Vorticity

David Randall

Introduction

The vorticity, $\boldsymbol{\omega}$, is a local measure of the angular velocity of a fluid. It plays a central role in atmospheric science, and in many other applications of fluid dynamics. It is the curl of the velocity:

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{V} . \tag{1}$$

Because the vorticity is a curl, it is non-divergent:

$$\nabla \cdot \boldsymbol{\omega} = 0 . \tag{2}$$

The absolute vorticity vector is $\boldsymbol{\omega} + 2\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the angular velocity vector associated with the Earth's rotation. We note that $\boldsymbol{\Omega}$, like $\boldsymbol{\omega}$, is non-divergent:

$$\nabla \cdot \boldsymbol{\Omega} = 0 . \tag{3}$$

The eastward velocity of a particle rotating with the Earth is $\mathbf{e}_\lambda \Omega r \cos\varphi$, where $\mathbf{e}_\lambda$ is a unit vector pointing toward the east, r is the distance from the center of the Earth, and φ is latitude. The vorticity associated with this Earth rotation is given by

$$\begin{aligned} \nabla \times \left(\mathbf{e}_\lambda \Omega r \cos\varphi\right) &= \left\{0, \frac{1}{r}\frac{\partial}{\partial r}\left(r\Omega r \cos\varphi\right), \frac{1}{r\cos\varphi}\left[-\frac{\partial}{\partial\varphi}\left(\Omega r \cos\varphi \cos\varphi\right)\right]\right\} \\ &= \left[0, \frac{2r\Omega\cos\varphi}{r}, \frac{\Omega r}{r\cos\varphi}\left(2\cos\varphi\sin\varphi\right)\right] \\ &= 2\Omega\left(\mathbf{e}_\varphi \cos\varphi + \mathbf{e}_r \sin\varphi\right), \end{aligned} \tag{4}$$

where $\mathbf{e}_\varphi$ is a unit vector pointing north, and $\mathbf{e}_r$ is a unit vector pointing outward from the center of the Earth. Note that the meridional component of $\nabla \times \left(\mathbf{e}_\lambda \Omega r \cos\varphi \right)$ comes from the *radial* derivative of r times $\mathbf{e}_\lambda \Omega r \cos\varphi$.

The vorticity equation

The equation of motion can be written as

$$\frac{\partial \mathbf{V}}{\partial t} + (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \mathbf{V} + \nabla \left(\frac{1}{2} \mathbf{V} \cdot \mathbf{V} + \phi \right) = -\alpha \nabla p - \mathbf{F} . \tag{5}$$

Here both gravity and the centripetal acceleration are included in the gradient of ϕ, which is defined by

$$\phi = \phi_a - \frac{1}{2} \left| \boldsymbol{\Omega} \times \mathbf{r} \right|^2 , \tag{6}$$

where ϕ_a is the gravitational potential, and $\mathbf{r}$ is a position vector that points from the center of the Earth to the point where (6) is applied. In other notation, α is the specific volume, p is pressure, and and $\mathbf{F}$ is the friction vector. A derivation of (5) is given in the *QuickStudy* about *Motion on the Sphere*.

The vorticity equation is derived by taking the curl of the equation of motion. The result is

$$\boxed{\frac{\partial}{\partial t}(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) + \nabla \times \left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \mathbf{V} \right] = \nabla p \times \nabla \alpha - \nabla \times \mathbf{F}} . \tag{7}$$

Here we have assumed that $\boldsymbol{\Omega}$ is independent of time, and used the identity

$$\nabla \times (\nabla \psi) = 0 , \tag{8}$$

where ψ is an arbitrary scalar, to eliminate the gradient terms of the momentum equation:

$$\nabla \times \left[\nabla \left(\frac{1}{2} \mathbf{V} \cdot \mathbf{V} \right) \right] = \nabla \times (\nabla \phi) = 0 . \tag{9}$$

We have also used (8) and the identity

$$\nabla \times (\psi \mathbf{A}) = \nabla \psi \times \mathbf{A} + \psi (\nabla \times \mathbf{A}), \tag{10}$$

where $\mathbf{A}$ is an arbitrary vector, to write

$$\begin{aligned} -\nabla \times (\alpha \nabla p) &= -\nabla \alpha \times \nabla p \\ &= \nabla p \times \nabla \alpha \ . \end{aligned} \tag{11}$$

