Vorticity and Angular Momentum

David Randall

Definitions

The vertical component of the absolute vorticity is given by

$$\begin{aligned}\eta &\equiv \mathbf{k}\cdot(2\boldsymbol{\Omega}+\nabla\times\mathbf{V}) \\ &= 2\Omega\sin\varphi+\frac{1}{a\cos\varphi}\left[\frac{\partial v}{\partial\lambda}-\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\ &\equiv f+\zeta\ ,\end{aligned} \tag{1}$$

where $\boldsymbol{\Omega}$ is the angular velocity vector associated with the Earth's rotation, $\mathbf{k}$ is a unit vector pointing upward, $\mathbf{V}\equiv(u,v)$ is the horizontal wind vector,

$$\begin{aligned}f &\equiv \mathbf{k}\cdot 2\boldsymbol{\Omega} \\ &= 2\Omega\sin\varphi\end{aligned} \tag{2}$$

is the Coriolis parameter,

$$\begin{aligned}\zeta &\equiv \mathbf{k}\cdot(\nabla\times\mathbf{V}) \\ &= \frac{1}{a\cos\varphi}\left[\frac{\partial v}{\partial\lambda}-\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\end{aligned} \tag{3}$$

is the relative vorticity, λ is longitude, φ is latitude, and a is the radius of the Earth.

The component of the angular momentum vector (per unit mass) that is parallel to the Earth's rotation vector is

$$\begin{aligned}M &= a\cos\varphi(\Omega a\cos\varphi+u) \\ &= M_{\Omega}+M_{rel}\ ,\end{aligned}, \tag{4}$$

where

$$M_{\Omega} = \Omega a^2 \cos^2\varphi \ . \tag{5}$$

is the part of M that comes from the Earth's rotation, and the relative angular momentum is

$$M_{rel} = ua\cos\varphi \ . \tag{6}$$

A relationship in terms of zonal averages

Both ζ and M are measures of the "rotational" quality of the winds, so it stands to reason that they should be related in some simple way. One simple connection can be seen by taking the meridional derivative of M:

$$\frac{1}{a\cos\varphi}\frac{\partial M}{\partial\varphi} = \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi) - 2\Omega a\sin\varphi \tag{7}$$

Comparing with (1), we see that

$$\zeta + f = \frac{-1}{a\cos\varphi}\frac{\partial M}{\partial\varphi} + \frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda} \ . \tag{8}$$

Averaging with respect to longitude, we obtain

$$\boxed{\frac{1}{2\pi}\int_0^{2\pi}(\zeta + f)d\lambda = \frac{-1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{1}{2\pi}\int_0^{2\pi} M d\lambda\right)} \ . \tag{9}$$

This demonstrates that the zonal mean of the absolute vorticity is proportional to the zonal mean of the meridional derivative of the angular momentum.

A relationship in terms of global averages

We now show that an additional simple relationship holds in terms of global means. An area average over the sphere, denoted here by an overbar, is defined by

$$A\overline{(\ )} = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi}(\ )a^2\cos\varphi d\lambda d\varphi \ , \tag{10}$$

where

$$A = 4\pi a^2 \tag{11}$$

is the area of the sphere. The global area integral of $\sin\varphi$ times the Coriolis parameter is

$$A\overline{(f\sin\varphi)} = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\left(2\Omega\sin^2\varphi\right)a^2\cos\varphi d\lambda d\varphi$$
$$= 4\pi\Omega a^2\left(\frac{\sin^3\varphi}{3}\right)_{-\pi/2}^{\pi/2}$$
$$= \frac{2}{3}\Omega A\ . \tag{12}$$

Note that the global mean of satisfies

$$A\overline{M_\Omega} = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\left(\Omega a^2\cos^2\varphi\right)a^2\cos\varphi d\lambda d\varphi$$
$$= 2\pi\Omega a^4\int_{-\pi/2}^{\pi/2}\cos^3\varphi d\varphi$$
$$= \frac{\Omega A a^2}{2}\left[\frac{3}{4}\sin\varphi + \frac{1}{12}\sin(3\varphi)\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{\Omega A a^2}{2}\left(\frac{3}{2} - \frac{1}{6}\right)$$
$$= \frac{2\Omega A a^2}{3}\ . \tag{13}$$

Comparison of (12) and (13) shows that

$$\boxed{\overline{M_\Omega} = a^2\overline{(f\sin\varphi)}}\ . \tag{14}$$

Now consider the area average of $\zeta\sin\varphi$:

$$A\overline{(\zeta\sin\varphi)} = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi}(\zeta\sin\varphi)a^2\cos\varphi d\lambda d\varphi$$
$$= a^2\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\left\{\frac{1}{a\cos\varphi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\right\}\sin\varphi\cos\varphi d\lambda d\varphi$$
$$= a\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\sin\varphi d\lambda d\varphi$$
$$= -a\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\sin\varphi d\lambda d\varphi$$
$$= -\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\frac{\partial M_{rel}}{\partial\varphi}\sin\varphi d\lambda d\varphi\ . \tag{15}$$

Integrating (15) by parts with respect to latitude, we see that

$$A\overline{(\zeta\sin\varphi)} = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} M_{rel}\cos\varphi d\lambda d\varphi$$
$$= \frac{A\overline{M_{rel}}}{a^2}, \tag{16}$$

or

$$\boxed{\overline{M_{rel}} = a^2\overline{(\zeta\sin\varphi)}}\,. \tag{17}$$

This demonstrates that the globally averaged relative angular momentum is proportional to the $\sin\varphi$-weighted global mean of the relative vorticity. (Recall that the global mean of the vorticity itself is exactly zero.)

Finally, we see from (1), (14) and (17) that

$$\boxed{\overline{M} = a^2\overline{(\eta\sin\varphi)}}\,. \tag{18}$$

Diffusion of vorticity and angular momentum

Diffusion tries to homogenize the diffused quantity. Angular momentum cannot be homogenized unless its mean is zero, because a finite angular momentum at a pole would imply an (impossible) infinite vorticity there. Vorticity can be homogenized without creating any problems of this type although, of course, we are guaranteed by a mathematical identity that the global mean of the vorticity is zero, so if the vorticity is homogenized it will be zero everywhere.