Vectors, Vector Calculus, and Coordinate Systems

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1 Physical laws and coordinate systems

For the present discussion, we define a "coordinate system" as a tool for describing positions in space. Coordinate systems are human inventions, and therefore are not part of physics, although they can be used in a discussion of physics. For obvious reasons, spherical coordinates are particularly useful in geophysics.

Any physical law should be expressible in a form that is invariant with respect to our choice of coordinate systems; we certainly do not expect that the laws of physics change when we switch from spherical coordinates to cartesian coordinates! It follows that we should be able to express physical laws without making reference to any coordinate system. Nevertheless, it is useful to understand how physical laws can be expressed in different coordinate systems, and in particular how various quantities "transform" as we change from one coordinate system to another.

2 Scalars, vectors, and tensors

Tensors can be defined without reference to any particular coordinate system. A tensor is simply "out there," and has a meaning that is the same whether we happen to be working in spherical coordinates, or Cartesian coordinates, or whatever. Tensors are, therefore, just what we need to formulate physical laws.

The simplest kind of tensor, called a "tensor of rank 0," is a scalar, which is represented by a single number – essentially a magnitude with no direction. An example of a scalar is temperature. Not all quantities that are represented by a single number are scalars, because not all of them are defined without reference to any particular coordinate system. An example of a (single) number that is not a scalar is the longitudinal component of the wind, which is defined with respect to a particular coordinate system, i.e., spherical coordinates.

A scalar is expressed in exactly the same way regardless of what coordinate system may be in use to describe non-scalars in a problem. For example, if someone tells you the temperature in Fort Collins, you don't have to ask whether they are using spherical coordinates or some other coordinate system, because it makes no difference at all.

Vectors are "tensors of rank 1;" a vector can be represented by a magnitude and one direction. An example is the wind vector. In atmospheric science, vectors are normally either three dimensional or two dimensional, but in principle they have any number of dimensions. A scalar can be considered to be a vector in a one-dimensional space.

A vector can be expressed in a particular coordinate system by an ordered list of numbers, which are called the "components" of the vector. The components have meaning only with respect to the particular coordinate system. More or less by definition, the number of components needed to describe a vector is equal to the number of dimensions in which the vector is "embedded."

We can define "unit vectors" that point in each of the coordinate directions. A vector can then be written as the vector sum of each of the unit vectors times the "component" associated with the unit vector. In general, the directions in which the unit vectors point depend on position.

Unit vectors are always non-dimensional; here we are using the word "dimension" to refer to physical quantities, such as length, time, and mass. Because the unit vectors are non-dimensional, all components of a vector must have the same dimensions as the vector itself.

Spatial coordinates may or may not have the dimensions of length. In the familiar Cartesian coordinate system, the three coordinates, (x, y, z), each have dimensions of length. In spherical coordinates, (λ, φ, r), where λ is longitude, φ is latitude, and r is distance from the origin, the first two coordinates are non-dimensional angles, while the third has the dimension of length.

When we change from one coordinate system to another, an arbitrary vector $\mathbf{V}$ transforms according to

$$\mathbf{V}' = \mathsf{\mathbf{M}}\mathbf{V}. \tag{1}$$

Here $\mathbf{V}$ is the representation of the vector in the first coordinate system (i.e., $\mathbf{V}$ is the list of the components of the vector in the first coordinate system), $\mathbf{V}'$ is the representation the vector in the second coordinate system, and $\mathsf{\mathbf{M}}$ is a "rotation matrix" that maps $\mathbf{V}$ onto $\mathbf{V}'$. The rotation matrix used to transform a vector from one coordinate system to another is a property of the two coordinate systems in question; it is the same for all vectors, but it *does* depend on the particular coordinate systems involved, so it is not a tensor.

The transformation rule (1) is actually part of the definition of a vector, i.e., a vector must, by definition, transform from one coordinate system to another via a rule of the form (1) . It follows that not all ordered lists of numbers are vectors. For example, the list

(mass of the moon, distance from Fort Collins to Denver)

is not a vector.

