Separation of Variables

David A. Randall

Department of Atmospheric Science
Colorado State University, Fort Collins, Colorado 80523

We illustrate the method of “Separation of Varriables” with a simple example. The motion of a vibrating string is described by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \ . \tag{1}$$

Assume a solution of the form

$$u(x, t) = X(x)\, T(t) \ , \tag{2}$$

i.e., a function of x only multiplied by a function of t only. Then substitution gives

$$XT'' = a^2 X'' T \ , \tag{3}$$

which can be written as

$$\frac{X''}{X} = \frac{T''}{a^2 T} \ . \tag{4}$$

Since X is independent of t, and T is independent of x, we conclude that

$$\frac{X''}{X} = -\lambda^2 \text{ and } \frac{T''}{a^2 T} = -\lambda^2 \ . \tag{5}$$

where λ^2 is a *constant*, called the “separation constant,” independent of *both* x and t.

Adopt boundary conditions

$$u(0, t) = u(1, t) = 0 \ . \tag{6}$$

This means the string is clamped on both ends. Then

$$X = A\sin(n\pi x) \ , \tag{7}$$

where n is an integer. We see that

$$(n\pi)^2 = \lambda^2 \quad . \tag{8}$$

Moreover,

$$T = e^{i\sigma t} \quad . \tag{9}$$

where

$$\sigma^2 = a^2\, \lambda^2 (an\pi)^2 \quad . \tag{10}$$

The full solution of (1) is thus

$$u(x,\, t) = A\sin(n\pi x)\, e^{\pm ian\pi t} \quad . \tag{11}$$