
Static energies

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The mechanical energy equation

The kinetic energy equation can be derived from the three-dimensional equation of motion in the form

$$\frac{\partial \mathbf{V}}{\partial t} + (\nabla \times \mathbf{V} + 2\boldsymbol{\Omega}) \times \mathbf{V} + \nabla(K + \phi) = -\alpha \nabla p - \alpha \nabla \cdot \mathbf{F}, \quad (1)$$

where

$$K \equiv \frac{1}{2} \mathbf{V} \cdot \mathbf{V} \quad (2)$$

is the kinetic energy per unit mass. Here we are using height coordinates. Dotting (1) with \mathbf{V} , we find that

$$\frac{\partial K}{\partial t} + \mathbf{V} \cdot \nabla(K + \phi) = -\alpha \mathbf{V} \cdot \nabla p - \alpha \mathbf{V} \cdot \nabla \cdot \mathbf{F}. \quad (3)$$

Since ϕ is independent of time in height coordinates, we can rewrite (3) as

$$\frac{D(K + \phi)}{Dt} = -\alpha \mathbf{V} \cdot \nabla p - \alpha \mathbf{V} \cdot (\nabla \cdot \mathbf{F}). \quad (4)$$

We refer to $K + \phi$ as the mechanical energy per unit mass, and Eq. (4) is sometimes called the mechanical energy equation.

In (4), the rate at which work is done by the pressure force, per unit mass, is represented by $-\alpha \mathbf{V} \cdot \nabla p$. This expression can be manipulated as follows:

$$\begin{aligned}
-\alpha \mathbf{V} \cdot \nabla p &= -\alpha \nabla \cdot (p \mathbf{V}) + \alpha (p \nabla \cdot \mathbf{V}) \\
&= -\alpha \nabla \cdot (p \mathbf{V}) + p \frac{D\alpha}{Dt} \\
&= -\alpha \nabla \cdot (p \mathbf{V}) + \frac{D}{Dt}(p\alpha) - \frac{Dp}{Dt} \alpha \\
&= -\alpha \nabla \cdot (p \mathbf{V}) + \frac{D}{Dt}(RT) - \omega \alpha .
\end{aligned}
\tag{5}$$

On the second line of (5), we have used the continuity equation to eliminate $\alpha \nabla \cdot \mathbf{V}$, and on the fourth line we have used

$$\omega \equiv \frac{Dp}{Dt} ,
\tag{6}$$

and the equation of state in the form

$$p\alpha = RT ,
\tag{7}$$

where R is the gas constant.

Similarly, the friction term of (3) can be expanded to reveal two physically distinct parts, as follows:

$$-\alpha \mathbf{V} \cdot (\nabla \cdot \mathbf{F}) = -\alpha \nabla \cdot (\mathbf{F} \cdot \mathbf{V}) - \delta ,
\tag{8}$$

where

$$\delta \equiv -\alpha (\mathbf{F} \cdot \nabla) \cdot \mathbf{V}
\tag{9}$$

is the rate of kinetic energy dissipation per unit volume.

Substitution of (5) and (8) into (4) gives the mechanical energy equation in the form

$$\frac{D(K + \phi - RT)}{Dt} = -\alpha \nabla \cdot (p \mathbf{V} + \mathbf{F} \cdot \mathbf{V}) - \omega \alpha - \delta .
\tag{10}$$

This is somewhat easier to interpret when we use continuity to convert to flux form:

$$\frac{\partial}{\partial t}[\rho(K + \phi) - p] + \nabla \cdot [\rho \mathbf{V}(K + \phi) + \mathbf{F} \cdot \mathbf{V}] = -\rho(\omega\alpha) - \rho\delta . \quad (11)$$

Continuity can be used again to rewrite (11) as

$$\frac{D(K + \phi)}{Dt} = -\left(\omega - \frac{\partial p}{\partial t}\right)\alpha - \alpha\nabla \cdot (\mathbf{F} \cdot \mathbf{V}) - \delta . \quad (12)$$