Now let $\mathbf{V}$ be the a vector representing the three-dimensional velocity of a particle in the atmosphere. The Cartesian and spherical representations of are

$$\mathbf{V} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{2}$$

$$\mathbf{V} = \dot{\lambda} r \cos\varphi \mathbf{e}_\lambda + r\dot{\varphi}\mathbf{e}_\varphi + \dot{r}\mathbf{e}_r \tag{3}$$

Here a "dot" denotes a Lagrangian time derivative, i.e., a time derivative following a moving particle, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the cartesian coordinate system, and $\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$ are unit vectors in the spherical coordinate system. Eqs. (2) and (3) both describe the same vector, $\mathbf{V}$, i.e., the meaning of $\mathbf{V}$ is independent of the coordinate system that is chosen to represent it.

Vectors are considered to be tensors of rank one, and scalars are tensors of rank zero. The number of directions associated with a tensor is called the "rank" of the tensor. In principle, the rank can be arbitrarily large, but we rarely meet tensors with ranks higher than two in atmospheric science.

A tensor of rank 2 that is important in atmospheric science is the flux of momentum . The momentum flux, also called a "stress," and equivalent to a force per unit area, has a magnitude and "two directions." One of the directions is associated with the force vector itself, and the other is associated with the normal vector to the unit area in question. The momentum flux tensor can be written as , where is the density of the air, is the wind vector, and is the "outer" or "dyadic" product that accepts two vectors as input and delivers a rank-2 tensor as output.

Like a vector, a tensor of rank 2 can be expressed in a particular coordinate system, i.e., we can define the "components" of the tensor with respect to a particular coordinate system. The components of a tensor of rank 2 can be arranged in the form of a two-dimensional matrix, in contrast to the components of a (column or row) vector, which form an ordered one-dimensional list. When we change from one coordinate system to another, a tensor of rank 2 transforms according to

$$\mathbf{T}' = \mathbf{MTM}^{-1} \tag{4}$$

where $\mathbf{T}$ is the representation of a rank-2 tensor in the first coordinate system, $\mathbf{T}'$ is the representation of the same tensor in the second coordinate system, $\mathbf{M}$ is the matrix introduced in Eq. (1) above, and $\mathbf{M}^{-1}$ is its inverse.

3 Differential operators

Several familiar differential operators can be defined without reference to any coordinate system. These operators are more fundamental than, for example, $\partial/\partial x$, where x is a particular spatial coordinate. The coordinate-independent operators that we need most often for atmospheric science (and for most other branches of physics too) are:

- the gradient, denoted by ∇A, where A is an arbitrary scalar;

(5)

- the divergence, denoted by $\nabla \cdot \mathbf{V}$, where $\mathbf{V}$ is an arbitrary vector;

(6)

- the curl, denoted by $\nabla \times \mathbf{V}$, and

(7)

- the Laplacian, given by $\nabla^2 A = \nabla \cdot (\nabla A)$.

(8)

Note that the gradient and curl are vectors, while the divergence is a scalar. The gradient operator accepts scalars as "input," while the divergence and curl operators consume vectors.

In discussions of two-dimensional motion, it is often convenient to introduce an additional operator called the Jacobian, denoted by

$$\begin{aligned} J(\alpha,\beta) &\equiv \mathbf{k}\cdot(\nabla\alpha\times\nabla\beta) \\ &= \mathbf{k}\cdot\nabla\times(\alpha\nabla\beta) \\ &= -\mathbf{k}\cdot\nabla\times(\beta\nabla\alpha)\,. \end{aligned} \tag{9}$$

Here the gradient operators are understood to produce vectors in the two-dimensional space, α and β are arbitrary scalars, and $\mathbf{k}$ is a unit vector perpendicular to the two-dimensional surface. The second and third lines of (9) can be derived with the use of vector identities found in a table later in this QuickStudy.

