determine Π . From ρ_θ and $z(\theta)$, we can find the density ρ . Then the equation of state in the form

$$\Pi = c_p \left(\frac{\rho R \theta}{p_0} \right)^{\frac{\kappa}{1-\kappa}} \tag{36}$$

can be used to diagnose Π .

The quasi-static limit

In the quasi-static limit, the equation of vertical motion, (32), reduces to the hydrostatic equation in the form

$$\frac{\partial s}{\partial \theta} - \Pi = 0 , \tag{37}$$

and Eq. (23) for the pseudo-density can be rewritten as

$$\rho_\theta = -\frac{1}{g} \frac{\partial p}{\partial \theta} . \tag{38}$$

To obtain (38) we have used the hydrostatic equation in the form

$$\frac{\partial p}{\partial z} = -\rho g . \tag{39}$$

Even in the quasi-static limit, the pseudo-density is still predicted using (33). Using the known value of the pseudo-density, we can vertically integrate (38) to obtain the pressure and Π as functions of θ . We can then use (36) to determine ρ as a function of θ . Finally, we use (39) to compute the height of each θ surface, so that (34) is not needed and should not be used. The procedure outlined above can be streamlined somewhat.

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