Stretching and twisting

It is conventional to manipulate (7) into an alternative form. The identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \ . \tag{12}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two arbitrary vectors, allows us to write

$$\nabla \times \left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \mathbf{V} \right] = (\boldsymbol{\omega} + 2\boldsymbol{\Omega})(\nabla \cdot \mathbf{V}) - \mathbf{V}\left[\nabla \cdot (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \right] - \left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla \right]\mathbf{V} + (\mathbf{V} \cdot \nabla)(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \ . \tag{13}$$

Using (2) and (3), Eq. (13) can be simplified to

$$\nabla \times \left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \mathbf{V} \right] = (\boldsymbol{\omega} + 2\boldsymbol{\Omega})(\nabla \cdot \mathbf{V}) - \left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla \right]\mathbf{V} + (\mathbf{V} \cdot \nabla)(\boldsymbol{\omega} + 2\boldsymbol{\Omega}). \tag{14}$$

After substitution from (14), we can write (7) as

$$\frac{D}{Dt}(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) + (\boldsymbol{\omega} + 2\boldsymbol{\Omega})(\nabla \cdot \mathbf{V}) = \left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla \right]\mathbf{V} + \nabla p \times \nabla \alpha - \nabla \times \mathbf{F}, \tag{15}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla) \tag{16}$$

is the Lagrangian time derivative, and we assume that the Earth's angular velocity is independent of time.

Next, we use continuity in the form

$$\frac{1}{\alpha}\frac{D\alpha}{Dt} = \nabla \cdot \mathbf{V} \tag{17}$$

to eliminate $\nabla \cdot \mathbf{V}$ in (15). This gives

$$\alpha \frac{D}{Dt}(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) + (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\frac{D\alpha}{Dt} = \alpha\left\{\left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\right]\mathbf{V} + \nabla p \times \nabla\alpha - \nabla \times (\alpha \nabla \cdot \mathbf{F})\right\} . \tag{18}$$

Combining terms on the left-hand side of (18), we find that

$$\boxed{\frac{D}{Dt}\left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega})\alpha\right] = \alpha\left\{\left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\right]\mathbf{V} + \nabla p \times \nabla\alpha - \nabla \times \mathbf{F}\right\}} . \tag{19}$$

The vector $(\boldsymbol{\omega} + 2\boldsymbol{\Omega})\alpha$ is materially conserved when the right-hand side of (18) vanishes. If the specific volume increases, the magnitude of the vorticity tends to decrease, and vice versa.

The first term on the right-hand side of (19) represents both (i.e., the combination of) stretching and tilting. It can be written as

$$\left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\right]\mathbf{V} = |\boldsymbol{\omega} + 2\boldsymbol{\Omega}|\frac{\partial \mathbf{V}}{\partial s} , \tag{20}$$

where s is a curvilinear coordinate that points in the direction of $\boldsymbol{\omega} + 2\boldsymbol{\Omega}$. We can divide $|\boldsymbol{\omega} + 2\boldsymbol{\Omega}|\frac{\partial \mathbf{V}}{\partial s}$ into two parts:

- Stretching, given by $|\boldsymbol{\omega} + 2\boldsymbol{\Omega}|\frac{\partial}{\partial s}(V_s \mathbf{e}_s)$, where V_s is the component of $\mathbf{V}$ in the direction of $\boldsymbol{\omega} + 2\boldsymbol{\Omega}$, and $\mathbf{e}_s$ is a unit vector parallel to $\boldsymbol{\omega} + 2\boldsymbol{\Omega}$. See Fig. 1. Positive stretching $\left(\frac{\partial V_s}{\partial s} > 0\right)$ favors $\frac{\partial |\boldsymbol{\omega}|}{\partial t} > 0$. The effects of stretching on the vorticity can be understood in terms of conservation of the angular momentum of a rotating cylinder that is oriented parallel to the vorticity vector. Positive stretching is due to divergence along the axis of the cylinder, which pulls mass in towards the axis. Angular momentum conservation then leads to an increase in the cylinder's angular velocity. Negative stretching does the same thing in reverse.

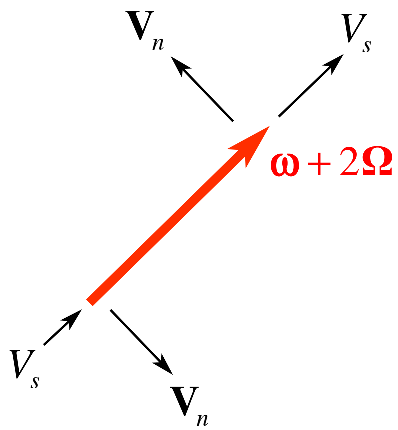

Figure 1: The components of the velocity normal and tangent to the vorticity vector, and the stretching and twisting processes associated with these velocity components. The vorticity vector is shown in red.