A definition of the gradient operator that does not make reference to any coordinate system is:

$$\nabla A \equiv \lim_{S\to 0}\left[\frac{1}{V}\oint_S \mathbf{n}A\,dS\right], \tag{10}$$

where S is the surface bounding a volume V, and $\mathbf{n}$ is the outward normal on S. Here the terms "volume" and "bounding surface" are used in the following generalized sense: In a three-dimensional space, "volume" is literally a volume, and "bounding surface" is literally a surface. In a two-dimensional space, "volume" means an area, and "bounding surface" means the curve bounding the area. In a one-dimensional space, "volume" means a curve, and "bounding surface" means the end points of the curve. The limit in (10) is one in which the volume and the area of its bounding surface shrink to zero.

As an example, consider a Cartesian coordinate system on a plane, with unit vectors and in the and directions, respectively. Consider a "box" of width Δx and height Δy, as shown in Figure 1. We can write

$$\begin{aligned}\nabla A \equiv \lim_{(\Delta x,\Delta y)\to 0} \Bigg\{ \frac{1}{\Delta x \Delta y} \Bigg[& A\left(x_0 + \frac{\Delta x}{2}, y_0\right)\Delta y \mathbf{i} + A\left(x_0, y_0 + \frac{\Delta y}{2}\right)\Delta x \mathbf{j} \\ & - A\left(x_0 - \frac{\Delta x}{2}, y_0\right)\Delta y \mathbf{i} - A\left(x_0, y_0 - \frac{\Delta y}{2}\right)\Delta x \mathbf{j} \Bigg] \Bigg\} \\ = {} & \frac{\partial A}{\partial x}\mathbf{i} + \frac{\partial A}{\partial y}\mathbf{j}\,.\end{aligned} \tag{11}$$

This is the answer that we expect.

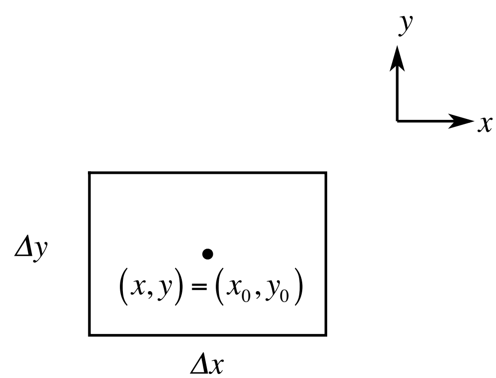

Figure 1: A rectangular box in a planar two-dimensional space, with center at (x_0, y_0), width Δx, and height Δy.

Definitions of the divergence and curl operators that do not make reference to any coordinate system are:

$$\nabla \cdot \mathbf{Q} \equiv \lim_{S\to 0}\left[\frac{1}{V}\oint_S \mathbf{n}\cdot\mathbf{Q}\,dS\right] \tag{12}$$

$$\nabla \times \mathbf{Q} \equiv \lim_{S\to 0}\left[\frac{1}{V}\oint_S \mathbf{n}\times\mathbf{Q}\,dS\right] \tag{13}$$

It is possible to work through exercises similar to (11) for these operators too. You might want to try it yourself, to see if you understand.

Finally, the Jacobian on a two-dimensional surface can be defined by

$$J(A,B) = \lim_{A \to 0} \left[\oint_C A \nabla B \cdot \mathbf{t} dl \right], \tag{14}$$

where $\mathbf{t}$ is a unit vector that is tangent to the bounding curve C.

