Taylor Series

David A. Randall

Department of Atmospheric Science
Colorado State University, Fort Collins, Colorado 80523

A sufficiently differentiable function can be represented by a power series that is referred to an arbitrary point :

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f''(a)}{3!}(x-a)^3 + \dots \ . \tag{1}$$

Here primes denote differentiation. This expansion, which is called a “Taylor series,” can be derived without any assumptions or approximations except that the indicated derivatives exist (Arfken, 1985). On the right-hand side of (1), f and its various derivatives are evaluated at the point $x = a$.

We will use Taylor series to show how a function behaves at a distance h from a fixed point a. If we write $x = a + h$ in the above, so that $x - a = h$, we get

$$\begin{aligned} f(a+h) = f(a) + f'(a)\,h + \frac{f''(a)}{2!}h^2 + \dots + \frac{f^{(n)}(a)}{n!}h^n \\ + \frac{f^{(n+1)}(a)}{(n+1)!}h^{n+1} + \dots \ . \end{aligned} \tag{2}$$

For a function of two variables, $f(x, y)$, the total change of the function, at a neighboring point in the (x, y) plane, can be due to changes in either of x or y:

$$df = dx\,\frac{\partial f}{\partial x} + dy\,\frac{\partial f}{\partial y} \ . \tag{3}$$

The analogy to advection is obvious, but should not be taken literally. We consider dx and dy to be constant here; their ratio essentially specifies a direction in the (x, y) plane.

Continuing in an analogous manner, the second total differential of f, i.e., the total differential of the first total differential of f, is

$$
\begin{aligned}
d^2 f &= dx \frac{\partial}{\partial x}(df) + dy \frac{\partial}{\partial y}(df) \\
&= dx \frac{\partial}{\partial x}\left(dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y}\right) + dy \frac{\partial}{\partial y}\left(dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y}\right) \\
&= \left[(dx)^2 \frac{\partial^2 f}{\partial x^2} + dx\,dy \frac{\partial^2 f}{\partial x \partial y}\right] + \left[dx\,dy \frac{\partial^2 f}{\partial x \partial y} + (dy)^2 \frac{\partial^2 f}{\partial y^2}\right] \qquad (4) \\
&= (dx)^2 \frac{\partial^2 f}{\partial x^2} + 2\,dx\,dy \frac{\partial^2 f}{\partial x \partial y} + (dy)^2 \frac{\partial^2 f}{\partial y^2} \\
&= \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^2 f(x, y) \ .
\end{aligned}
$$

Note the cross-derivative. In general,

$$
d^n f = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^n f(x, y) \ . \qquad (5)
$$

For convenient reference, we note that

$$
d^3 f = (dx)^3 f_{xxx} + 3\,(dx)^2\,dy f_{xxy} + 3\,dx(dy)^2 f_{xyy} + (dy)^3 f_{yyy} \ , \qquad (6)
$$

$$
\begin{aligned}
d^4 f = (dx)^4 f_{xxxx} + 4\,(dx)^3\,dy f_{xxxy} + 6\,(dx)^2\,(dy)^2 f_{xxyy} \\
+4\,dx(dy)^3 f_{xyyy} + (dy)^4 f_{yyyy} \ . \qquad (7)
\end{aligned}
$$

Here the subscripts denote differentiation.

We can now write the Taylor series expansion of $f(x, y)$ for a point in the neighborhood of the point (a, b):

$$
\begin{aligned}
f(x, y) = f(a, b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f \\
+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f + \frac{1}{3!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^3 f + \dots \ . \qquad (8)
\end{aligned}
$$

References & Bibliography

Arfken, G., 1985: *Mathematical methods for physicists*. Academic Press, San Diego, 985 pp.

Thomas, G. B., Jr., 1972: *Calculus and analytic geometry*. Addison-Wesley Publ. Co. Inc., 1034 pp.