- Twisting, $|\boldsymbol{\omega}+2\boldsymbol{\Omega}|\frac{\partial \mathbf{V}_n}{\partial s}$, where $\mathbf{V}_n$ is the vector portion of $\mathbf{V}$ that lies in the plane normal to $\boldsymbol{\omega}+2\boldsymbol{\Omega}$. Twisting changes the direction of $\boldsymbol{\omega}$, but not its magnitude.

The effects of density variations

The term $\nabla p \times \nabla \alpha$ on the right-hand side of (19) vanishes if the density is spatially uniform, in which case the pressure drops out of the vorticity equation altogether. For the interpretation of $\nabla p \times \nabla \alpha$, it is very useful to distinguish between horizontal and vertical derivatives. We write

$$\boxed{\begin{aligned}\nabla p \times \nabla \alpha &= \left[\left(\nabla_r p + \frac{\partial p}{\partial r}\mathbf{e}_r\right)\times\left(\nabla_r \alpha + \frac{\partial \alpha}{\partial r}\mathbf{e}_r\right)\right] \\ &= \left(\nabla_r p \times \nabla_r \alpha\right) + \left(\nabla_r p \times \frac{\partial \alpha}{\partial r}\mathbf{e}_r + \frac{\partial p}{\partial r}\mathbf{e}_r \times \nabla_r \alpha\right).\end{aligned}} \tag{21}$$

On the second line of (21), there is no term involving the product $\frac{\partial \alpha}{\partial r}\frac{\partial p}{\partial r}$, because $\mathbf{e}_r \times \mathbf{e}_r = 0$. The first term on the second line is oriented vertically, and the second and third terms lie in the horizontal plane. The term $\frac{\partial p}{\partial r}\mathbf{e}_r \times \nabla_r \alpha$ can represent the effects of buoyancy. To see this, use the hydrostatic approximation in the form $\frac{\partial p}{\partial r} \cong -\frac{g}{\alpha}$. We can then write

$\frac{\partial p}{\partial r}\mathbf{e}_r \times \nabla_r \alpha \cong -\mathbf{e}_r \times \frac{g\nabla_r \alpha}{\alpha}$, which shows that vorticity can be generated by horizontal variations of the density, in the presence of gravity. This is what happens on the side of a buoyant thermal or plume.

The vertical component of the vorticity, and the horizontal vorticity vector

It is useful to separate the three-dimensional vorticity vector into the horizontal vorticity vector, denoted by $\boldsymbol{\eta}$, and the vertical component of the vorticity, denoted by ζ:

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{V} = \boldsymbol{\eta} + \zeta \mathbf{e}_r \,. \tag{22}$$

From (2), we know that

$$\nabla_r \cdot \boldsymbol{\eta} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\zeta\right) = 0 \,. \tag{23}$$

Similarly, we can separate $\boldsymbol{\Omega}$ into horizontal and vertical vectors:

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_h + \boldsymbol{\Omega}_r \text{, where } \boldsymbol{\Omega}_h = \left(\Omega \cos\varphi\right)\mathbf{e}_\varphi \text{ and } \boldsymbol{\Omega}_r = \left(\Omega \sin\varphi\right)\mathbf{e}_r \,. \tag{24}$$

Finally, we separate the velocity vector into the horizontal velocity vector, denoted by $\mathbf{V}_h$, and the vertical component of the velocity, denoted by w:

$$\mathbf{V} = \mathbf{V}_h + w\mathbf{e}_r \,. \tag{25}$$

It can be shown that

$$\boldsymbol{\eta} \equiv \mathbf{e}_r \times \left(\frac{\partial \mathbf{V}_h}{\partial r} - \nabla_r w\right), \tag{26}$$

and

$$\begin{aligned} \zeta &\equiv \mathbf{e}_r \cdot \boldsymbol{\omega} \\ &= \mathbf{e}_r \cdot \left(\nabla_r \times \mathbf{V}_h\right). \end{aligned} \tag{27}$$

