The Quasi-Static Approximation

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For resting air, the equation of vertical motion reduces to

$$\frac{\partial p}{\partial z} = -\rho g \; . \tag{1}$$

This is called the hydrostatic approximation. With an appropriate boundary condition, (1) allows us to compute $p(z)$ from $\rho(z)$. Even when the air is moving, (1) gives an exellent approximation to $p(z)$, simply because $\frac{Dw}{Dt}$ is small compared to g, even in a strong convective updraft. The hydrostatic equation itself is almost always a good approximation.

The quasi-static approximation is harder to justify. It consists of using *the approximate* $p(z)$ *obtained from (1) to compute the pressure-gradient force in the equation of horizontal motion*? The issue is that the horizontally varying part of the pressure, which is what matters for the horizontal pressure-gradient force, is usually quite small compared to the total pressure. Even through the total pressure is almost always well approximated by (1), the horizontally varying part of the pressure as determined from (1) may contain large errors.

To investigate the range of validity of the quasi-static approximation, define

$$\begin{aligned} p &= p_s(z) + \delta p, \\ \alpha &= \alpha_s(z) + \delta\alpha, \end{aligned} \tag{2}$$

where the subscript s denotes a "reference sounding" that varies only with height. *Define* the reference state in such a way that it is hydrostatically balanced, i.e.,

$$\alpha_s \frac{\partial p_s}{\partial z} = -g \; . \tag{3}$$

With these definitions, the equation of vertical motion can be written as

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$$\frac{Dw}{Dt} = -\alpha_s \frac{\partial p_s}{\partial z} - \delta\alpha \frac{\partial p_s}{\partial z} - \alpha_s \frac{\partial \delta p}{\partial z} - \delta\alpha \frac{\partial \delta p}{\delta z} - g \; . \tag{4}$$

Here we neglect the effects of rotation and friction. The first and last terms on the right-hand-side of (4) cancel, because of (3). The second-to-last term on the right-hand side of (4) can be neglected because it contains the product of two δ 's. This is justified if $\alpha_s >> |\delta\alpha|$, and $p_s >> |\delta p|$. We scale the remaining terms of (4) as follows:

$$\frac{Dw}{Dt} \cong \frac{\delta\alpha}{\alpha_s} g - \alpha_s \frac{\partial}{\partial z}(\delta p) \; . \tag{5}$$

$$NW \qquad\qquad \alpha_s \frac{\delta p}{D}$$

Here N^{-1} is an advective time scale, W is a vertical velocity scale, and D is a depth scale.

Again neglecting friction, the horizontal momentum equation can be written as

$$\frac{D\mathbf{V}_H}{Dt} = -\alpha_s \nabla_H (\delta p) - f\mathbf{k} \times \mathbf{V}_H \; . \tag{6}$$

$$NV \qquad\qquad \alpha_s \frac{\delta p}{L} \qquad\qquad fV$$

Here V is a horizontal velocity scale, and L is a horizontal length scale.

Consider two cases:

1. $N \geq f$ ("short" advective time scale; advection faster than rotation)

This is the case in which the advective time scale, N^{-1}, is small compared to an inertial period, which is usually true for *small-scale motions*. If $N \sim V / L$, then $N \geq f$ equivalent to $Ro \geq 1$, where $Ro \equiv \frac{V}{fL}$ is the Rossby number. For this case, we find from (6) that

$$\alpha_s \delta p \sim NLV \; . \tag{7}$$

Therefore, to have $\left|\frac{Dw}{Dt}\right| << \left|\alpha \frac{\partial}{\partial z}(\delta p)\right|$, we need

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$$NW << \frac{NLV}{D}, \tag{8}$$

or

$$\left(\frac{W}{D}\cdot\frac{L}{V}\right)\left(\frac{D}{L}\right)^2 << 1. \tag{9}$$

Normally (from continuity), we have

$$\frac{W}{D}\cdot\frac{L}{V} \le 1, \tag{10}$$

provided that the stratification is stable. Then (9) shows that "fast" motions can be quasi-static if they are "shallow" in the sense that $\left(D/L\right)^2 << 1$, but not if they are "deep." Deep convective circulations do not satisfy (9).

2. $N \le f$ ("long" advective time scale; rotation faster than advection)

This is the case in which the advective time scale is relatively long, which is usually true for *large-scale motions*. If we assume that $N \sim V/L$, then $N \le f$ is equivalent to $Ro \le 1$. For this case, (6) leads to

$$\alpha_s \delta p \sim fLV, \tag{11}$$

which is essentially an expression of geostrophic balance. To have $\left|\frac{Dw}{Dt}\right| << \left|\alpha_s \frac{\partial}{\partial z}(\delta p)\right|$, we need

$$NW << \frac{fLV}{D}, \tag{12}$$

or

$$\left(\frac{W}{D}\cdot\frac{L}{V}\right)\left(\frac{D}{L}\right)^2 \frac{N}{f} << 1. \tag{13}$$

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Compare this with (9). The only difference is the factor of N/f, which by assumption is less than one. Large-scale, geostrophically balanced circulations normally satisfy

$$\left(\frac{D}{L}\right)^2 << 1 , \tag{14}$$

so they are expected to satisfy quasi-static balance.

When the quasi-static approximation is made, the effective kinetic energy is due entirely to the horizontal motion; the vertical component of the velocity, w, makes no contribution. For large-scale motions, $w << |\mathbf{V}_H|$, so that this quasi-static kinetic energy is very close to the true kinetic energy.

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