Wave-mean flow interactions

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Interactions and non-interactions of gravity waves with the mean flow

We have seen how eddies can affect the mean flow, through eddy flux divergences and energy conversions. Despite the presence of such terms in the equations, however, it turns out that under surprisingly general conditions the eddies are actually powerless to affect the mean flow. There are several related theorems that demonstrate this "non-interaction" of the eddies with the mean flow. They are called, reasonably enough, "non-interaction theorems." The earliest such ideas were published by Eliassen and Palm (1961), and the following discussion is based on their paper. The same material is also discussed in more detail, and in somewhat more general form, in Chapter 8 of Lindzen's (1990) book.

Consider the equation of zonal motion in the simplified form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} . \tag{1}$$

We have omitted rotation, sphericity, friction, and meridional motions (i.e., v) and variations (i.e., $\frac{\partial}{\partial y}$). Eq. (1) can apply, for example, to small-scale gravity waves forced by flow over topography. Let

$$\begin{aligned} u &= U + u',\ U = U(z), \\ w &= w', \\ p &= \overline{p} + p',\ \overline{p} = \overline{p}(z), \\ \rho &= \overline{\rho} + \rho',\ \overline{\rho} = \overline{\rho}(z) . \end{aligned} \tag{2}$$

We interpret the primed quantities as small-amplitude wave-like perturbations with zero means. Recall that

$$\rho \frac{\partial U}{\partial t} \sim -\frac{\partial}{\partial z} \rho \overline{w'u'} . \tag{3}$$

We are interested in what determines the wave momentum flux divergence, $\frac{\partial}{\partial z} \rho \overline{w'u'}$.

Substitute (2) into (1) and linearize, to obtain

$$\bar{\rho} \frac{\partial u'}{\partial t} = -\left(\bar{\rho} U \frac{\partial u'}{\partial x} + \bar{\rho} w' \frac{\partial U}{\partial z} + \frac{\partial p'}{\partial x} \right). \tag{4}$$

Assume that the perturbations are steady, so that

$$\frac{\partial u'}{\partial t} = 0 . \tag{5}$$

This implies both that the waves are neutral, i.e., neither amplifying or decaying, and also that they are stationary, i.e., their phase speed is zero. The latter assumption is reasonable, e.g., for mountain waves. Then (4) reduces to

$$\begin{aligned} 0 &= \bar{\rho} U \frac{\partial u'}{\partial x} + \bar{\rho} w' \frac{\partial U}{\partial z} + \frac{\partial p'}{\partial x} \\ &= \frac{\partial}{\partial x} \left(\bar{\rho} U u' + p' \right) + \bar{\rho} w' \frac{\partial U}{\partial z} . \end{aligned} \tag{6}$$

This is the form of the steady-state equation of motion that we will use.

Next, multiply (6) by $\left(\bar{\rho} U u' + p' \right)$, to obtain

$$0 = \frac{\partial}{\partial x} \left[\frac{\left(\bar{\rho} U u' + p' \right)^2}{2} \right] + \bar{\rho}^2 U \frac{\partial U}{\partial z} w'u' + \bar{\rho} \frac{\partial U}{\partial z} w'p' . \tag{7}$$

The term involving $\frac{\partial}{\partial x}$ vanishes when integrated over the whole domain, leaving

$$\frac{\partial U}{\partial z} \left(U \int_{-\infty}^{\infty} \bar{\rho} w'u' \, dx + \int_{-\infty}^{\infty} w'p' \, dx \right) = 0 , \tag{8}$$

which can be simplified to

$$U\int_{-\infty}^{\infty}\bar{\rho}w'u'\,dx+\int_{-\infty}^{\infty}w'p'\,dx=0\ , \tag{9}$$

provided that $\frac{\partial U}{\partial z}\neq 0$.

Eq. (9) is an important result. It shows that the wave momentum flux, $\int_{-\infty}^{\infty}\bar{\rho}w'u'\,dx$, and the wave energy flux, $\int_{-\infty}^{\infty}w'p'\,dx$, are closely related. At a "critical" level, where $U=0$, the wave energy flux must vanish; the only other possibility is that our assumptions, e.g., a steady state with no friction, do not apply at the critical level. For a wave forced by flow over a mountain, the energy flux is, of course, upward, but (9) shows that it goes to zero at a critical level. This means that the wave does not exist above the critical level. The upward propagation of the wave is blocked at the critical level.

Eq. (9) also shows that a wave with an upward energy flux will produced a downward momentum flux in westerlies and an upward momentum flux in easterlies. In either case, the wave is driving the mean flow towards zero, i.e., it is exerting a drag on the mean flow.

Let e_E be the total eddy energy per unit mass associated with the wave (the sum of the eddy kinetic, eddy internal, and eddy potential energies). It can be shown that e_E satisfies

$$\frac{\partial}{\partial x}\left(\bar{\rho}e_E U+p'u'\right)+\frac{\partial}{\partial z}\left(p'w'\right)=-\bar{\rho}u'w'\frac{\partial U}{\partial z}\ . \tag{10}$$

The right-hand side of (10) is a "gradient production" term that represents conversion of the kinetic energy of the mean state into the total eddy energy, e_E . Eq. (10) simply says that the production term on the right-hand side is balanced by the transport terms on the left-hand side. Integration over the domain gives

$$\frac{\partial}{\partial z}\int_{-\infty}^{\infty}p'w'\,dx=-\frac{\partial U}{\partial z}\int_{-\infty}^{\infty}\bar{\rho}u'w'\,dx\ . \tag{11}$$

This means that the wave energy flux divergence balances conversion to or from the kinetic energy of the mean flow.

By combining (9) and (11) we can show that

$$U\frac{\partial}{\partial z}\left(\int_{-\infty}^{\infty}\bar{\rho}u'w'\,dx\right)=0\,.$$

(12)

Therefore, *when* $U\neq 0$ *, the wave momentum flux* $\int_{-\infty}^{\infty}\bar{\rho}u'w'\,dx$ *is independent of height.* This is very important because, as shown by (3), it implies that the wave momentum flux has no effect on $U(z)$, except at the critical level where $U=0$. The wave momentum flux is absorbed at the critical level. From (3), it follows that U will tend to change with time at the critical level, so U will become different from zero. Therefore, the critical level will move.

If we allowed the phase speed c to be non-zero, we would find $U-c$ everywhere in place of U. The momentum would be absorbed at the critical level where $U=c$.

Since (12) tells us that $\int_{-\infty}^{\infty}\bar{\rho}u'w'\,dx$ is independent of height (where $U\neq 0$), we see from (9) that the wave energy flux is just proportional to U. Alternatively, we can combine (9) and (12) to write

$$\frac{1}{U}\int_{-\infty}^{\infty}w'p'\,dx=\text{constant.}$$

(13)

The conserved quantity $\frac{1}{U}\int_{-\infty}^{\infty}w'p'\,dx$ is called the "wave action." Eq. (9) can be written as "wave action plus wave momentum flux = zero."

Since the mid-1980s, there has been a lot of interest in the effects of gravity wave momentum fluxes on the general circulation; because the waves act to decelerate the mean flow, these interactions are referred to as "gravity wave drag" (McFarlane, 1987). Most of the discussion so far has been on gravity waves forced by flow over topography, although recently gravity waves forced by convective storms are receiving a lot of attention (e.g., Fovell et al., 1992).

Fig. 1 shows the deceleration of the zonally averaged zonal wind induced by gravity-wave drag in a general circulation model, as reported by McFarlane (1987). Here the gravity wave drag has been parameterized using methods that we will not discuss, which are based on the assumption that the waves are produced by flow over mountains. The plot shows the "tendency" of the zonally averaged zonal wind due to this orographic gravity-wave drag, for northern-winter conditions. The actual response of the zonally averaged zonal wind is shown in

Fig. 2. The changes are very large. In order for thermal wind balance to be maintained, there must be corresponding changes in the zonally averaged temperature; these are shown in Fig. 3. The polar troposphere has warmed dramatically, to be consistent with the weaker westerly jet. The changes shown in Fig. 2 and Fig. 3 make the model results more realistic than before, suggesting that gravity-wave drag is an important process in nature.

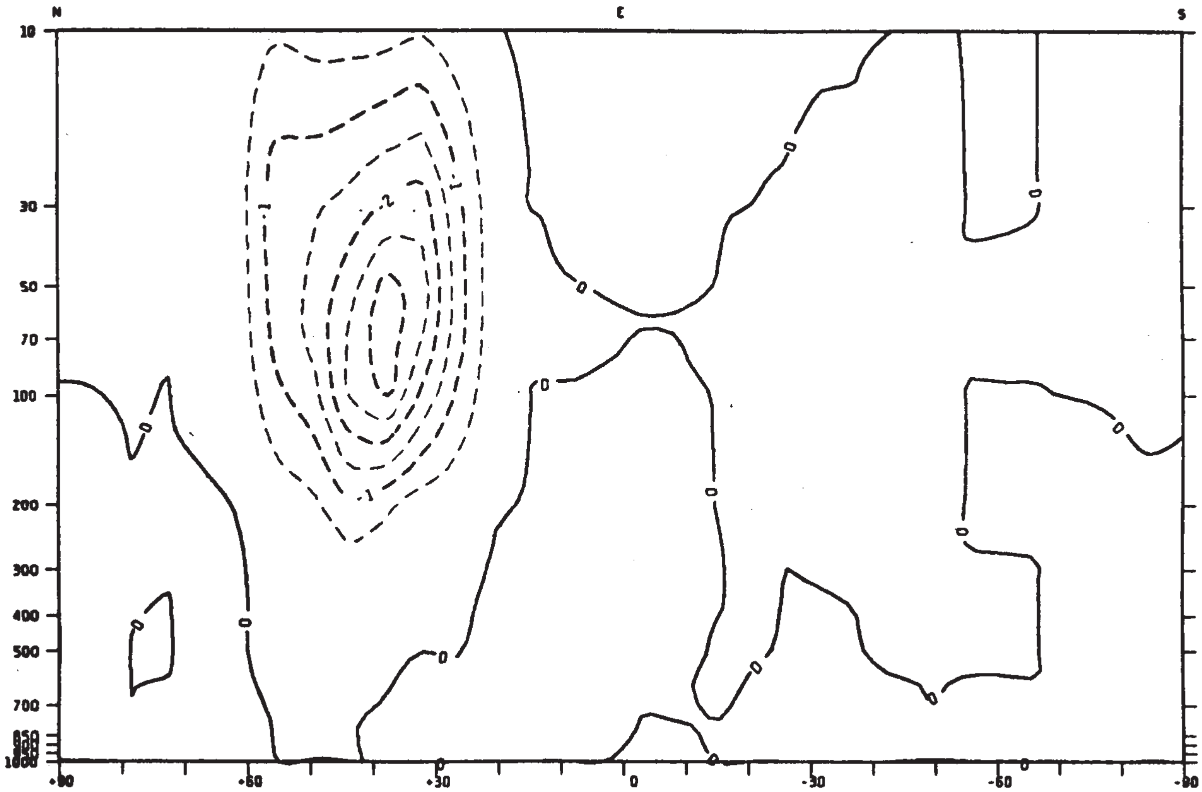

Figure 1: The deceleration of the zonally averaged zonal flow, induced by orographically forced gravity waves, as simulated with a general circulation model. The units are m s^{-1} day^{-1}. From McFarlane (1987).

Vertical propagation of planetary waves

The following discussion is based on the famous paper by Charney and Drazin (1961), which deals with the vertically propagating planetary waves. Closely related work can be found in Dickinson (1968 a) and Matsuno (1970).