We now rewrite the second term on the left-hand side of (7) as

$$
\begin{aligned}
\nabla\times\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\times\mathbf{V}\right] &= \nabla\times\left\{\left[(\boldsymbol{\eta}+\zeta\mathbf{e}_r)+2(\boldsymbol{\Omega}_h+\Omega_r\mathbf{e}_r)\right]\times(\mathbf{V}_h+w\mathbf{e}_r)\right\} \\
&= \nabla\times\left\{\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)+(\zeta+2\Omega_r)\mathbf{e}_r\right]\times(\mathbf{V}_h+w\mathbf{e}_r)\right\} \\
&= \nabla\times\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\times\mathbf{V}_h+(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\times(w\mathbf{e}_r)+(\zeta+2\Omega_r)\mathbf{e}_r\times\mathbf{V}_h\right] \\
&= \nabla\times\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\times\mathbf{V}_h\right]+\nabla\times\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\times(w\mathbf{e}_r)\right]+\nabla\times\left[(\zeta+2\Omega_r)\mathbf{e}_r\times\mathbf{V}_h\right].
\end{aligned}
\tag{28}
$$

To go further, we use the identity (12) to expand the second term on the right-hand side of (28) as

$$
\begin{aligned}
&\nabla\times\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\times(w\mathbf{e}_r)\right] \\
&=(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\left[\nabla\cdot(w\mathbf{e}_r)\right]-(w\mathbf{e}_r)\left[\nabla\cdot(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]-\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\cdot\nabla\right](w\mathbf{e}_r)+\left[(w\mathbf{e}_r)\cdot\nabla\right](\boldsymbol{\eta}+2\boldsymbol{\Omega}_h) \\
&=(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\frac{\partial w}{\partial r}-(w\mathbf{e}_r)\left[\nabla\cdot(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]-\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\cdot\nabla\right](w\mathbf{e}_r)+w\frac{\partial}{\partial r}(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h) \\
&=\frac{\partial}{\partial r}\left[w(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]-(w\mathbf{e}_r)\left[\nabla\cdot(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]-\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\cdot\nabla\right](w\mathbf{e}_r) \\
&=\frac{\partial}{\partial r}\left[w(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]-(w\mathbf{e}_r)\left[\nabla\cdot(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]-\mathbf{e}_r\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\cdot\nabla\right]w-w\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\cdot\nabla\right]\mathbf{e}_r \\
&=\left\{\frac{\partial}{\partial r}\left[w(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]-w\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\cdot\nabla\right]\mathbf{e}_r\right\}-\mathbf{e}_r\left\{w\left[\nabla_r\cdot(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]+\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\cdot\nabla\right]w\right\} \\
&=\left\{\frac{\partial}{\partial r}\left[w(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]-w\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\cdot\nabla\right]\mathbf{e}_r\right\}-\mathbf{e}_r\nabla_r\cdot\left[w(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]
\end{aligned}
\tag{29}
$$

It can be shown that

$$
\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\cdot\nabla\right]\mathbf{e}_r=\frac{-(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)}{r}.
\tag{30}
$$

Using (30) in (29), we finally obtain

$$
\begin{aligned}
\nabla\times\left[(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\times(w\mathbf{e}_r)\right] &= \left\{\frac{\partial}{\partial r}\left[w(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]+\frac{w(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)}{r}\right\}-\mathbf{e}_r\nabla_r\cdot\left[w(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right] \\
&=\frac{1}{r}\frac{\partial}{\partial r}\left[rw(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right]-\mathbf{e}_r\nabla_r\cdot\left[w(\boldsymbol{\eta}+2\boldsymbol{\Omega}_h)\right].
\end{aligned}
\tag{31}
$$

On a plane, with vertical coordinate z and vertical unit vector $\mathbf{k}$, we would find in place of (31) that

$$\nabla \times \left[\left(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h \right) \times \left(w\mathbf{k} \right) \right] = \frac{\partial}{\partial z} \left[w\left(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h \right) \right] - \mathbf{k}\nabla_r \cdot \left[w\left(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h \right) \right]. \tag{32}$$