As an approximation, we can write

$$\omega - \frac{\partial p}{\partial t} \cong \omega . \quad (13)$$

For example, this is justified when $\left|w \frac{\partial p}{\partial z}\right| \gg \left|\frac{\partial p}{\partial t}\right|$, which would be the case in a cumulus updraft or downdraft. With the use of (13), we can approximate (12) as

$$\boxed{\frac{D(K + \phi)}{Dt} \cong -\omega\alpha - \alpha\nabla \cdot (\mathbf{F} \cdot \mathbf{V}) - \delta} . \quad (14)$$

Conservation of thermodynamic energy

The thermodynamic energy equation can be written in the form

$$\boxed{\frac{D}{Dt}(c_p T) = \omega\alpha - \alpha\nabla \cdot (\mathbf{R} + \mathbf{F}_s) + LC + \delta} . \quad (15)$$

Here T is temperature, c_p is the specific heat of air at constant pressure, \mathbf{F}_s is the vector flux of internal energy due to molecular diffusion; \mathbf{R} is the vector flux of energy due to radiation, L is the latent heat of condensation, and C is the rate of condensation per unit mass. The dissipation rate appears in (15) as a source of internal energy. We ignore the ice phase for now.

Conservation of total energy

When we add (14)-(15), the $\omega\alpha$ and dissipation terms cancel, and we obtain

$$\frac{D}{Dt}(K + c_p T + \phi) = -\alpha \nabla \cdot (\mathbf{F} \cdot \mathbf{V} + \mathbf{R} + \mathbf{F}_h) + LC . \quad (16)$$

This is a form of the total energy equation, although we have not yet included the latent heat as part of the total energy.

Dry static energy

We now make two approximations in (16):

$$\text{Neglect of } \frac{DK}{Dt} , \quad (17)$$

$$\text{Neglect of the friction term.} \quad (18)$$

Then (16) reduces to

$$\boxed{\frac{Ds}{Dt} \cong -\alpha \nabla \cdot (\mathbf{R} + \mathbf{F}_h) + LC} , \quad (19)$$

where

$$s \equiv c_p T + \phi \quad (20)$$

is the dry static energy.

An alternative derivation

If we expand the Lagrangian time derivative in the thermodynamic energy equation, (15) , using pressure coordinates, and also use the hydrostatic approximation in the form

$$\alpha \cong -\frac{\partial \phi}{\partial p} , \quad (21)$$

we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_h \cdot \nabla_p\right)(c_p T) + \omega \frac{\partial}{\partial p}(c_p T + \phi) = -\alpha \nabla \cdot (\mathbf{R} + \mathbf{F}_s) + LC + \delta. \quad (22)$$

To obtain conservation of dry static energy, i.e. (19), from (22), we need the following two approximations, *in addition to* the hydrostatic approximation already used:

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_h \cdot \nabla_p\right)(c_p T) \cong \left(\frac{\partial}{\partial t} + \mathbf{V}_h \cdot \nabla_p\right)(c_p T + \phi), \quad (23)$$

Neglect of the dissipation term of (22). (24)

In summary, three approximations, namely (21), (23), and (24), are needed to obtain conservation of dry static energy from the thermodynamic energy equation. These can be weighed against (17) and (18), which are the two approximations needed to obtain conservation of dry static energy from the total energy equation. It is thus slightly simpler to obtain conservation of dry static energy as an approximation to total energy conservation.

Moist static energy

Now we introduce the water vapor equation in the form

$$\frac{Dq_v}{Dt} = -C - \alpha \nabla \cdot \mathbf{F}_{q_v}, \quad (25)$$

where q_v is the mixing ratio of water vapor, and \mathbf{F}_{q_v} is the vector flux of water vapor due to molecular diffusion. Next, multiply (24) by the latent heat of condensation, and neglect the variation of L with temperature. This leads to

$$\boxed{\frac{D}{Dt}(Lq_v) = -LC - \alpha L \nabla \cdot \mathbf{F}_{q_v}}. \quad (26)$$

Adding (24) and (17), we obtain

$$\boxed{\frac{Dh}{Dt} \cong -\alpha \nabla \cdot (\mathbf{R} + \mathbf{F}_h)} \quad (27)$$

where

$$h \equiv c_p T + \phi + Lq_v \quad (28)$$

is the moist static energy, and $\mathbf{F}_h = \mathbf{F}_s + L\mathbf{F}_{q_v}$ is the molecular flux of moist static energy.

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