4 Vector identities

Many useful identities relate the divergence, curl, and gradient operators. Most of the following identities can be found in any mathematics reference manual, e.g., Beyer (1984). As before, let α and β be arbitrary scalars, let $\mathbf{V}$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be arbitrary vectors, and let $\mathbf{T}$ be an arbitrary tensor of rank 2. Then:

$$\nabla \times (\nabla \alpha) = 0 \tag{15}$$

$$\nabla \cdot (\nabla \times \mathbf{V}) = 0 \tag{16}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{17}$$

$$\nabla \cdot (\alpha \mathbf{V}) = \alpha (\nabla \cdot \mathbf{V}) + \mathbf{V} \cdot \nabla \alpha \tag{18}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A} \tag{19}$$

$$\nabla \times (\alpha \mathbf{V}) = \nabla \alpha \times \mathbf{V} + \alpha (\nabla \times \mathbf{V}) \tag{20}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \tag{21}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{C} \cdot \mathbf{A}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) \tag{22}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \tag{23}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \tag{24}$$

$$\begin{aligned} J(\alpha, \beta) \equiv \mathbf{k} \cdot (\nabla\alpha \times \nabla\beta) &= \quad \mathbf{k} \cdot \nabla \times (\alpha \nabla \beta) \\ &= -\mathbf{k} \cdot \nabla \times (\beta \nabla \alpha) \\ &= -\mathbf{k} \cdot (\nabla\beta \times \nabla\alpha) \end{aligned} \tag{25}$$

$$\nabla^2 \mathbf{V} \equiv (\nabla \cdot \nabla)\mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla \times (\nabla \times \mathbf{V}) \tag{26}$$

$$\nabla \cdot (\mathbf{A} \otimes \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \tag{27}$$

$$\nabla \cdot (\alpha \mathsf{T}) = (\nabla\alpha) \cdot \mathsf{T} + \alpha(\nabla \cdot \mathsf{T}) \tag{28}$$

In (27), $\otimes$ denotes the outer or dyadic product of two vectors, which yields a tensor of rank 2.

A special case of (24) is

$$\frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) = (\mathbf{V} \cdot \nabla)\mathbf{V} + \mathbf{V} \times (\nabla \times \mathbf{V}) \tag{29}$$

This identity is used to write the advection terms of the momentum equation in alternative forms.

Identity (26) says that the Laplacian of a vector is the gradient of the divergence of the vector, minus the curl of the curl of the vector. The first term involves only the divergent part of the wind field, and the second term involves only the rotational part. Eq. (26) can be used, for example, in a parameterization of momentum diffusion.

5 Spherical coordinates

5.1 Vector operators in spherical coordinates

The gradient, divergence, curl, Laplacian, and Jacobian operators can be expressed in spherical coordinates as follows:

$$\nabla\alpha = \left(\frac{1}{r\cos\varphi}\frac{\partial\alpha}{\partial\lambda},\ \frac{1}{r}\frac{\partial\alpha}{\partial\varphi},\ \frac{\partial\alpha}{\partial r}\right) \tag{30}$$

$$\nabla\cdot\mathbf{V} = \frac{1}{r\cos\varphi}\frac{\partial V_\lambda}{\partial\lambda} + \frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}\left(V_\varphi\cos\varphi\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(V_r r^2\right) \tag{31}$$

$$\nabla\times\mathbf{V} = \left\{\frac{1}{r}\left[\frac{\partial V_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rV_\varphi\right)\right],\ \frac{1}{r}\frac{\partial}{\partial r}\left(rV_\lambda\right) - \frac{1}{r\cos\varphi}\frac{\partial V_r}{\partial\lambda},\ \frac{1}{r\cos\varphi}\left[\frac{\partial V_\varphi}{\partial\lambda} - \frac{\partial}{\partial\varphi}\left(V_\lambda\cos\varphi\right)\right]\right\} \tag{32}$$

$$\nabla^2\alpha = \frac{1}{r^2\cos\varphi}\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{\cos\varphi}\frac{\partial\alpha}{\partial\lambda}\right) + \frac{\partial}{\partial\varphi}\left(r^2\cos\varphi\frac{\partial\alpha}{\partial r}\right)\right] \tag{33}$$

$$J(\alpha,\beta) = \frac{1}{r^2\cos\varphi}\left(\frac{\partial\alpha}{\partial\lambda}\frac{\partial\beta}{\partial\varphi} - \frac{\partial\beta}{\partial\lambda}\frac{\partial\alpha}{\partial\varphi}\right) \tag{34}$$

Here α is an arbitrary scalar, and $\mathbf{V}$ is an arbitrary vector.