Similarly, we expand the third term on the right-hand side of (28) as

$$\begin{aligned}
&\nabla \times \left[\left(\zeta + 2\Omega_r \right) \mathbf{e}_r \times \mathbf{V}_h \right] \\
&= \left(\zeta + 2\Omega_r \right) \mathbf{e}_r \left(\nabla \cdot \mathbf{V}_h \right) - \mathbf{V}_h \left[\nabla \cdot \left(\zeta + 2\Omega_r \right) \mathbf{e}_r \right] - \left[\left(\zeta + 2\Omega_r \right) \mathbf{e}_r \cdot \nabla \right] \mathbf{V}_h + \left(\mathbf{V}_h \cdot \nabla \right) \left[\left(\zeta + 2\Omega_r \right) \mathbf{e}_r \right] \\
&= \left(\zeta + 2\Omega_r \right) \mathbf{e}_r \left(\nabla \cdot \mathbf{V}_h \right) - \mathbf{V}_h \frac{\partial}{\partial r} \left(\zeta + 2\Omega_r \right) - \left(\zeta + 2\Omega_r \right) \frac{\partial \mathbf{V}_h}{\partial r} + \left(\mathbf{V}_h \cdot \nabla \right) \left[\left(\zeta + 2\Omega_r \right) \mathbf{e}_r \right] \\
&= -\frac{\partial}{\partial r} \left[\mathbf{V}_h \left(\zeta + 2\Omega_r \right) \right] + \left(\zeta + 2\Omega_r \right) \mathbf{e}_r \left(\nabla \cdot \mathbf{V}_h \right) + \left(\mathbf{V}_h \cdot \nabla \right) \left[\left(\zeta + 2\Omega_r \right) \mathbf{e}_r \right] \\
&= -\frac{\partial}{\partial r} \left[\mathbf{V}_h \left(\zeta + 2\Omega_r \right) \right] + \left(\zeta + 2\Omega_r \right) \mathbf{e}_r \left(\nabla \cdot \mathbf{V}_h \right) + \mathbf{e}_r \left(\mathbf{V}_h \cdot \nabla \right) \left(\zeta + 2\Omega_r \right) + \left(\zeta + 2\Omega_r \right) \left(\mathbf{V}_h \cdot \nabla \right) \mathbf{e}_r \\
&= \left\{ -\frac{\partial}{\partial r} \left[\mathbf{V}_h \left(\zeta + 2\Omega_r \right) \right] + \left(\zeta + 2\Omega_r \right) \left(\mathbf{V}_h \cdot \nabla \right) \mathbf{e}_r \right\} + \mathbf{e}_r \left[\left(\zeta + 2\Omega_r \right) \left(\nabla \cdot \mathbf{V}_h \right) + \left(\mathbf{V}_h \cdot \nabla \right) \left(\zeta + 2\Omega_r \right) \right] \\
&= \left\{ -\frac{\partial}{\partial r} \left[\mathbf{V}_h \left(\zeta + 2\Omega_r \right) \right] + \left(\zeta + 2\Omega_r \right) \left(\mathbf{V}_h \cdot \nabla \right) \mathbf{e}_r \right\} + \mathbf{e}_r \nabla_r \cdot \left[\mathbf{V}_h \left(\zeta + 2\Omega_r \right) \right].
\end{aligned} \tag{33}$$

It can be shown that

$$\left(\mathbf{V}_h \cdot \nabla \right) \mathbf{e}_r = \frac{-\mathbf{V}_h}{r} \tag{34}$$

Using (34) in (33), we finally obtain

$$\begin{aligned}
\nabla \times \left[\left(\zeta + 2\Omega_r \right) \mathbf{e}_r \times \mathbf{V}_h \right] &= \left\{ -\frac{\partial}{\partial r} \left[\mathbf{V}_h \left(\zeta + 2\Omega_z \right) \right] - \frac{\left(\zeta + 2\Omega_r \right) \mathbf{V}_h}{r} \right\} + \mathbf{e}_r \nabla_r \cdot \left[\mathbf{V}_h \left(\zeta + 2\Omega_r \right) \right] \\
&= -\frac{1}{r} \frac{\partial}{\partial r} \left[r\mathbf{V}_h \left(\zeta + 2\Omega_z \right) \right] + \mathbf{e}_r \nabla_r \cdot \left[\mathbf{V}_h \left(\zeta + 2\Omega_r \right) \right].
\end{aligned} \tag{35}$$

On a plane, we would find in place of (35) that

$$\nabla \times \left[\left(\zeta + 2\Omega_r \right) \mathbf{e}_r \times \mathbf{V}_h \right] = -\frac{\partial}{\partial z} \left[\mathbf{V}_h \left(\zeta + 2\Omega_z \right) \right] + \mathbf{k} \nabla \cdot \left[\mathbf{V}_h \left(\zeta + 2\Omega_r \right) \right]. \tag{36}$$

Substituting from (32) and (35), we can write (28) as

$$\nabla \times \left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \mathbf{V} \right]$$
$$= \nabla \times \left[(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) \times \mathbf{V}_h \right] + \frac{1}{r}\frac{\partial}{\partial r}\left[rw(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) - r\mathbf{V}_h(\zeta + 2\Omega_z) \right] + \mathbf{e}_r \nabla_r \cdot \left[\mathbf{V}_h(\zeta + 2\Omega_r) - w(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) \right] . \tag{37}$$