Let $T_S(p)$ be a basic-state temperature profile, and define α_S, θ_S, and ρ_S accordingly. The quasi-geostrophic form of the potential vorticity equation is

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla_p\right) q = 0 \,, \tag{14}$$

where

$$q = f + \zeta_g + \frac{\partial}{\partial p}\left(\frac{f_0}{S_p}\frac{\partial \phi}{\partial p}\right) \tag{15}$$

is the quasi-geostrophic pseudo-potential vorticity, $S_p \equiv -\frac{\alpha_S}{\theta_S}\frac{\partial \theta_S}{\partial p}$ is the static stability, and in the last term of (15) f has been replaced by f_0. We are working on a β -plane, such that $f = f_0 + \beta y$. Note that q is essentially determined by the absolute vorticity and the change of temperature with height, and that (14) does not contain a vertical advection term. [See Chapter 8 of Holton (1992).]

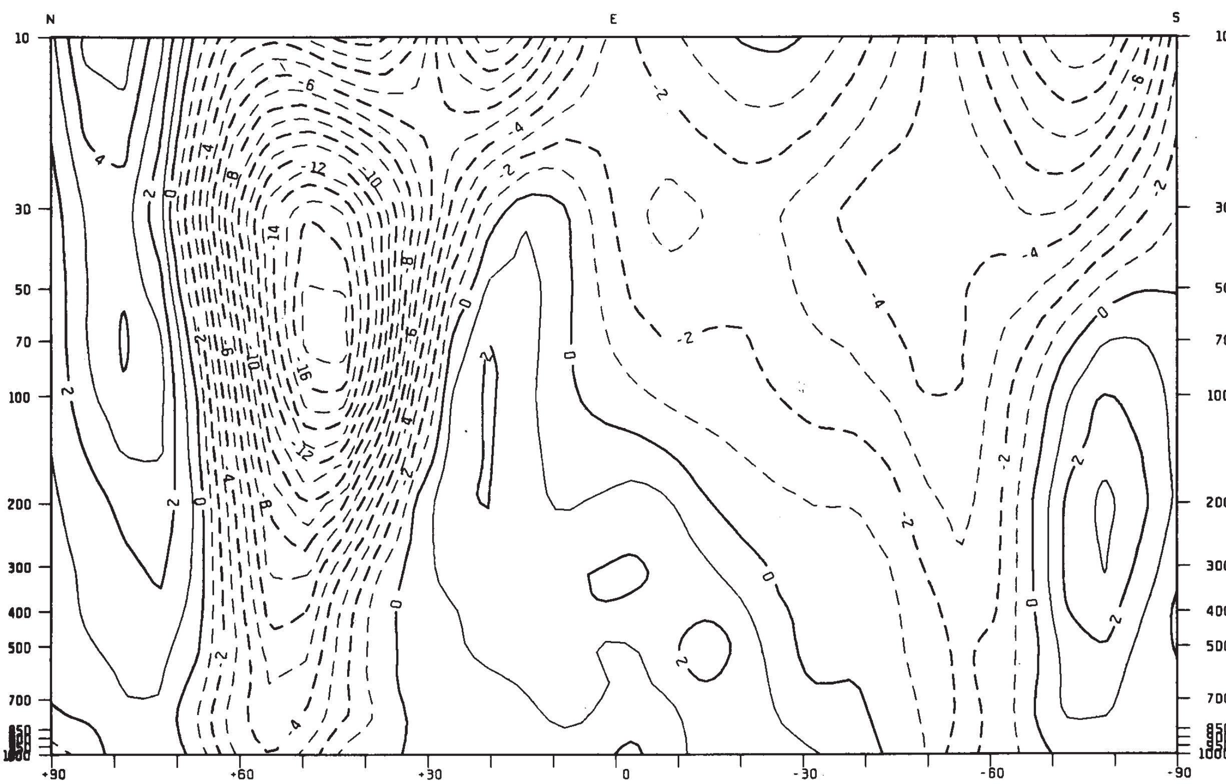

Figure 2: The actual change in the zonally averaged wind caused by the introduction of gravity wave drag in a general circulation model, as inferred by comparison with a control run. The units are m s^{-1}. From McFarlane (1987).

From (14) we can derive

$$\frac{\partial}{\partial t}[q] = -\frac{\partial}{\partial y}\left[v_g^* q^*\right]. \tag{16}$$

Exercise: Show that

$$\begin{aligned}\left[v_g^* q^*\right] &= \left[v_g^* \zeta_g^*\right] - \frac{\partial}{\partial p}\left(\frac{Rf_0}{pS_p}\left[v_g^* T^*\right]\right) \\ &= -\frac{\partial}{\partial y}\left[u_g^* v_g^*\right] - \frac{\partial}{\partial p}\left(\frac{Rf_0}{pS_p}\left[v_g^* T^*\right]\right).\end{aligned} \tag{17}$$

As we will see later, the expression on the right-hand side of (17) is the divergence of the Eliassen-Palm flux. Eq. (17) expresses a very important relationship. It says that the meridional eddy flux of potential vorticity is related to the convergence of the meridional eddy flux of zonal

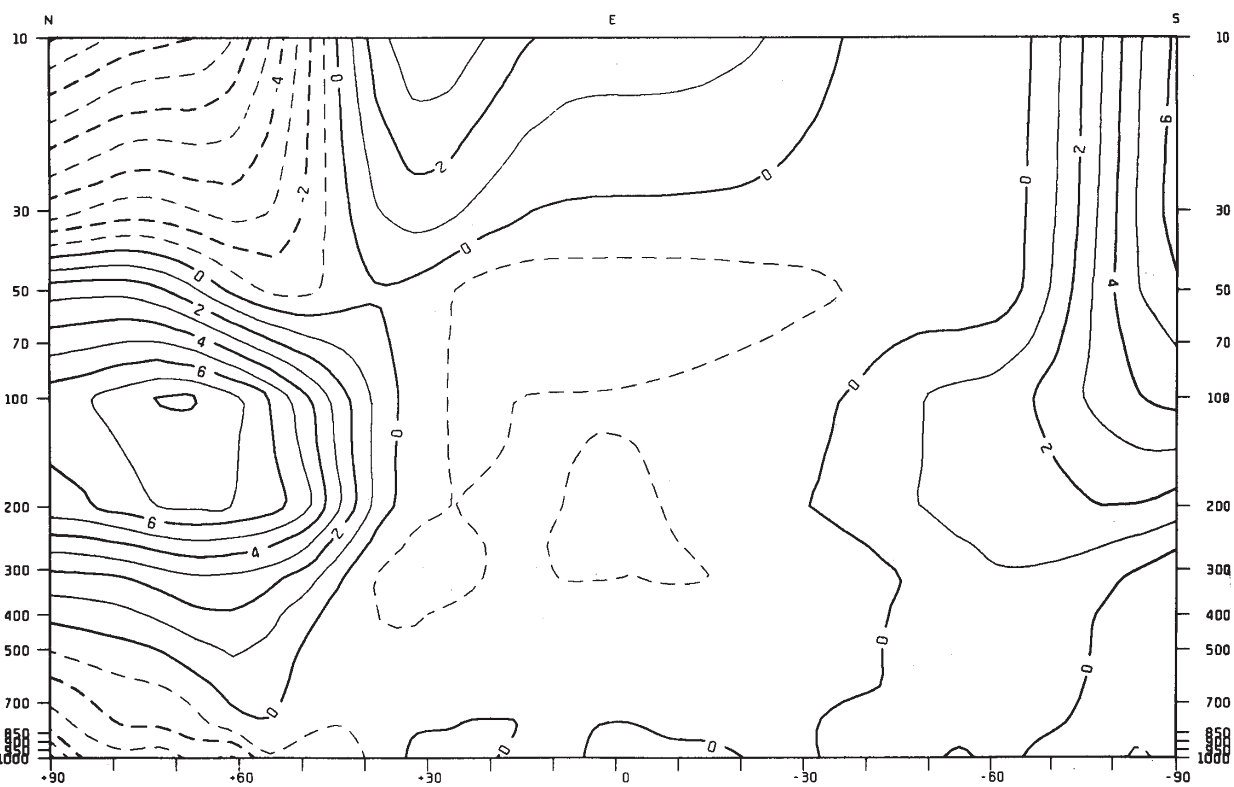

Figure 3: The actual change in the zonally averaged temperature caused by the introduction of gravity wave drag in a general circulation model, as inferred by comparison with a control run. The units are K[1]. From McFarlane (1987).

momentum, and to the rate of change with height of the meridional eddy sensible heat flux.

When we form the convergence of the eddy potential vorticity flux, i.e., $-\frac{\partial\left[v_g^* q^*\right]}{\partial y}$, (17) will give us $\frac{\partial}{\partial y}\left\{-\frac{\partial\left[u_g^* v_g^*\right]}{\partial y}\right\}$. This affects the meridional shear of $[u]$. We will also get a term proportional to $\frac{\partial}{\partial p}\left\{-\frac{\partial\left[v_g^* T^*\right]}{\partial y}\right\}$. This affects the static stability.

We adopt the "log pressure" coordinate

$$z(p) \equiv -\left(\frac{RT_0}{g}\right)\ln\left(\frac{p}{p_0}\right), \tag{18}$$

where T_0 is a constant reference temperature. With the use of (18), (15) can be rewritten as

$$q = f + \nabla^2\psi + \frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\rho_S \frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right), \tag{19}$$

where

$$\psi \equiv \frac{\phi}{f_0} \tag{20}$$

is called the "geostrophic stream function," and the Brunt-Vaisala frequency N, satisfies

$$N^2 \equiv \frac{g}{\theta_S}\frac{\partial\theta_S}{\partial z}. \tag{21}$$

Note that

$$v_g = \frac{\partial\psi}{\partial x} \text{ and } u_g = -\frac{\partial\psi}{\partial y}. \tag{22}$$

Linearizing (14) about the zonal-mean state gives

$$\left(\frac{\partial}{\partial t}+[u]\frac{\partial}{\partial x}\right)q^{*}+v_{g}^{*}\frac{\partial[q]}{\partial y}=0\ . \tag{23}$$

We look for solutions of the form

$$\psi^{*}=\mathrm{Re}\left\{\hat{\psi}(y,z)e^{ik(x-ct)}\right\}, \tag{24}$$

$$q^{*}=\mathrm{Re}\left\{\hat{q}(y,z)e^{ik(x-ct)}\right\}. \tag{25}$$

Substitution of (19), (24), and (25) into (23) gives

$$\left([u]-c\right)\hat{q}+\hat{\psi}\frac{\partial[q]}{\partial y}=0\,, \tag{26}$$

where

$$\hat{q}=-k^{2}\hat{\psi}+\frac{\partial^{2}\hat{\psi}}{\partial y^{2}}+\frac{1}{\rho_{S}}\frac{\partial}{\partial z}\left(\rho_{S}\frac{f_{0}^{2}}{N^{2}}\frac{\partial\hat{\psi}}{\partial z}\right). \tag{27}$$

Using (27), we can rewrite (26) as

$$\frac{\partial^{2}\hat{\psi}}{\partial y^{2}}+\frac{1}{\rho_{S}}\frac{\partial}{\partial z}\left(\rho_{S}\frac{f_{0}^{2}}{N^{2}}\frac{\partial\hat{\psi}}{\partial z}\right)=-\left(\frac{1}{[u]-c}\frac{\partial[q]}{\partial y}-k^{2}\right)\hat{\psi}\,. \tag{28}$$

This is a fairly general form of the quasi-geostrophic wave equation that we want to analyze, but we will simplify it considerably before doing so.

