

Why the Dissipation Rate is Positive

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A simplified proof that the dissipation rate is non-negative is as follows: Recall that the dissipation rate is given by

$$\delta = -(\mathbf{F} \bullet \nabla) \bullet \mathbf{V} , \tag{1.1}$$

where $\mathbf{F}$ is the stress tensor, which can be represented in matrix form by

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & \rho\overline{w'u'} \\ 0 & 0 & \rho\overline{w'v'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & 0 \end{bmatrix} , \tag{1.2}$$

and $\overline{\mathbf{V}}$ is the mean horizontal wind vector. Here we have used the turbulent fluxes, rather than the viscous fluxes, and we have ignored the horizontal fluxes of momentum for the usual reason. Then

$$\mathbf{F} \bullet \nabla = \left(\rho\overline{w'u'}\frac{\partial}{\partial z}, \rho\overline{w'v'}\frac{\partial}{\partial z}, 0 \right), \tag{1.3}$$

and

$$(\mathbf{F} \bullet \nabla) \bullet \overline{\mathbf{V}} = \rho\overline{w'u'}\frac{\partial}{\partial z}\bar{u} + \rho\overline{w'v'}\frac{\partial}{\partial z}\bar{v} . \tag{1.4}$$

If we now assume that the turbulent momentum fluxes are *downgradient*, i.e. from regions of stronger wind to regions of weaker wind, so that

$$\rho\overline{w'u'} = -K\frac{\partial}{\partial z}\bar{u} , \text{ and } \rho\overline{w'v'} = -K\frac{\partial}{\partial z}\bar{v} , \tag{1.5}$$

where K is a *positive* “eddy viscosity,” then we find by substitution above that

$$\delta = -(\mathbf{F} \bullet \nabla) \bullet \overline{\mathbf{V}} = K\left[\left(\frac{\partial}{\partial z}\bar{u}\right)^2 + \left(\frac{\partial}{\partial z}\bar{v}\right)^2\right] \geq 0 . \tag{1.6}$$

The conclusion is that downgradient turbulent momentum fluxes lead to dissipation of the kinetic energy of the mean flow. A similar argument can be given in terms of the effects of the molecular viscosity on the small-scale motion field.

The assumption of downgradient turbulent momentum fluxes is by no means well justified; countergradient fluxes are routine. For the effects of the molecular viscosity, however, downgradient momentum exchange is very well justified, so the conclusion that molecular viscosity leads to positive dissipation of kinetic energy is very sound.

References and Bibliography

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