Using (37) and (21), we can separate the three-dimensional (3D) vector vorticity equation, (7), into a two-dimensional (2D) equation that governs the horizontal vorticity vector, and a second equation for the vertical component of the vorticity. The results are:

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) + \nabla \times \left[(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) \times \mathbf{V}_h \right] - \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\left[\mathbf{V}_h(\zeta + 2\Omega_r) - w(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) \right] \right\} \\ &= \left(\nabla_r p \times \frac{\partial \alpha}{\partial r}\mathbf{e}_r + \frac{\partial p}{\partial r}\mathbf{e}_r \times \nabla_r \alpha \right) - (\nabla \times \mathbf{F})_h , \end{aligned}} \tag{38}$$

$$\boxed{\frac{\partial}{\partial t}(\zeta + 2\Omega_r) + \nabla_r \cdot \left[\mathbf{V}_h(\zeta + 2\Omega_r) - w(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) \right] = \mathbf{e}_r \cdot \left[(\nabla_r p \times \nabla_r \alpha) - (\nabla \times \mathbf{F}) \right]} . \tag{39}$$

Eq. (39) is attractive because it involves only a horizontal divergence on its left-hand side and the vertical component of the curl of the forces on its right-hand side. Eq. (38) is also pleasingly simple. The non-divergence of the 3D vorticity vector implies that *the horizontal divergence of (38) is equivalent to minus* $\frac{1}{r}\frac{\partial}{\partial r}\left[r(.) \right]$ *applied to (39).*

In (38), the term $\nabla \times \left[(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) \times \mathbf{V}_h \right]$ is the curl of the cross product of two horizontal vectors; it is, therefore, the curl of a vertical (i.e., radial) vector, which we will call $d\mathbf{e}_r$, so that $\nabla \times \left[(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) \times \mathbf{V}_h \right] = \nabla \times (d\mathbf{e}_r)$. From the form of the curl in spherical coordinates, we see that $\nabla \times \left[(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h) \times \mathbf{V}_h \right] = -\mathbf{e}_r \times (\nabla_r d)$.

Potential vorticity

Let Λ be a scalar, such that

$$\frac{D\Lambda}{Dt} = S_\Lambda . \tag{40}$$

We can write

$$\alpha\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]\frac{D\Lambda}{Dt}=\alpha(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\left[\frac{D(\nabla\Lambda)}{Dt}\right]+\alpha\left\{\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]\mathbf{V}\right\}\cdot\nabla\Lambda\ , \tag{41}$$

which can be rearranged to

$$\alpha(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\left[\frac{D(\nabla\Lambda)}{Dt}\right]=-\alpha\left\{\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]\mathbf{V}\right\}\cdot\nabla\Lambda+\alpha\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]\frac{D\Lambda}{Dt}\ . \tag{42}$$

We now use (40) to eliminate $\frac{D\Lambda}{Dt}$ in (42), giving

$$\alpha(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\left[\frac{D(\nabla\Lambda)}{Dt}\right]=-\alpha\left\{\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]\mathbf{V}\right\}\cdot\nabla\Lambda+\alpha\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]S_\Lambda\ . \tag{43}$$

When we add (43) to $(\nabla\Lambda)\cdot$ (19), the stretching-twisting terms on the right-hand side of (19) cancel with the corresponding terms on the right-hand side of (43), and we obtain

$$\frac{D\left[\alpha(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot(\nabla\Lambda)\right]}{Dt}=\alpha(\nabla\Lambda)\cdot(\nabla p\times\nabla\alpha)-\alpha(\nabla\Lambda)\cdot(\nabla\times\mathbf{F})+\alpha\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]S_\Lambda\ . \tag{44}$$

The pressure-gradient term of (19) is still visible in (44). However, if we choose Λ to be a thermodynamic function that depends only on α and p, then

$$(\nabla\Lambda)\cdot(\nabla p\times\nabla\alpha)=0\ , \tag{45}$$

and (44) reduces to

$$\frac{D\left[\alpha(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot(\nabla\Lambda)\right]}{Dt}=-\alpha(\nabla\Lambda)\cdot(\nabla\times\mathbf{F})+\alpha\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]S_\Lambda\ . \tag{46}$$

Here $\alpha(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot(\nabla\Lambda)$ is the *potential vorticity*, which is materially conserved when there is no friction and $S_\Lambda=0$. The most common choice is $\Lambda\equiv\theta$, the potential temperature. With that choice, (46) can be written as

$$\frac{Dq}{Dt} = \alpha\left\{-(\nabla\theta)\cdot(\nabla\times\mathbf{F}) + \left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]\dot{\theta}\right\}, \tag{47}$$

where

$$\boxed{q \equiv \alpha(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\theta}\ . \tag{48}$$

According to (48), the potential vorticity depends on the component of $\boldsymbol{\omega}+2\boldsymbol{\Omega}$ that is perpendicular to the isentropic surface. It does not involve the part of $\boldsymbol{\omega}+2\boldsymbol{\Omega}$ that is parallel to the isentropic surface.