As wave energy propagates up to higher levels, it encounters decreasing values of ρ_S . The energy-density (energy per unit volume) scales like $\rho_S\left(k\psi\right)^2$, so if the energy density is constant with height, $\hat{\psi}$ must increase like $\frac{1}{\sqrt{\rho_S}}$. Because of this effect, the equations become simpler if we introduce a scaled value of $\hat{\psi}$:

$$\psi \equiv \frac{\sqrt{\rho_S}}{N}\hat{\psi} \; . \tag{29}$$

Note that here ψ (no hat) is the scaled value; the meaning of ψ now departs from that used in (20). We also note that

$$\begin{aligned}
\frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\rho_S \frac{f_0^2}{N^2}\frac{\partial \hat{\psi}}{\partial z}\right) &= \frac{f_0^2}{\rho_S}\frac{\partial}{\partial z}\left\{\frac{\sqrt{\rho_S}}{N}\frac{\partial}{\partial z}\left(\frac{\sqrt{\rho_S}}{N}\hat{\psi}\right) - \frac{\sqrt{\rho_S}}{N}\hat{\psi}\frac{\partial}{\partial z}\left(\frac{\sqrt{\rho_S}}{N}\right)\right\} \\
&= \frac{f_0^2}{\rho_S}\left\{\frac{\partial}{\partial z}\left(\frac{\sqrt{\rho_S}}{N}\right)\frac{\partial}{\partial z}\left(\frac{\sqrt{\rho_S}}{N}\hat{\psi}\right) + \frac{\sqrt{\rho_S}}{N}\frac{\partial^2}{\partial z^2}\left(\frac{\sqrt{\rho_S}\hat{\psi}}{N}\right)\right. \\
&\quad \left. - \frac{\partial}{\partial z}\left(\frac{\sqrt{\rho_S}\hat{\psi}}{N}\right)\frac{\partial}{\partial z}\left(\frac{\sqrt{\rho_S}}{N}\right) - \left(\frac{\sqrt{\rho_S}\hat{\psi}}{N}\right)\frac{\partial^2}{\partial z^2}\left(\frac{\sqrt{\rho_S}}{N}\right)\right\} \\
&= \frac{f_0^2}{\rho_S}\left[\frac{\sqrt{\rho_S}}{N}\frac{\partial^2}{\partial z^2}\psi - \psi\frac{\partial^2}{\partial z^2}\left(\frac{\sqrt{\rho_S}}{N}\right)\right] \; .
\end{aligned} \tag{30}$$

Substituting from (29) and (30), we can rewrite (28) as

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{f_0^2}{N^2}\frac{\partial^2 \psi}{\partial z^2} = -\left(\frac{f_0^2}{N^2}\frac{n^2}{4H_0^2}\right)\psi , \tag{31}$$

where

$$n^2 \equiv \frac{4N^2 H_0^2}{f_0^2}\left\{\frac{1}{[u]-c}\frac{\partial [q]}{\partial y} - k^2 - \frac{f_0^2}{\sqrt{\rho_S}N}\frac{\partial^2}{\partial z^2}\left(\frac{\sqrt{\rho_S}}{N}\right)\right\} \tag{32}$$

is called the "index of refraction." Here $H_0 \equiv \frac{RT_0}{g}$, where T_0 is a reference temperature. Eq. (32) is a form of the quasi-geostrophic wave equation. When $n^2 > 0$, ψ is oscillatory (propagating), and when $n^2 < 0$, ψ is "evanescent" (exponentially decaying away from the source of excitation).

Comparing (31) - (32) with (28), it seems that the left-hand side of (31) has been simplified but the expression for the index of refraction is pretty complicated. Through some idealizations we can simplify it drastically without altering the basic meaning. Consider an isothermal atmosphere with $T_S(p) \cong T_0 =$ constant. This is not unrealistic for the lower stratosphere. For this case, we can show that $N \cong$constant and $\rho_S \sim e^{-\frac{z}{H_0}}$, so that (32) reduces to

$$n^2 \cong \frac{4N^2 H_0^2}{f_0^2}\left(\frac{1}{[u]-c}\frac{\partial[q]}{\partial y} - k^2\right) - 1 . \tag{33}$$

Inspection of (33) shows that $[u]-c>0$ is necessary for $n^2>0$, i.e., for propagation.

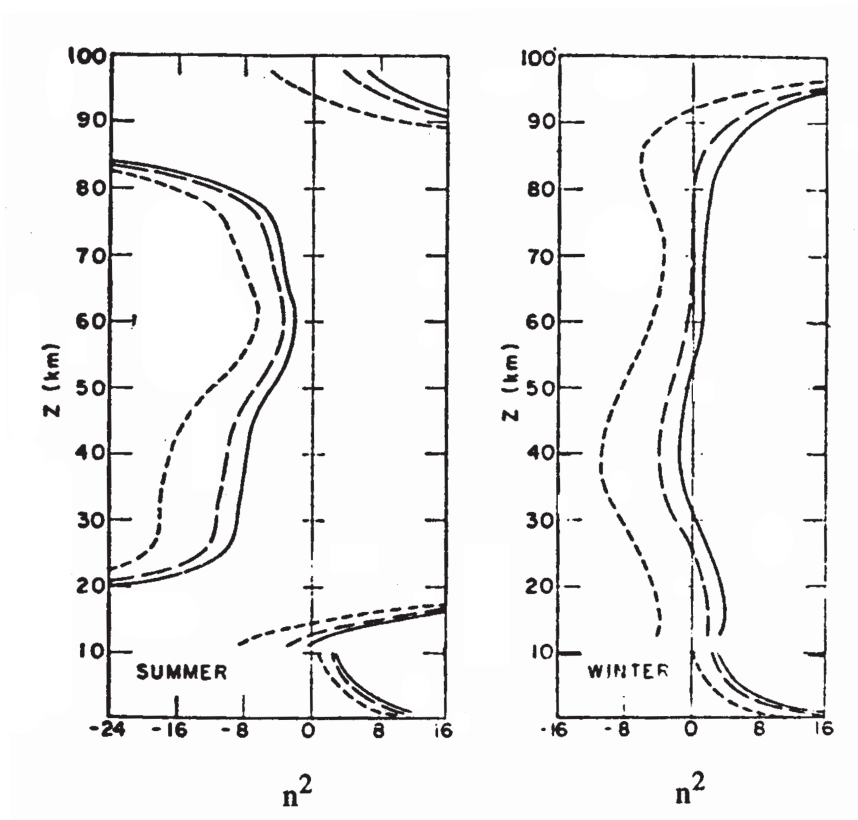

Figure 4: The square of the index of refraction for summer and winter, averaged between 30° and 60°N, for waves of different wavelengths, *L*. The short-dashed lines correspond to *L* = 6,000 km, the long-dashed lines correspond to *L* = 10,000 km, and the solid lines correspond to *L* = 14,000 km. From Charney and Drazin (1961).

Now we concentrate on stationary waves, for which the phase speed, c, is zero. This type of wave can be forced by orography, for example, as discussed in Chapter 8. Then (31) and (33) become

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{f_0^2}{N^2}\frac{\partial^2 \psi}{\partial z^2} = -\left(\frac{f_0^2}{N^2}\frac{n^2}{4H_0^2}\right)\psi \,, \tag{34}$$

$$n^2 = \frac{4N^2 H_0^2}{f_0^2}\left(\frac{1}{[u]}\frac{\partial[q]}{\partial y} - k^2\right) - 1 \,. \tag{35}$$

To simplify n^2 even further, note from (19) that

$$\frac{\partial[q]}{\partial y} = \beta - \frac{\partial^2 [u]}{\partial y^2} - \frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\rho_S \frac{f_0^2}{N^2}\frac{\partial[u]}{\partial z}\right), \tag{36}$$

where $\beta \equiv \frac{df}{dy}$. When the meridional and vertical shears of $[u]$ are not too strong,

$$\frac{\partial[q]}{\partial y} \cong \beta \geq 0 \,. \tag{37}$$

Using (37), we finally obtain

$$n^2 \cong \frac{4N^2 H_0^2}{f_0^2}\left(\frac{\beta}{[u]} - k^2\right) - 1 \,. \tag{38}$$

From (38), we see the following:

- To have vertical propagation ($n^2 > 0$), we need $\beta / [u] > 0$. Because $\beta > 0$, $[u]$ must be positive (westerly). Stationary Rossby waves cannot exist in easterlies, simply because they propagate westward relative to the air, so that easterlies cannot hold them in one place, as discussed in Chapter 8. Recall that the summer hemisphere stratosphere is dominated by easterlies, while the winter hemisphere stratosphere is dominated by westerlies. Note, however, that large positive $[u]$ also makes $n^2 < 0$. Stationary waves cannot propagate through very strong westerlies, because they would be swept downstream. Fig. 4, from Charney and Drazin (1961), shows the vertical distribution of n^2 for summer and winter, averaged over the Northern Hemisphere middle latitudes, for stationary waves with three different wavelengths.

- Even when $\beta / [u] > 0$, for a given $[u]$ waves with large k (i.e., sufficiently short wavelength) cannot propagate. Short waves are, therefore, "trapped" near their excitation levels. Since $[u]$ has a maximum near the tropopause in middle latitudes, many short waves are trapped in the troposphere, even in winter. Only longer waves can propagate to great heights. This suggests that long waves will dominate in the stratosphere and mesosphere even more than they do in the troposphere.
- A level where $[u] = 0$ is called a "critical level" for stationary waves. Suppose that $[u] > 0$ below a critical level, and $[u] < 0$ above. Then, for waves excited at the lower boundary (e.g., by flow over topography), upward propagation is completely blocked at the critical level.