5.2 Horizontal and vertical vectors in spherical coordinates

The unit vectors in spherical coordinates are denoted by $\mathbf{e}_\lambda$ pointing towards the east, $\mathbf{e}_\varphi$ pointing towards the north, and $\mathbf{e}_r$ pointing outward from the origin (in geophysics, outward from the center of the Earth).

A useful result that is a special case of (23) is

$$\mathbf{e}_r\cdot\left[\nabla\times\left(\mathbf{e}_r\times\mathbf{H}\right)\right] = \nabla\cdot\mathbf{H}, \tag{35}$$

where $\mathbf{e}_r$ is the unit vector pointing upward, and $\mathbf{H}$ is an arbitrary horizontal vector. In words, the curl of $\mathbf{e}_r \times \mathbf{H}$ is equal to the divergence of $\mathbf{H}$. Similarly, a useful special case of (19) is

$$\nabla \cdot (\mathbf{e}_r \times \mathbf{H}) = -\mathbf{e}_r \cdot (\nabla \times \mathbf{H}) \tag{36}$$

This means that the divergence of $\mathbf{e}_r \times \mathbf{H}$ is equal to minus the curl of $\mathbf{H}$.

If $\mathbf{V}$ is separated into a horizontal vector and a vertical vector, as in

$$\mathbf{V} = \mathbf{V}_h + V_r \mathbf{e}_r, \tag{37}$$

then (32)) can be written as

$$\boxed{\nabla \times (\mathbf{V}_h + V_r \mathbf{e}_r) = \nabla_r \times \mathbf{V}_h + \mathbf{e}_r \times \left[\frac{1}{r} \frac{\partial}{\partial r} (r\mathbf{V}_h) - \nabla_r V_r \right].} \tag{38}$$

In case $\mathbf{V}$ is the velocity, the first term on the right-hand side of (38) is the vertical component of the vorticity, and the second term is the horizontal vorticity vector. Eq. (38) shows that the curl of a purely vertical vector is minus $\mathbf{e}_r$ crossed with the horizontal gradient of the magnitude of that vector. The three-dimensional curl of a purely horizontal vector has both a vertical part, given by $\nabla_r \times \mathbf{V}_h$, and a horizontal part, given by $\mathbf{e}_r \times \left[\frac{1}{r} \frac{\partial}{\partial r} (r\mathbf{V}_h) - \nabla_r V_r \right]$. The *two-dimensional* curl of a horizontal vector has only a vertical component, namely $\nabla_r \times \mathbf{V}_h$.

5.3 Derivation of the gradient operator in spherical coordinates

Consider how the two-dimensional version of (30) can be derived from (10). Figure 2 illustrates the problem. Here we have replaced r by a, the radius of the Earth. The angle θ depicted in the figure arises from the gradual rotation of $\mathbf{e}_\lambda$ and $\mathbf{e}_\varphi$, the unit vectors associated with the spherical coordinates, as the longitude changes; the directions of $\mathbf{e}_\lambda$ and $\mathbf{e}_\varphi$ in the center of the area element, where ∇A is defined, are different from their respective directions on either east-west wall of the area element. Inspection of Figure 2 shows that θ satisfies

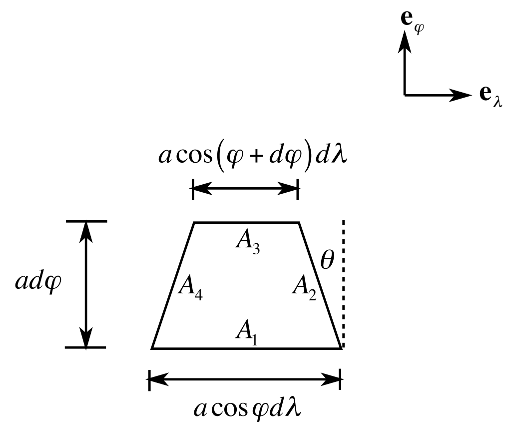

Figure 2: A patch of the sphere, with longitudinal width $a\cos\varphi d\lambda$, and latitudinal height $ad\varphi$.