In view of (2) and (3), we can rewrite (48) as

$$q \equiv \alpha\nabla\cdot\left[\theta(\boldsymbol{\omega}+2\boldsymbol{\Omega})\right]. \tag{49}$$

Finally, we use vector identities to write the heating and friction terms of (47) as divergences, following Haynes and McIntyre (1987):

$$\begin{aligned}(\nabla\theta)\cdot(\nabla\times\mathbf{F}) &= \nabla\cdot(\nabla\theta\times\mathbf{F}) + \mathbf{F}\cdot\left(\cancel{\nabla\times\nabla\theta}\right)\\ &= \nabla\cdot(\nabla\theta\times\mathbf{F}),\end{aligned} \tag{50}$$

$$\begin{aligned}\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\right]\dot{\theta} &= \nabla\cdot\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\dot{\theta}\right] - \dot{\theta}\left[\cancel{\nabla\cdot(\boldsymbol{\omega}+2\boldsymbol{\Omega})}\right]\\ &= \nabla\cdot\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\dot{\theta}\right].\end{aligned} \tag{51}$$

Then (47) becomes

$$\frac{Dq}{Dt} = \alpha\nabla\cdot\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\dot{\theta} - \nabla\theta\times\mathbf{F}\right]. \tag{52}$$

This is the potential vorticity equation. The flux form corresponding to (52) is

$$\boxed{\frac{\partial}{\partial t}(\rho q) + \nabla\cdot\left[\rho\mathbf{V}q + \nabla\theta\times\mathbf{F} - (\boldsymbol{\omega}+2\boldsymbol{\Omega})\dot{\theta}\right] = 0}\ . \tag{53}$$

This says that the time rate of change of the mass-weighted potential vorticity is due only to the divergence of a flux.

The potential enstrophy equation can be obtained by multiplying (52) by q . The result can be written as

$$\frac{D}{Dt}\left(\frac{q^2}{2}\right)=\alpha\nabla\cdot\left\{q\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\dot{\theta}-\nabla\theta\times\mathbf{F}\right]\right\}-\alpha\left[(\boldsymbol{\omega}+2\boldsymbol{\Omega})\dot{\theta}-\nabla\theta\times\mathbf{F}\right]\cdot\nabla q\ . \tag{54}$$

Potential vorticity in isentropic coordinates

We can think of q , given by (49), as the sum of a "horizontal" part and a "vertical" part:

$$q=\alpha\left[\left(\boldsymbol{\eta}+\mathbf{e}_\varphi 2\Omega\cos\varphi\right)\cdot\nabla_r\theta+\left(\zeta+2\Omega\sin\varphi\right)\frac{\partial\theta}{\partial r}\right]. \tag{55}$$

Recall from (26) and (27) that $\boldsymbol{\eta}$ and ζ were defined using horizontal derivatives on surfaces of constant r . It is useful to transform (55) into a form that involves horizontal derivatives along surfaces of constant θ . Keep in mind that the meanings of $\mathbf{V}_h$, w , and the various unit vectors are exactly the same in radial coordinates and θ -coordinates.

As a first step, we note that, for an arbitrary scalar A ,

$$\nabla_r A=\nabla_\theta A-\frac{\partial A}{\partial r}\nabla_\theta r\ , \tag{56}$$

and for an arbitrary horizontal vector $\mathbf{H}$

$$\nabla_r\times\mathbf{H}=\nabla_\theta\times\mathbf{H}+\frac{\partial\mathbf{H}}{\partial r}\times\nabla_\theta r\ . \tag{57}$$

For the special case $A\equiv\theta$, (56) reduces to

$$\nabla_r\theta=-\frac{\partial\theta}{\partial r}\nabla_\theta r\ . \tag{58}$$

Use of (58) in (55) gives

$$q = \alpha\left[-\left(\boldsymbol{\eta} + \mathbf{e}_{\varphi} 2\Omega\cos\varphi\right)\cdot\nabla_{\theta} r + \left(\zeta + 2\Omega\sin\varphi\right)\right]\frac{\partial\theta}{\partial r}. \tag{59}$$

Although (56) and (57) are valid for an arbitrary coordinate transformation (in which θ can be any vertical coordinate, not necessarily potential temperature), Eq. (59) applies *only* to the case in which θ is the same thermodynamic variable used in the definition of q.