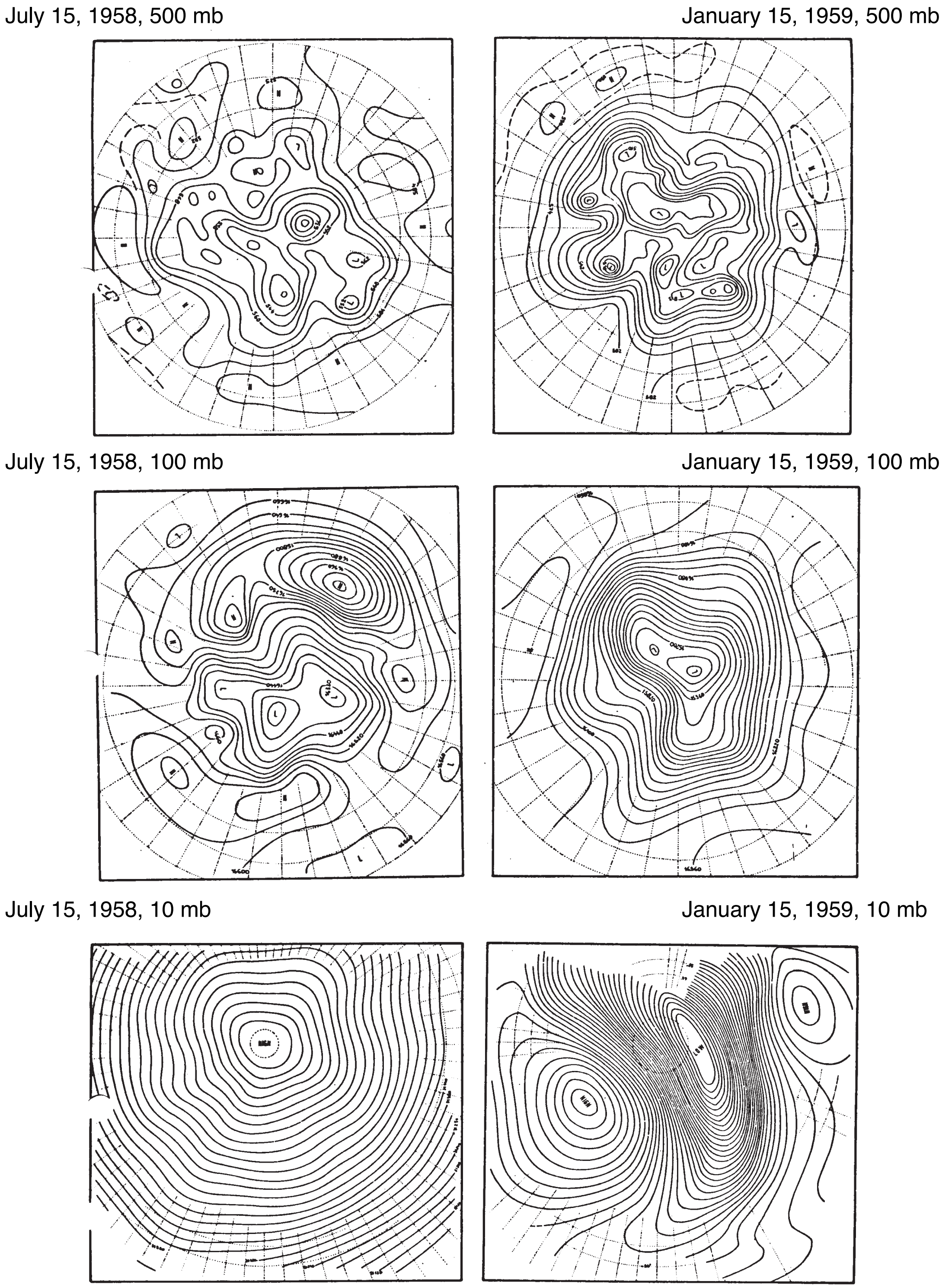

Figure 5: These Northern Hemisphere data were collected during the International Geophysical Year. Geopotential heights for July 15, 1958 are shown on the left, and those for January 15, 1959 are shown on the right. The levels plotted are 500 mb, 100 mb, and 10 mb. From Charney (1973).

Fig. 5 provides evidence that the theory is correct. It shows the geopotential height fields at 500 mb, 100 mb, and 10 mb, for Northern Hemisphere summer and winter. In winter, planetary waves clearly propagate upward to the 10 mb level, while in summer they do not. Note that the apparent horizontal scale of the dominant eddies increases upward, in winter. This is consistent with the theory, which predicts that the shorter modes are trapped at lower levels while longer modes can continue to propagate upward to great heights.

Waves can also be trapped at critical latitudes where $[u]=0$. We could therefore define critical surfaces in the *y*-*z* plane.

If we allowed $c \neq 0$, we would find that the critical surfaces are those for which $[u]-c=0$.

Matsuno (1970) used for the Northern Hemisphere winter to compute $\frac{\partial[q]}{\partial\varphi}$, the index of refraction, and the energy flow in the latitude-height plane for zonal wave number 1. His results

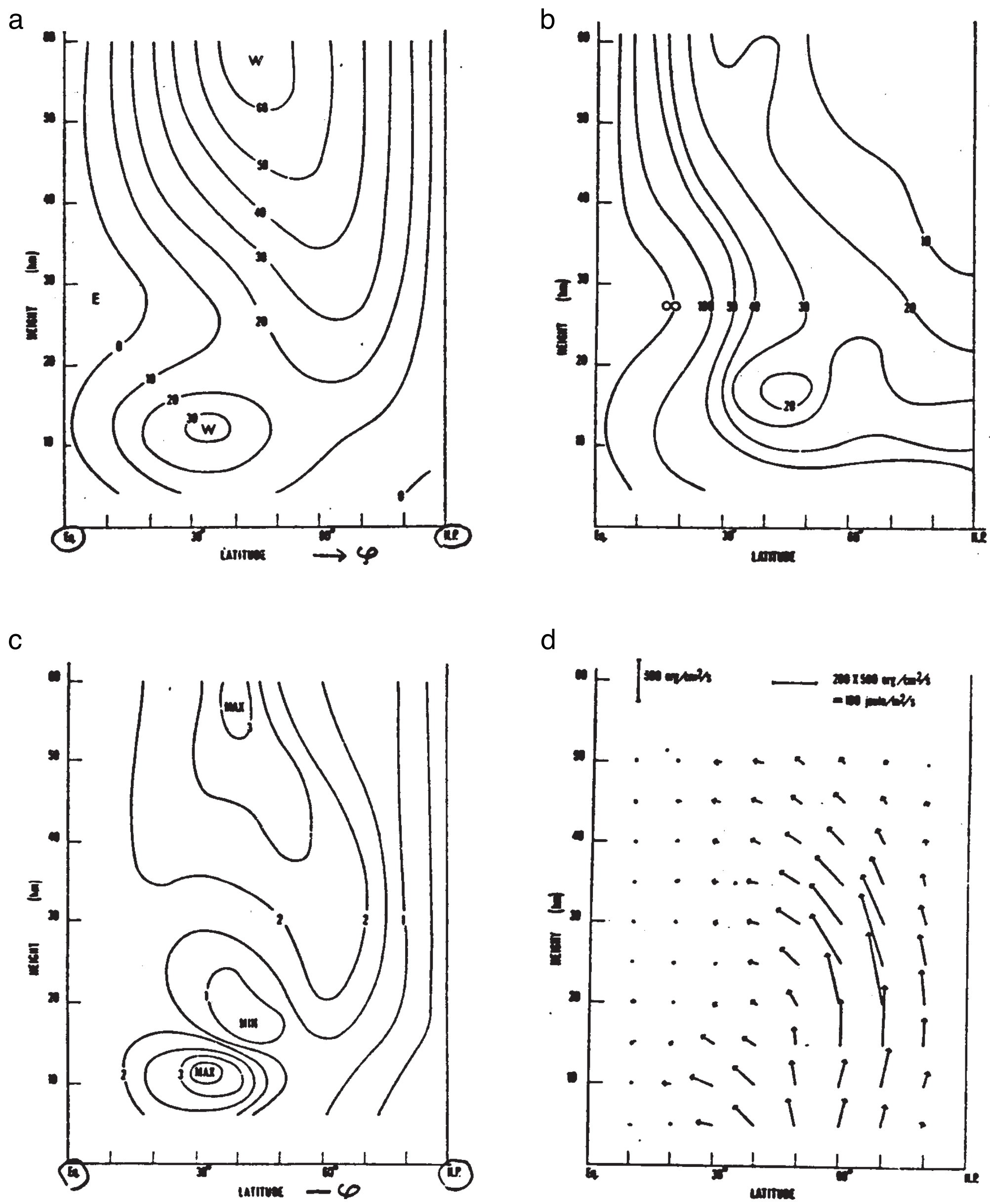

Figure 6: a) An idealized basic state zonal wind distribution (in m s-1) for the Northern Hemisphere winter. b) The refractive index square n^2 , for the $k = 0$ wave. c) The latitudinal gradient of the potential vorticity, $\partial[q]/\partial\varphi$, expressed as a multiple of the Earth's rotation rate. d) Computed distribution of energy flow in the meridional plane associated with zonal wave number 1. From Matsuno (1970).

are shown in Fig. 6. The upward-propagating waves are directed equatorward by the variations of the index of refraction.

Vertical and meridional fluxes due to planetary waves

Now we investigate under what conditions planetary waves can transport energy and momentum. The quasi-geostrophic form of the thermodynamic energy equation is (e.g., Holton, 1992)

$$\left(\frac{\partial}{\partial t}+\mathbf{V}_g\cdot\nabla\right)\frac{\partial\phi}{\partial p}+S_p\omega=0\ .$$

(39)

Here $\mathbf{V}_g$ is the geostrophic wind, which is important for the following discussion, and ∇ is ∇_p. Eq. (39) can be written as

$$\left(\frac{\partial}{\partial t}+\mathbf{V}_g\cdot\nabla\right)\psi_z+\frac{N^2}{f_0}w=0\,,$$

(40)

where w is defined by $-\omega/\left(\rho_S g\right)$. Here $\psi_z\equiv\partial\psi/\partial z$, and z is the "log-*p*" coordinate defined by (18). Linearizing (40) gives

$$\left(\frac{\partial}{\partial t}+[u]\frac{\partial}{\partial x}\right)\psi_z^*-v^*\frac{\partial[u]}{\partial z}+\frac{N^2}{f_0}w^*=0\ .$$

(41)

Here we have used the thermal wind equation. Multiplying (41) by ψ_z^*, we obtain a form of the "temperature variance equation:"

$$\left(\frac{\partial}{\partial t}+[u]\frac{\partial}{\partial x}\right)\left\{\frac{1}{2}\left(\psi_z^*\right)^2\right\}-v^*\psi_z^*\frac{\partial[u]}{\partial z}+\frac{N^2}{f_0}w^*\psi_z^*=0\ .$$

(42)

Note the two gradient production terms.

Take the zonal mean of (42), so that the $[u]\frac{\partial}{\partial x}$ term drops out. Rearrange to isolate the meridional energy flux by itself on the left-hand side:

$$\left[v^*\psi_z^*\right]\frac{\partial[u]}{\partial z}=\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\psi_z^*\right)^2\right]+N^2\left[\frac{w^*\psi_z^*}{f_0}\right].$$

(43)

Note that $\left[w^* \psi_z^*\right] / f_0 > 0$ implies an upward temperature flux, in either hemisphere. Also $\left[v^* \psi_z^*\right] > 0$ implies a poleward temperature flux, in either hemisphere.

First, consider a baroclinically amplifying wave, for which $\frac{\partial}{\partial t}\left[\left(\psi_z^*\right)^2\right] > 0$ and the wave temperature flux is upward. From (43), we see that *a baroclinically amplifying wave produces a poleward temperature flux (in either hemisphere) when* $\frac{\partial [u]}{\partial z} > 0$*, i.e., when the temperature is decreasing towards the pole*. Such a temperature flux is downgradient, so the gradient-production term is positive.

Next, consider a *neutral wave* of the form $e^{ik(x-ct)}$, for which $\frac{\partial}{\partial t} = -c\frac{\partial}{\partial x}$, where c is real. Multiply (41) by ψ^* and take the zonal mean, to obtain

$$\left([u]-c\right)\left[v^* \psi_z^*\right] = N^2 \left[\frac{w^* \psi^*}{f_0}\right]. \tag{44}$$

Note that $\left[w^* \psi^*\right] / f_0 > 0$ means an upward propagation of wave energy in either hemisphere. Recall also that $[u] - c > 0$ is needed in order for the wave to propagate. It follows that *an upward-propagating neutral wave transports energy poleward.* Such a wave might be forced, for example, by flow over mountains.

In summary, poleward energy transport is produced by either a baroclinically amplifying wave with $\frac{\partial [u]}{\partial z} > 0$ or a neutral wave that propagates upward.