$$\begin{aligned}\sin\theta =& \frac{-\frac{1}{2}\left[a\cos(\varphi + d\varphi) - a\cos\varphi\right]d\lambda}{ad\varphi} \\ &\rightarrow -\frac{1}{2}\left(\frac{\partial}{\partial\varphi}\cos\varphi\right)d\lambda \\ =& \frac{1}{2}\sin\varphi d\lambda.\end{aligned} \tag{39}$$

The angle θ is of "differential" or infinitesimal size. Nevertheless, it is needed in the derivation of (30). The line integral in (10) can be expressed as

$$\begin{aligned}\frac{1}{\text{Area}}\oint A\mathbf{n}dl =& \frac{1}{a^2\cos\varphi d\lambda d\varphi}\left[-\mathbf{e}_\varphi A_1 a\cos\varphi d\lambda + \mathbf{e}_\lambda A_2\cos\theta ad\varphi + \mathbf{e}_\varphi A_2\sin\theta ad\varphi\right. \\ &\left. + \mathbf{e}_\varphi A_3 a\cos(\varphi + d\varphi) - \mathbf{e}_\lambda A_4\cos\theta ad\varphi + \mathbf{e}_\varphi A_4\sin\theta ad\varphi\right] \\ =& \mathbf{e}_\lambda\frac{(A_2 - A_4)\cos\theta}{a\cos\varphi d\lambda} \\ &+ \mathbf{e}_\varphi\left\{\frac{\left[A_3\cos(\varphi + d\varphi) - A_1\cos\varphi\right]d\lambda + (A_2 + A_4)\sin\theta d\varphi}{a\cos\varphi d\lambda d\varphi}\right\}\end{aligned} \tag{40}$$

Note how the angle θ has entered here. Put $\cos\theta \to 1$ and $\sin\theta \to \frac{1}{2}\sin\varphi d\lambda$ to obtain

$$\begin{aligned}\frac{1}{\text{Area}}\oint \alpha\mathbf{n}dl =&\mathbf{e}_\lambda \frac{(A_2-A_4)}{a\cos\varphi d\lambda}+\mathbf{e}_\varphi\left\{\left[\frac{A_3\cos(\varphi+d\varphi)-A_1\cos\varphi}{a\cos\varphi d\varphi}\right]+\left(\frac{A_2+A_4}{2}\right)\frac{\sin\varphi}{a\cos\varphi}\right\}\\ \to&\mathbf{e}_\lambda\frac{1}{a\cos\varphi}\frac{\partial A}{\partial\lambda}+\mathbf{e}_\varphi\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(A\cos\varphi)+\frac{A\sin\varphi}{a\cos\varphi}\right]\\ =&\mathbf{e}_\lambda\frac{1}{a\cos\varphi}\frac{\partial A}{\partial\lambda}+\mathbf{e}_\varphi\frac{1}{a}\frac{\partial A}{\partial\varphi}\ ,\end{aligned} \tag{41}$$

which agrees with the two-dimensional version of (30).

Similar (but more straightforward) derivations can be given for (31) - (34).