Eq. (59) is not very convenient because it involves horizontal derivatives on both r-surfaces (to compute ζ and $\boldsymbol{\eta}$ from the winds) and θ-surfaces (to compute $\nabla_{\theta} r$). To obtain a more useful expression for q, use (57) in (27) to obtain

$$\begin{aligned}\zeta &= \mathbf{e}_r\cdot\left(\nabla_r\times\mathbf{V}_h\right)\\ &= \mathbf{e}_r\cdot\left(\nabla_{\theta}\times\mathbf{V}_h + \frac{\partial\mathbf{V}_h}{\partial r}\times\nabla_{\theta} r\right)\\ &= \zeta_{\theta} + \mathbf{e}_r\cdot\left(\frac{\partial\mathbf{V}_h}{\partial r}\times\nabla_{\theta} r\right),\end{aligned} \tag{60}$$

where

$$\boxed{\zeta_{\theta} \equiv \mathbf{e}_r\cdot\left(\nabla_{\theta}\times\mathbf{V}_h\right)}. \tag{61}$$

A vector identity can be used to show that

$$\mathbf{e}_r\cdot\left(\frac{\partial\mathbf{V}_h}{\partial r}\times\nabla_{\theta} r\right) = \left(\mathbf{e}_r\times\frac{\partial\mathbf{V}_h}{\partial r}\right)\cdot\nabla_{\theta} r, \tag{62}$$

which allows us to write

$$\boxed{\zeta = \zeta_{\theta} + \left(\mathbf{e}_r\times\frac{\partial\mathbf{V}_h}{\partial r}\right)\cdot\nabla_{\theta} r}. \tag{63}$$

Similarly, we can use (56) in (26) to write

$$\boldsymbol{\eta} = \mathbf{e}_r\times\left(\frac{\partial\mathbf{V}_h}{\partial r} - \nabla_{\theta} w + \frac{\partial w}{\partial r}\nabla_{\theta} r\right). \tag{64}$$

Substitute (63) and (64) into (59), to obtain

$$q=\alpha\left\{-\left[\mathbf{e}_r\times\left(\frac{\partial\mathbf{V}_h}{\partial r}-\nabla_\theta w+\frac{\partial w}{\partial r}\nabla_\theta r\right)+\mathbf{e}_\varphi 2\Omega\cos\varphi\right]\cdot\nabla_\theta r+\left[\zeta_\theta+\left(\mathbf{e}_r\times\frac{\partial\mathbf{V}_h}{\partial r}\right)\cdot\nabla_\theta r+2\Omega\sin\varphi\right]\right\}\frac{\partial\theta}{\partial r}$$
$$=\alpha\left\{\left[\mathbf{e}_r\times\left(\nabla_\theta w-\frac{\partial w}{\partial r}\nabla_\theta r\right)-\mathbf{e}_\varphi 2\Omega\cos\varphi\right]\cdot\nabla_\theta r+\left(\zeta_\theta+2\Omega\sin\varphi\right)\right\}\frac{\partial\theta}{\partial r}.$$

(65)

This can be simplified using

$$\left[\mathbf{e}_r\times\left(\frac{\partial w}{\partial r}\nabla_\theta r\right)\right]\cdot\nabla_\theta r=0,$$

(66)

and the definition

$$\boxed{\boldsymbol{\eta}_\theta\equiv-\mathbf{e}_r\times\nabla_\theta w}.$$

(67)

We finally obtain

$$\boxed{q=\alpha\left[-\left(\boldsymbol{\eta}_\theta+\mathbf{e}_\varphi 2\Omega\cos\varphi\right)\cdot\nabla_\theta r+\left(\zeta_\theta+2\Omega\sin\varphi\right)\right]\frac{\partial\theta}{\partial r}}.$$

(68)

When the isentropic surfaces are "flat" in the sense that $\nabla_\theta r=0$, Eq. (68) reduces to $q=\alpha\left(\zeta_\theta+2\Omega\sin\varphi\right)\frac{\partial\theta}{\partial r}$.

Use of (67) allows us to rewrite (64) as

$$\boxed{\boldsymbol{\eta}=\boldsymbol{\eta}_\theta+\mathbf{e}_r\times\left(\frac{\partial\mathbf{V}_h}{\partial r}+\frac{\partial w}{\partial r}\nabla_\theta r\right)}.$$

(69)

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