Applying the eddy PV equation (23) to a neutral wave gives

$$\left([u]-c\right)\frac{\partial q_*}{\partial x} + v^* \frac{\partial [q]}{\partial y} = 0\,. \tag{45}$$

Multiply (45) by ψ^* and take the zonal mean to show that

$$\left[v^* q^*\right] = 0 \text{ except where } [u] = c \tag{46}$$

(a critical line). This very important result shows that neutral waves produce no potential vorticity flux except at a critical line. It follows from (16) that neutral waves do not affect $[q]$ except at a critical level or critical latitude. *This is a non-interaction theorem for planetary waves*, analogous to the non-interaction theorem for gravity waves obtained by Eliassen and Palm (1961).

From (17), $\left[v^* q^*\right] = 0$ means

$$-\frac{\partial\left[u^* v^*\right]}{\partial y} + \frac{f_0^2}{\rho_S}\frac{\partial}{\partial z}\left(\frac{\rho_S}{N^2}\left[v^* \psi_z^*\right]\right) = 0 \,. \tag{47}$$

As mentioned earlier, the expression equal to zero in (47) is the divergence of the Eliassen-Palm flux. Vertically integrate (47) through the depth of the atmosphere to obtain

$$-\int_0^{p_S} \frac{\partial}{\partial y}\left[u^* v^*\right]\frac{dp}{g} = f_0^2 \frac{\rho_S}{N^2}\left[v^* \psi_z^*\right]_S \tag{48}$$

for the neutral waves. The left-hand side represents the vertically integrated convergence of meridional momentum flux, and the right-hand side represents the near-surface value of the eddy meridional energy flux. Recall from (44) that an upward propagating neutral wave produces a poleward energy flux, i.e. $\left[v^* \psi_z^*\right] > 0$. It follows from (48) that

$$-\int_0^{p_S} \frac{\partial}{\partial y}\left[u^* v^*\right]\frac{dp}{g} > 0 \,. \tag{49}$$

This means that the vertically integrated meridional momentum flux convergence tends to accelerate the vertically integrated $[u]$*. In other words, the eddies feed the jet!* This is consistent with the observation that $KE \rightarrow KZ$. If the waves are also transporting temperature poleward, they will tend to reduce the meridional temperature gradient and so tend to reduce the strength of the westerlies. The momentum flux and heat flux thus have opposing effects on the mean flow.

An upward propagating neutral wave in westerly shear tends to produce a downward momentum flux at the Earth's surface. To see this, consider the angular momentum equation,

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial x}(uM) + \frac{\partial}{\partial y}(vM) + \frac{1}{\rho_S}\frac{\partial}{\partial z}(\rho_S w M) = -\frac{\partial \phi}{\partial \lambda} \,. \tag{50}$$

We have neglected the metric term and assumed no friction above the boundary layer. Taking the zonal mean of (50) gives

$$\frac{\partial [M]}{\partial t}+\frac{\partial}{\partial y}\left[v^{*}M^{*}\right]+\frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\rho_S\left[w^{*}M^{*}\right]\right)=0 \; . \tag{51}$$

Here advection of $\left[M\right]$ by $\left[v\right]$ and $\left[w\right]$ is neglected; this is justified in the midlatitude winter. To the extent that $\left[v\right]$ is geostrophic, it vanishes anyway. Now assume that $\frac{\partial [M]}{\partial t}=0$, which is consistent with $\frac{\partial [q]}{\partial t}=0$. This leads to

$$\frac{\partial}{\partial y}\left[v^{*}M^{*}\right]=-\frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\rho_S\left[w^{*}M^{*}\right]\right). \tag{52}$$

Integrating (52) vertically with respect to mass, and employing (49), we find that

$$\int_0^{\infty}\frac{\partial}{\partial z}\left(\rho_S\left[w^{*}M^{*}\right]\right)dz>0 \; . \tag{53}$$

We know that $\rho_S\left[w^{*}M^{*}\right]$ must vanish at great height, so we conclude that

$$\rho_S\left[w^{*}M^{*}\right]_S<0 \; . \tag{54}$$

This shows that, near the lower boundary, friction and/or mountain torque must carry angular momentum into the Earth's surface, in the presence of an upward propagating planetary wave. An alternative interpretation is that frictional and/or mountain torque, in a belt of westerlies where (54) is satisfied, will produce an upward-propagating planetary wave that transports energy poleward.

Compare (49) and (54). The meridional momentum flux accelerates the westerlies, while the vertical momentum flux decelerates them.

Eliassen-Palm Theorem-Reprise

Previously we discussed non-interaction theorems for pure gravity waves and for quasi-geostrophic waves on a β -plane. It was discovered during the 1970's that non-interaction theorems can be derived for very general balanced flows. The following discussion provides an example. The discussion is based on Andrews et al. (1987).

The zonally averaged momentum equations in spherical coordinates can be written as

$$\frac{\partial[M]}{\partial t}+[v]\frac{\partial[M]}{\partial\varphi}+[w]\frac{\partial[M]}{\partial z}-[F_x]a\cos^2\varphi=\frac{-1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\left[v^*M^*\right]\cos\varphi\right)-\frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\rho_S\left[w^*M^*\right]\right) \tag{55}$$

$$\frac{\partial[v]}{\partial t}+\frac{1}{a}[v]\frac{\partial[v]}{\partial\varphi}+[w]\frac{\partial[v]}{\partial z}+[u]\left(f+\frac{[u]\tan\varphi}{a}\right)+\frac{1}{a}\frac{\partial[\phi]}{\partial\varphi}-\left[F_y\right]$$
$$=\frac{-1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\left[\left(v^*\right)^2\right]\cos\varphi\right)-\frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\rho_S\left[v^*w^*\right]\right)-\frac{\left[\left(u^*\right)^2\right]\tan\varphi}{a} \quad . \tag{56}$$

Here $z\equiv -H\log\left(p/p_0\right)$ is the vertical coordinate, and $w\equiv Dz/Dt$. The scale height H is $\frac{RT_0}{g}$, where T_0 is a constant. The zonally averaged thermodynamic energy equation is

$$\frac{\partial[\theta]}{\partial t}+\frac{[v]}{a}\frac{\partial[\theta]}{\partial\varphi}+[w]\frac{\partial[\theta]}{\partial z}-[Q]=\frac{-1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\left[\theta^*v^*\right]\cos\varphi\right)-\frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\rho_S\left[w^*\theta^*\right]\right) \tag{57}$$

Here Q represents a heating process. Finally, we will need the zonally averaged continuity equation,

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\rho_S[v]\cos\varphi\right)+\frac{\partial}{\partial z}\left(\rho_S[w]\right)=0\,, \tag{58}$$

and hydrostatics:

$$\frac{\partial[\phi]}{\partial z}-\frac{R[\theta]}{H}e^{-\frac{\kappa z}{H}}=0\,. \tag{59}$$

In the above equations, $\rho_S(z)\equiv\rho_0 e^{-\frac{z}{H}}$, where ρ_0 is a constant. We assume that (56) can be approximated by gradient wind balance, so that it simplifies drastically to

$$[u]\left(f+[u]\frac{\tan\varphi}{a}\right)+\frac{1}{a}\frac{\partial[\phi]}{\partial\varphi}=0\ . \tag{60}$$

This assumed balance is essential to the following argument.

We *define* a "residual mean meridional circulation" $(0,V,W)$ by

$$V\equiv[v]-\frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\frac{\rho_S\left[v^*\theta^*\right]}{\partial[\theta]/\partial z}\right), \tag{61}$$

$$W\equiv[w]+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\left[v^*\theta^*\right]\cos\varphi}{\partial[\theta]/\partial z}\right). \tag{62}$$

In the absence of eddies, $V=[v]$ and $W=[w]$. Substitution shows that V and W satisfy a continuity equation analogous to (58). Use of (61) and (62) to eliminate $[v]$ and $[w]$ in favor of V and W allows us to rewrite (55) and (57) as:

$$\rho_S\left\{\frac{\partial[M]}{\partial t}+V\frac{\partial[M]}{\partial\varphi}+W\frac{\partial[u]}{\partial z}-a\cos\varphi\left[F_x\right]\right\}=\nabla\cdot\mathbf{EPF}, \tag{63}$$

and

$$\frac{\partial[\theta]}{\partial t}+\frac{V}{a}\frac{\partial[\theta]}{\partial\varphi}+W\frac{\partial[\theta]}{\partial z}-[Q]=\frac{-1}{\rho_S}\frac{\partial}{\partial z}\left\{\left(\frac{\partial[\theta]}{\partial z}\right)^{-1}\rho_S\left[v^*\theta^*\right]\frac{1}{a}\frac{\partial[\theta]}{\partial\varphi}+\rho_S\left[w^*\theta^*\right]\right\}, \tag{64}$$

respectively, where

$$\mathbf{EPF}\equiv\left[0,(EPF)_{\varphi},(EPF)_{z}\right] \tag{65}$$

is the "*Eliassen-Palm flux*," whose components are

$$(EPF)_\varphi \equiv \rho_S \left\{ \frac{\partial [M]}{\partial z} \cdot \frac{[v^* \theta^*]}{\partial [\theta] / \partial z} - [M^* v^*] \right\},$$

(66)

and

$$(EPF)_z \equiv -\rho_S \left(\left\{ \frac{1}{a} \frac{\partial [M]}{\partial \varphi} \right\} \frac{[v^* \theta^*]}{\partial [\theta] / \partial z} + [M^* w^*] \right).$$

(67)

In (66), the $[M^* v^*]$ term is dominant, and in (67) the $[v^* \theta^*]$ term is dominant. Compare (66) and (67) with (17) and (47). When the **EPF** points upward, the meridional energy flux is in control. When it points in the meridional direction, the meridional flux of zonal momentum is in control. From (63) we see that a positive Eliassen-Palm flux divergence tends to increase $[M]$.

The preceding derivation appears to be nothing more than an algebraic shuffle. We wrote down (61) and (62) without any explanation or motivation. What is the point of all this? The point is that *for steady linear waves with* $F_x = F_y = 0$ *and* $Q = 0$*, it can be shown that*

$$\nabla \cdot (\mathbf{EPF}) = 0 .$$

(68)

Recall that this follows essentially from $[v^* q^*] = 0$. It turns out that the eddy term of (64) is zero under the same conditions, i.e.,

$$\frac{1}{\rho_S} \frac{\partial}{\partial z} \left\{ \left(\frac{\partial [\theta]}{\partial z} \right)^{-1} \rho_S [v^* \theta^*] \frac{1}{a} \frac{\partial [\theta]}{\partial \varphi} + \rho_S [w^* \theta^*] \right\} = 0 .$$

(69)

This follows essentially from our assumptions that: 1) $[u]$ does not change, and 2) thermal wind balance is maintained.