5.4 Applying vector operators to the unit vectors in spherical coordinates

Using (15) - (17), we can prove the following about the unit vectors in spherical coordinates:

$$\nabla\cdot\mathbf{e}_\lambda = 0, \tag{42}$$

$$\nabla\cdot\mathbf{e}_\varphi = -\frac{\tan\varphi}{r}, \tag{43}$$

$$\nabla\cdot\mathbf{e}_r = \frac{2}{r}, \tag{44}$$

$$\nabla\times\mathbf{e}_\lambda = \frac{\mathbf{e}_\varphi}{r}+\frac{\tan\varphi}{r}\mathbf{e}_r, \tag{45}$$

$$\nabla\times\mathbf{e}_\varphi = -\frac{\mathbf{e}_\lambda}{r}, \tag{46}$$

$$\nabla\times\mathbf{e}_r = 0. \tag{47}$$

The following relations are useful when working with the momentum equation in spherical coordinates:

$$(\mathbf{V}_h \cdot \nabla)\,\mathbf{e}_\lambda = \frac{u\sin\varphi}{r}\mathbf{e}_\varphi - \frac{u\cos\varphi}{r}\mathbf{e}_r\ , \tag{48}$$

$$(\mathbf{V}_h \cdot \nabla)\,\mathbf{e}_\varphi = -\,\frac{u\sin\varphi}{r}\mathbf{e}_\lambda - \frac{v\sin\varphi}{r}\mathbf{e}_r, \tag{49}$$

$$(\mathbf{V}_h \cdot \nabla)\,\mathbf{e}_r = \frac{\mathbf{V}_h}{r}\ . \tag{50}$$

Here $\mathbf{V}_h$ is the horizontal wind vector.

6 Solid body rotation

As an example of the application of (32), the vertical component of the vorticity is

$$\zeta = \frac{1}{r\cos\varphi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\varphi}\left(u\cos\varphi\right)\right] \tag{51}$$

For the case of pure solid body rotation of the atmosphere about the Earth's axis of rotation, we have

$$u = \dot{\lambda}\, r\cos\varphi \text{ and } v = 0, \tag{52}$$

where $\dot{\lambda}$ is independent of φ. Substitution of (53) into (51) gives

$$\begin{aligned}\zeta =& \frac{-1}{r\cos\varphi}\frac{\partial}{\partial\varphi}\left(\dot{\lambda} r\cos^2\varphi\right)\\ =& 2\dot{\lambda}\sin\varphi\ .\end{aligned} \tag{53}$$

This is the expected form of the vorticity associated with the vertical component of the Earth's rotation vector. In the conventional notation, $\dot{\lambda}$ is replaced by Ω.

7 Formulas that are useful for two-dimensional flow

Consider the special case of two-dimensional flow. Two useful identities are

$$\nabla_r \times (\mathbf{e}_r \times \nabla_r A) = \mathbf{e}_r \nabla_r^2 A, \tag{54}$$

and

$$\nabla_r \cdot (\mathbf{e}_r \times \nabla_r A) = 0. \tag{55}$$

Also for two-dimensional flow, the Laplacian of a vector can be written in a very simple way. Let $\zeta \mathbf{e}_r \equiv \nabla_r \times \mathbf{V}_h$ and $\Delta \equiv \nabla_r \mathbf{V}_h$. Then (26) reduces to

$$\nabla_r^2 \mathbf{V}_h = \nabla_r \Delta - \nabla_r \times (\zeta \mathbf{e}_r) \tag{56}$$

Using (32), we can write

$$\begin{aligned} \nabla_r \times (\zeta \mathbf{e}_r) &= \left\{ \frac{1}{r}\frac{\partial \zeta}{\partial \varphi},\ -\frac{1}{r\cos\varphi}\frac{\partial \zeta}{\partial \lambda},\ 0 \right\} \\ &= -\,\mathbf{e}_r \times \nabla_r \zeta \ . \end{aligned} \tag{57}$$

Then (56) becomes

$$\nabla_r^2 \mathbf{V}_h = \nabla_r \Delta + \mathbf{e}_r \times \nabla_r \zeta. \tag{58}$$

8 Conclusion

This brief overview is intended mainly as a refresher for students who learned these concepts once upon a time, but may have not thought about them for awhile. We have also included many useful equations that are not readily available elsewhere, even on the Web.

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