For the case of steady, linear waves, in the absence of friction and heating, our system of equations reduces to

$$
\begin{aligned}
&\frac{\partial[M]}{\partial t}+V[M]+W\frac{\partial[M]}{\partial z}=0,\\
&[u]\left(f+[u]\frac{\tan\varphi}{a}\right)+\frac{1}{a}\frac{\partial[\phi]}{\partial\varphi}=0,\\
&\frac{\partial[\theta]}{\partial t}+\frac{V}{a}\frac{\partial[\theta]}{\partial\varphi}+W\frac{\partial[\theta]}{\partial z}=0,\\
&\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\rho_S V\cos\varphi\right)+\frac{\partial}{\partial z}\rho_S W=0,\\
&\frac{\partial[\phi]}{\partial z}-H^{-1}R[\theta]e^{\frac{-\kappa z}{H}}=0\,.
\end{aligned}
\tag{70}
$$

This system has the following steady solution:

$$
\begin{aligned}
&\frac{\partial[M]}{\partial t}=0\;,\; u \quad \text{in gradient-wind balance,}\\
&V=0,\; W=0,\\
&\frac{\partial[\theta]}{\partial t}=0\,,\;[\theta]\quad \text{specified from the past history or radiative-convective equilibrium.}
\end{aligned}
\tag{71}
$$

From the definitions of V and W, we can find the mean meridional circulation implied by $V=0$ and $W=0$:

$$
\rho_S[v]=\frac{\partial}{\partial z}\left(\rho_S\frac{\left[v^*\theta^*\right]}{\partial[\theta]/\partial z}\right),
\tag{72}
$$

$$
[w]=-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\left[v^*\theta^*\right]\cos\varphi}{\partial[\theta]/\partial z}\right).
\tag{73}
$$

Consider two scenarios. First, suppose we have a solution with no eddies at all. "No eddies" certainly qualify as "steady linear waves." The above argument therefore applies, so we can get $[u]$ and $[\theta]$, and from (72) and (73) we conclude that the mean meridional circulation (MMC) will vanish.

In the second scenario, add steady linear eddies, *so that* $\nabla \cdot (\mathbf{EPF})$ *continues to be zero*. Then exactly the same $[u]$ and $[\theta]$ will satisfy the equations! Of course, V and W will be different, i.e., the MMC will be different. In fact, the MMC will have to be whatever it takes to ensure that $V = W = 0$, i.e., to satisfy (72) and (73). This MMC is said to be "induced by" the eddies. The system produces this MMC in order to prevent the eddies from disrupting the thermal wind balance. Perhaps a better way to say this is that the processes that act to maintain thermal wind balance (i.e., geostrophic and hydrostatic adjustment) accomplish this feat by using the "wave-induced" MMC as a tool.

The interpretation of this amazing result is that if you try to modify $[u]$ and $[\theta]$ by applying eddy forcing such that $\nabla \cdot (\mathbf{EPF}) = 0$ (no potential vorticity flux), you will be disappointed! All that will happen is that the MMC will change, in such a way that V and W continue to be zero. In effect, the eddies will induce an MMC that exactly cancels the direct effects of the eddies on $[u]$ and $[\theta]$.

When the eddies are *not* steady, the residual circulation is different from zero, and $[u]$ and $[\theta]$ are modified by the combined effects of the eddies and/or the eddy-induced MMC. Cancellation of the effects of the eddies and the MMC still tends to occur, but the cancellation is incomplete.

Edmon et al. (1980) discussed the quasi-geostrophic form of the non-interaction theorem, and used it to analyze the data of Oort and Rasmussen (1971). As a reminder [see (47)], the meridional component of the quasi-geostrophic **EPF** is

$$(EPF)_\varphi = -a\cos(\varphi)\left[\overline{u^* v^*}\right], \tag{74}$$

and the vertical component is

$$(EPF)_p = fa\cos(\varphi)\frac{\left[\overline{v^*\theta^*}\right]}{\partial\bar{\theta}/\partial p}. \tag{75}$$

[Note: Compare (74) and (75) with (66) and (67), respectively.] Fig. 10 shows the contribution of the transient eddies to the Eliassen-Palm fluxes. First consider the winter results, shown in the upper panel. Near the surface in middle latitudes, we see arrows pointing strongly upward, indicating an intense poleward potential temperature flux. Near the tropopause, the arrows curve over and become horizontal, pointing towards the tropics. This indicates a strong poleward eddy a Winter

b Summer

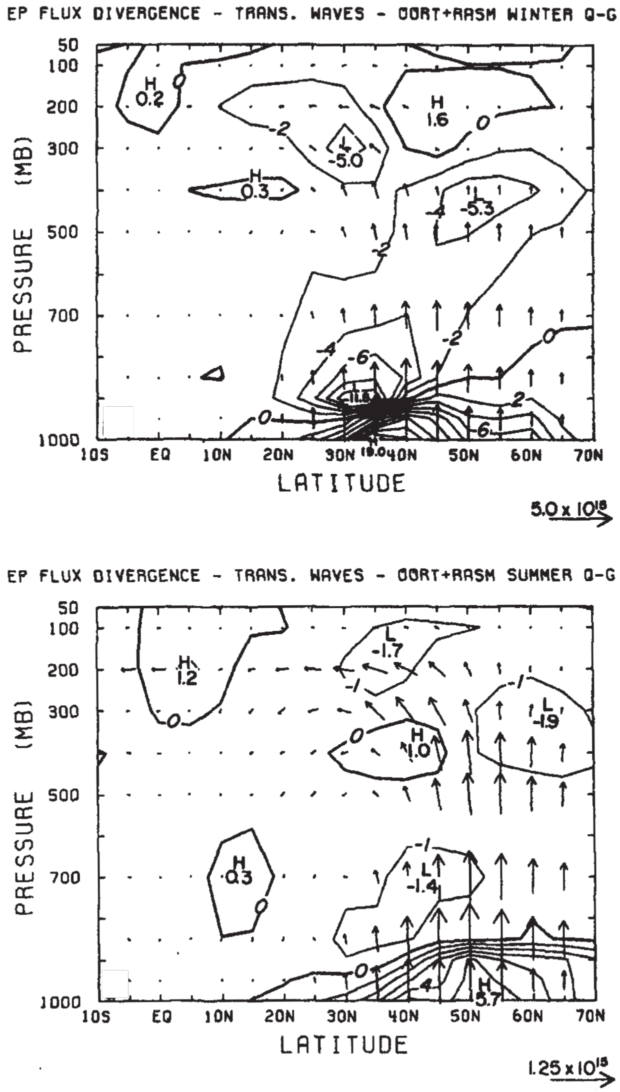

Figure 10: Contribution of *transient* eddies to the seasonally averaged Eliassen-Palm cross sections for the troposphere: (a) 5-year average from Oort and Rasmusson (1971) for winter; (b) the same for summer. The contour interval is 20 x 10^{15} m^3 for (a), and 1 x 10^{15} m^3 for (b). The horizontal arrow scale for the horizontal component in units of m^3 is indicated at bottom right; note that it is different from diagram to diagram. A vertical arrow of the same length represents the vertical component, in m^3 kPa, equal to that for the horizontal arrow multiplied by 80.4 kpa. From Edmon et al. (1980).

momentum flux. The contours in the figure show the divergence of the Eliassen-Palm flux. Keep in mind that $\nabla \cdot \mathbf{EPF} > 0$ means $\frac{\partial [M]}{\partial t} > 0$, i.e., a positive **EPF** divergence favors westerly acceleration. The negative divergence (i.e., convergence) near 200 mb at about 30° N indicates that the net effect of the eddies is to decelerate the jet. In fact, the westerlies are being a Winter

b Summer

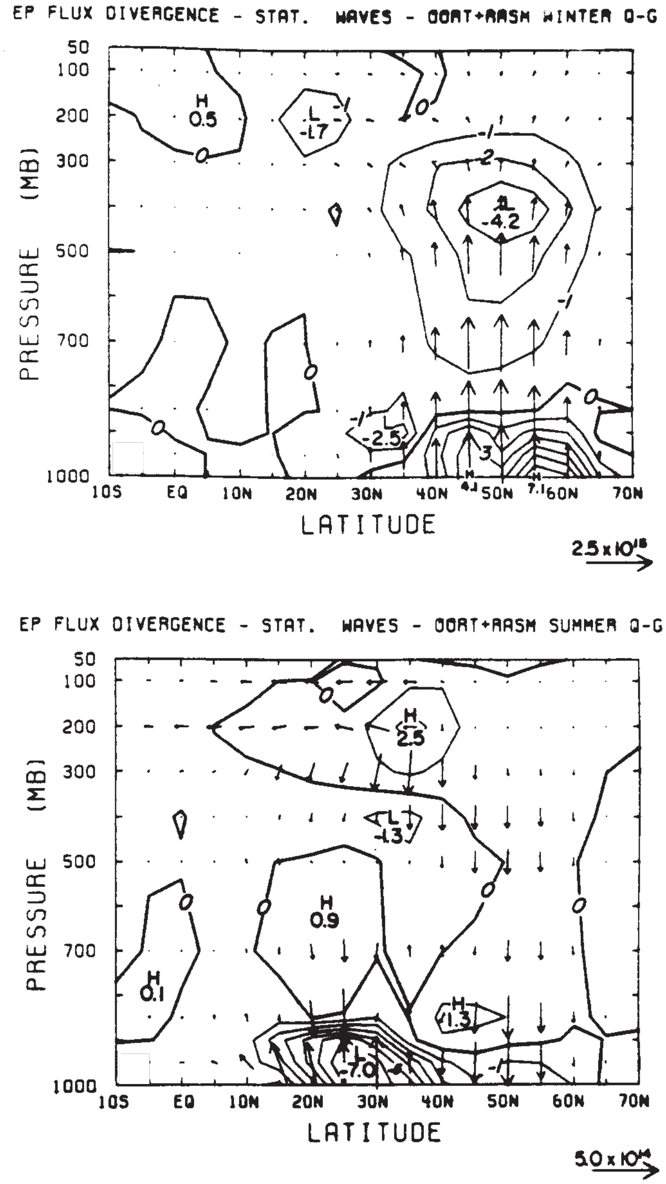

Figure 11: Contribution of *stationary* eddies to the seasonally averaged Eliassen-Palm cross sections for the troposphere: (a) 5-year average from Oort and Rasmusson (1971) for winter; (b) the same, respectively, for summer. The contour interval is 1 x 10^{15} m^3 for both panels. The horizontal arrow scale in units of m^3 is indicated at bottom right. From Edmon et al. (1980).

decelerated throughout middle latitudes, except near the surface. Note that this **EPF** convergence results mainly from the upward decrease of the upward component of the flux, i.e., it is mainly due to the energy flux.

The results for summer are quite similar, except that the action is generally weaker, and shifted poleward.

Fig. 11 shows the corresponding results for the stationary waves. In winter, the "strong"

a Winter

b Summer

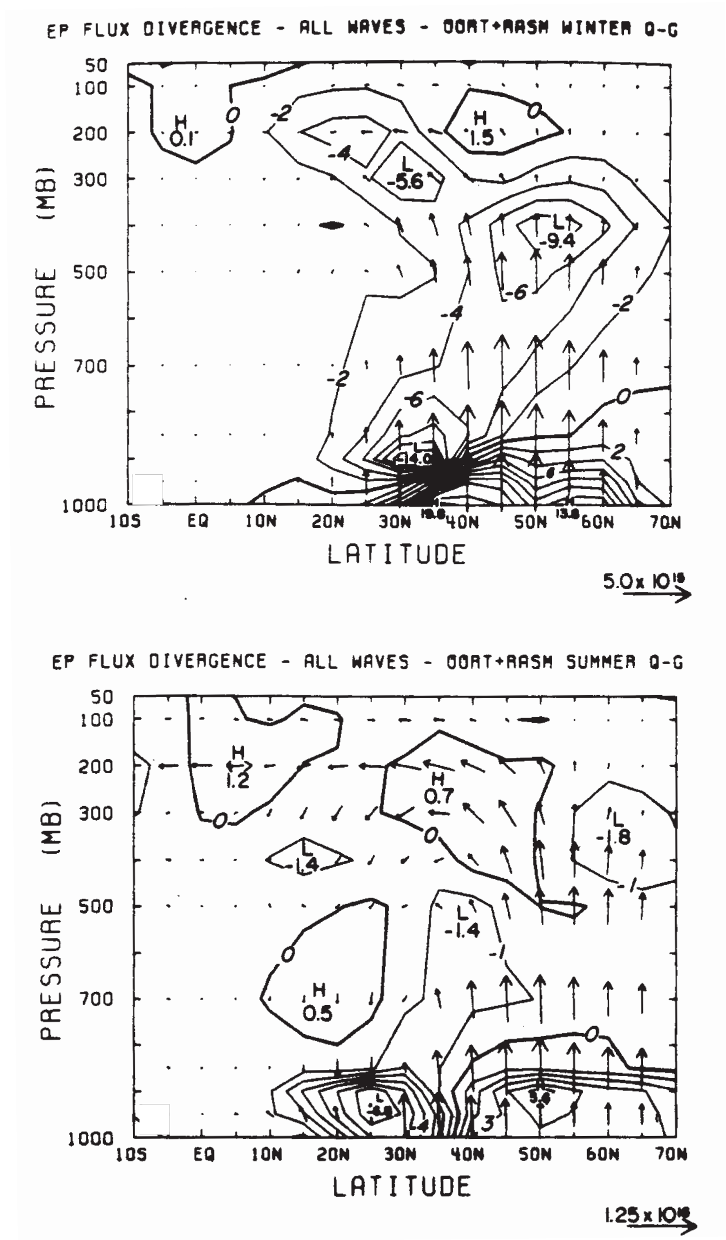

Figure 12: Total (transient plus stationary) Eliassen-Palm cross sections for the troposphere: (a) 5-year average from Oort and Rasmusson (1971) for winter; (b) the same, respectively, for summer. The contour interval is 2 x 10^{15} m^3 for (a), and 1 x 10^{15} m^3 for (b). The horizontal arrow scale in units of m^3 is indicated at bottom right. From Edmon et al. (1980).

arrows are pointing nearly straight up everywhere, indicating that the poleward eddy potential temperature flux is playing a much more important role than the eddy momentum flux. The westerlies are decelerated aloft, near 50° N, but they are accelerated near the surface. In summer the arrows point downward. The eddy momentum flux is important near the summer tropopause,

a Winter

b Summer

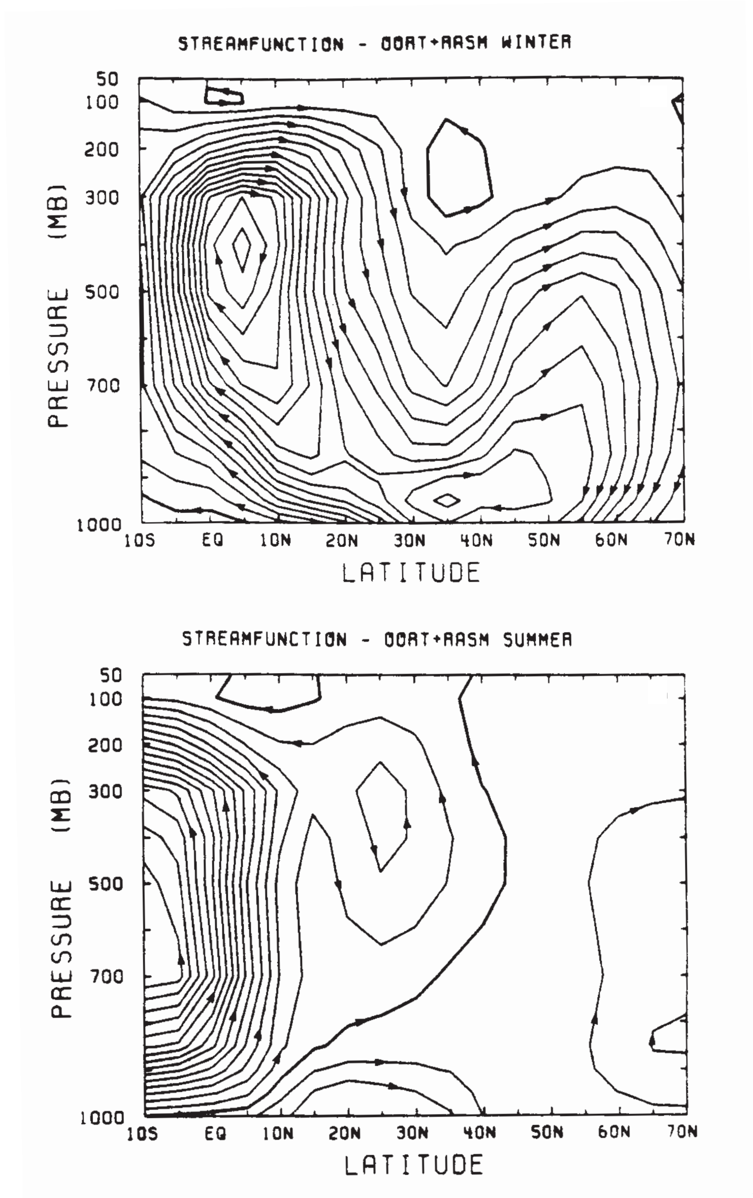

Figure 13: The stream function of the seasonally averaged residual meridional circulations. (a) 5-year average from Oort and Rasmusson (1971) for winter, and (b) the same for summer. The contour interval is 7.5 x 10^{16} m^2 s Pa. From Edmon et al. (1980).

but again the eddy potential temperature flux is more important overall. The westerlies are strongly decelerated near the surface in the subtropics, and they are actually accelerated at 200 mb near 35° N.

Fig. 12 shows the combined effects of the transient and stationary eddies. Note that the transient eddies dominate, in both seasons. Finally, Fig. 13 shows the residual circulation,

(V,W), for summer and winter. In winter, the residual circulation looks suspiciously like a giant Hadley Cell, extending from the tropics to the poles. This is reminiscent of the mean meridional circulation as seen in isentropic coordinates. In summer, we seem to see the northern edge of a Hadley Cell extending into the Southern Hemisphere. Clearly, we can regard the residual circulation as a response to heating.

The Eliassen-Palm theorem in isentropic coordinates

The Eliassen-Palm theorem is somewhat simpler and easier to interpret when we use isentropic coordinates. Following Andrews (1983), we begin with the flux form of the angular momentum equation in isentropic coordinates, neglecting friction:

$$\frac{\partial}{\partial t}(\rho_\theta M)+\frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\lambda}(\rho_\theta uM)+\frac{\partial}{\partial\varphi}(\rho_\theta vM\cos\varphi)\right]=-\rho_\theta\frac{\partial s}{\partial\lambda}-\frac{\partial}{\partial\theta}\left(\rho_\theta\dot{\theta}M\right). \tag{76}$$

Here ρ_θ is the pseudo-density:

$$\rho_\theta=-\frac{1}{g}\frac{\partial p}{\partial\theta}, \tag{77}$$

and

$$\begin{aligned} s &\equiv c_pT+gz \\ &=\Pi\theta+gz, \end{aligned} \tag{78}$$

where

$$\Pi\equiv c_p\left(\frac{p}{p_0}\right)^\kappa \tag{79}$$

is the Exner function. Using the hydrostatic equation in isentropic coordinates, i.e.,

$$\frac{\partial s}{\partial\theta}=\Pi, \tag{80}$$

we obtain

$$\begin{aligned}\rho_\theta \frac{\partial s}{\partial \lambda} &= -\frac{1}{g}\frac{\partial p \partial s}{\partial \theta \partial \lambda} \\ &= -\frac{1}{g}\frac{\partial}{\partial \theta}\left(p\frac{\partial s}{\partial \lambda}\right) + \frac{p}{g}\frac{\partial}{\partial \lambda}\left(\frac{\partial s}{\partial \theta}\right) \\ &= -\frac{1}{g}\frac{\partial}{\partial \theta}\left\{p\frac{\partial}{\partial \lambda}(\Pi\theta + gz)\right\} + \frac{p}{g}\frac{\partial \Pi}{\partial \lambda} \\ &= -\frac{1}{g}\frac{\partial}{\partial \theta}\left(\theta p\frac{\partial \Pi}{\partial \lambda} + pg\frac{\partial z}{\partial \lambda}\right) + \frac{p}{g}\frac{\partial \Pi}{\partial \lambda} \\ &= -\frac{1}{g}\frac{\partial}{\partial \theta}\left(\theta p\frac{\partial \Pi}{\partial \lambda}\right) + \frac{p}{g}\frac{\partial \Pi}{\partial \lambda} - \frac{\partial}{\partial \theta}\left(p\frac{\partial z}{\partial \lambda}\right) \\ &= -\frac{1}{g}\frac{\partial}{\partial \theta}\left(p\frac{\partial \Pi}{\partial \lambda}\right) - \frac{\partial}{\partial \theta}\left(p\frac{\partial z}{\partial \lambda}\right) \quad .\end{aligned} \tag{81}$$

Substituting back, the angular momentum equation becomes

$$\frac{\partial}{\partial t}(\rho_\theta M) + \frac{1}{a\cos\varphi}\left\{\frac{\partial}{\partial \lambda}(\rho_\theta u M) + \frac{\partial}{\partial \varphi}(\rho_\theta v M \cos\varphi)\right\} = \left\{\frac{\partial}{\partial \theta}\left(\frac{p}{g}\frac{\partial \Pi}{\partial \lambda}\right) + \frac{\partial}{\partial \theta}\left(p\frac{\partial z}{\partial \lambda}\right)\right\} - \frac{\partial}{\partial \theta}(\rho_\theta \dot{\theta} M) . \tag{82}$$

When we take the zonal mean of (82), the first term on the right-hand side vanishes (Why?) and we are left with

$$\frac{\partial}{\partial t}[\rho_\theta M] + \frac{1}{a\cos\varphi}\left\{\frac{\partial}{\partial \varphi}([\rho_\theta v M]\cos\varphi)\right\} = \frac{\partial}{\partial \theta}\left[p^*\frac{\partial z^*}{\partial \lambda}\right] - \frac{\partial}{\partial \theta}\left[\rho_\theta \dot{\theta} M\right]. \tag{83}$$

The pressure-gradient term of (83) has a very simple and interesting form: It is proportional to the change with θ of the zonal mean of the product of the pressure and the slope of the height of the isentropic surface. The expression $\left[p^*\frac{\partial z^*}{\partial \lambda}\right]$ can be interpreted as form drag on the isentropic surface, analogous to the form drag on mountains discussed earlier in the course. Here the "mountains" are upward bulges of the isentropic surfaces, associated with blobs of cold air at a given pressure level; the "valleys" are downward bulges of the isentropic surfaces, associated with blobs of warm air at a given pressure level. We can say that

$$\text{upward flux of zonal momentum due to the wave} = -\left[p^*\frac{\partial z^*}{\partial \lambda}\right]. \tag{84}$$

This shows that, from the perspective of isentropic coordinates, the upward flux of zonal momentum is associated with the pressure force, rather than with a covariance between the "vertical velocity" (which vanishes in isentropic coordinates in the absence of heating) and the zonal velocity. A layer of air confined between two isentropic surfaces will feel two momentum fluxes associated with the pressure force: one on its underside, and a second on its upper side. It is the difference between these two forces that tends to produce a net acceleration of the layer. That is why we see $\frac{\partial}{\partial\theta}\left[p^* \frac{\partial z^*}{\partial\lambda}\right]$ in (83). This form of the vertical momentum flux was discussed by Klemp and Lilly (1978).

The analogy between the form drag on a "wavy" isentropic surface and the form drag on topography is a very powerful aid to physical intuition. The form drag on a wavy isentropic surface is expected to be different from zero when the isentropic surface is moving relative to the mean flow, i.e., when $[u]-c \neq 0$, where c is the phase speed of the wave relative to the Earths' surface. This is analogous to the fact that a form drag on topography is expected when there is a low-level mean flow relative to the Earth's surface. A wavy isentropic surface moving to the east relative to the mean flow is expected to experience a form drag that pushes it back towards the west, i.e., that tries to slow it down relative to the mean flow. This corresponds to an upward flux of westerly momentum, because the air above the wavy surface is being pushed towards the east. Similarly, a wavy isentropic surface moving towards the west relative to the mean flow is expected to experience a form drag that pushes it back towards the west, and this corresponds to a downward flux of westerly momentum. *The sign of the (positive upward) wave momentum flux is therefore expected to be opposite to the sign of* $[u]-c$. Based on this argument, Kelvin waves are expected to produce an upward flux of westerly momentum, and Rossby waves are expected to produce a downward flux of westerly momentum.

Before completing our discussion of the Eliassen-Palm theorem in isentropic coordinates, it is useful to recall the form of the mechanical energy equation in isentropic coordinates, which was given earlier in the course and is repeated here for your convenience:

$$\left\{\frac{\partial}{\partial t}(\rho_\theta K)\right\}_\theta + \nabla_\theta \cdot \{\rho_\theta \mathbf{V}(K+\phi)\} + \frac{\partial}{\partial\theta}\{\rho_\theta \dot{\theta}(K+\phi)\} + \rho_\theta \alpha \nabla \cdot (\mathbf{F}\cdot\mathbf{V}) = -\frac{\partial}{\partial\theta}\left\{-z\left(\frac{\partial p}{\partial t}\right)_\theta\right\} - \rho_\theta \omega\alpha - \rho_\theta \delta \tag{85}$$

The first-term on the right-hand side of (85) represents the vertical transport of energy via "pressure-work." The zonally averaged upward flux of wave energy is, therefore, given by

$\left[-z^*\left(\frac{\partial p^*}{\partial t}\right)_\theta\right]$. Recall that for a neutral wave propagating zonally and vertically, with zonal phase velocity c, we can write $\frac{\partial}{\partial t}=\left(\frac{[u]-c}{a\cos\varphi}\right)\frac{\partial}{\partial\lambda}$. It follows that

$$\text{upward wave energy flux}=\left(\frac{[u]-c}{a\cos\varphi}\right)\left[p^*\left(\frac{\partial z^*}{\partial\lambda}\right)_\theta\right]. \tag{86}$$

Comparing (83) with (86), we conclude that

$$\text{upward wave energy flux}=-\left(\frac{[u]-c}{a\cos\varphi}\right)\text{ times the upward wave angular momentum flux.} \tag{87}$$

This shows that, for neutral waves propagating towards the east relative to the air, i.e., $-([u]-c)>0$, the momentum flux and the energy flux have the same sign, while for neutral waves propagating towards the west relative to the air, i.e., $-([u]-c)<0$, the momentum and energy fluxes have opposite signs. As we know, Rossby waves always propagate west relative to the air, so for Rossby waves the momentum flux is always opposite in direction to the energy flux.

Now we relate the preceding analysis to the Eliassen-Palm theorem, following Andrews (1983) and Andrews et al. (1987). Recall that

$$[\rho_\theta A]=[\rho_\theta][A]+\left[\rho_\theta^* A^*\right], \tag{88}$$

for an arbitrary variable A. Using (88) in the time-rate-of-change term of (83), we obtain

$$\frac{\partial}{\partial t}\left([\rho_\theta][M]\right)+\frac{1}{a\cos\varphi}\left\{\frac{\partial}{\partial\varphi}\left([\rho_\theta vM]\cos\varphi\right)\right\}=-\frac{\partial}{\partial t}\left[\rho_\theta^* M^*\right]+\frac{\partial}{\partial\theta}\left[p^*\frac{\partial z^*}{\partial\lambda}\right]-\frac{\partial}{\partial\theta}\left[\rho_\theta M\dot{\theta}\right]. \tag{89}$$

Here the "eddy part" of the time-rate-of-change term has been moved to the right-hand-side of the equals sign; this will be discussed later. We want to derive an "advective form" of (89), so we bring in the zonally averaged continuity equation in isentropic coordinates, which can be written as

$$\frac{\partial[\rho_\theta]}{\partial t}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([\rho_\theta v]\cos\varphi\right)=-\frac{\partial\left[\rho_\theta\dot{\theta}\right]}{\partial\theta}. \tag{90}$$

In order to obtain an "advective form," we subtract $[M]$ times (90) from (89), to obtain

$$\begin{aligned}&[\rho_\theta]\frac{\partial[M]}{\partial t}+\frac{1}{a\cos\varphi}\left\{\frac{\partial}{\partial\varphi}\left([\rho_\theta vM]\cos\varphi\right)\right\}-\frac{[M]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([\rho_\theta v]\cos\varphi\right)\\ &=-\frac{\partial\left[\rho_\theta^* M^*\right]}{\partial t}+\frac{\partial}{\partial\theta}\left[p^*\frac{\partial z^*}{\partial\lambda}\right]+[M]\frac{\partial\left[\rho_\theta\dot{\theta}\right]}{\partial\theta}-\frac{\partial\left[\rho_\theta M\dot{\theta}\right]}{\partial\theta}.\end{aligned} \tag{91}$$

We cannot yet combine terms to obtain an advective form. One more step is needed first. What we need to do is introduce a mass-weighted zonal mean, defined by

$$\left[\hat{A}\right]\equiv\frac{[\rho_\theta A]}{[\rho_\theta]}. \tag{92}$$

Using the definition (92), we can write

$$\begin{aligned}\rho_\theta A&=[\rho_\theta A]+(\rho_\theta A)^*\\ &=[\rho_\theta]\left[\hat{A}\right]+(\rho_\theta A)^*,\end{aligned} \tag{93}$$

and

$$\begin{aligned}[\rho_\theta AB]&=[\rho_\theta A][B]+\left[(\rho_\theta A)^* B^*\right]\\ &=[\rho_\theta]\left[\hat{A}\right][B]+\left[(\rho_\theta A)^* B^*\right],\end{aligned} \tag{94}$$

where B is a second arbitrary variable. As special cases of (94), we can write the zonally averaged meridional and vertical fluxes of B as

$$\begin{aligned}[\rho_\theta vB]&=[\rho_\theta v][B]+\left[(\rho_\theta v)^* B^*\right]\\ &=[\rho_\theta][\hat{v}][B]+\left[(\rho_\theta v)^* B^*\right],\end{aligned} \tag{95}$$

and

$$\begin{aligned}\left[\rho_\theta \dot{\theta} B\right] &= \left[\rho_\theta \dot{\theta}\right][B] + \left[\left(\rho_\theta \dot{\theta}\right)^* B^*\right] \\ &= [\rho_\theta]\left[\hat{\dot{\theta}}\right][B] + \left[\left(\rho_\theta \dot{\theta}\right)^* B^*\right].\end{aligned} \tag{96}$$

The "eddy meridional mass flux," $(\rho_\theta v)^*$, obviously has a zonal mean of zero. This means that it does not transport any mass on the average. It is a "mixing" or "diffusive" or "sloshing" mass flux. A similar comment applies to the eddy vertical mass flux, $\left(\rho_\theta \dot{\theta}\right)^*$.

Using (95) and (96), we can rewrite (91) as

$$\begin{aligned}&[\rho_\theta]\frac{\partial [M]}{\partial t} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{\left([\rho_\theta][\hat{v}][M] + \left[(\rho_\theta v)^* M^*\right]\right)\cos\varphi\right\} - \frac{[M]}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left([\rho_\theta][\hat{v}]\cos\varphi\right) \\ &= -\frac{\partial\left[\rho_\theta^* M^*\right]}{\partial t} + \frac{\partial}{\partial \theta}\left[p^* \frac{\partial z^*}{\partial \lambda}\right] + [M]\frac{\partial}{\partial \theta}\left([\rho_\theta]\left[\hat{\dot{\theta}}\right]\right) - \frac{\partial}{\partial \theta}\left\{[\rho_\theta]\left[\hat{\dot{\theta}}\right][M] + \left[\left(\rho_\theta \dot{\theta}\right)^* M^*\right]\right\}.\end{aligned} \tag{97}$$

The meridional and vertical derivatives can now be combined. We also divide by $[\rho_\theta]$, simplify, and rearrange, obtain

$$\begin{aligned}\frac{\partial [M]}{\partial t} + \frac{[\hat{v}]}{a}\frac{\partial [M]}{\partial \varphi} + \left[\hat{\dot{\theta}}\right]\frac{\partial [M]}{\partial \theta} &= -\frac{1}{[\rho_\theta]}\frac{\partial\left[\rho_\theta^* M^*\right]}{\partial t} + \frac{1}{[\rho_\theta]}\frac{\partial}{\partial \theta}\left[p^* \frac{\partial z^*}{\partial \lambda}\right] \\ &\quad - \frac{1}{[\rho_\theta]\cos\varphi}\frac{\partial}{\partial \varphi}\left\{\left[(\rho_\theta v)^* M^*\right]\cos\varphi\right\} - \frac{1}{[\rho_\theta]}\frac{\partial}{\partial \theta}\left[\left(\rho_\theta \dot{\theta}\right)^* M^*\right].\end{aligned} \tag{98}$$

Here all of the eddy terms have been collected on the right-hand side, and the non-eddy terms have been collected on the rather simple-looking left-hand side.

Now define the isentropic Eliassen-Palm flux as

$$\mathbf{EPF} \equiv \left(0, EPF_\varphi, EPF_\theta\right),$$

where

$$EPF_\varphi \equiv -\left[(\rho_\theta v)^* M^* \right], \text{ and } EPF_\theta \equiv \left[p^* \frac{\partial z^*}{\partial \lambda} \right] - \left[(\rho_\theta \dot{\theta})^* M^* \right]. \tag{99}$$

The meridional component is minus the eddy angular momentum flux. The vertical component is minus the "total" vertical eddy angular momentum flux, due to the combination of isentropic form drag and the vertical mass flux associated with heating. The divergence of the isentropic Eliassen-Palm flux is given by

$$\nabla \cdot \mathbf{EPF} = \frac{-1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{ \left[(\rho_\theta v)^* M^* \right]\cos\varphi \right\} + \frac{\partial}{\partial\theta}\left\{ \left[p^* \frac{\partial z^*}{\partial\lambda} \right] - \left[(\rho_\theta \dot{\theta})^* M^* \right] \right\}, \tag{100}$$

where it is understood that the meridional derivative is taken along an isentropic surface. With these definitions, (98) can be written as

$$\frac{\partial\left[\hat{M}\right]}{\partial t} + \frac{[\hat{v}]}{a}\frac{\partial[M]}{\partial\varphi} + \left[\hat{\dot{\theta}}\right]\frac{\partial[M]}{\partial\theta} = \frac{1}{[\rho_\theta]}\left(-\frac{\partial\left[\rho_\theta^* M^*\right]}{\partial t} + \nabla \cdot \mathbf{EPF} \right). \tag{101}$$

Consider a steady state (or time average) with no heating. Then the continuity equation (90) reduces to

$$\frac{\partial}{\partial\varphi}\left([\rho_\theta v]\cos\varphi \right) = 0 \text{ , for steady flow without heating or friction.} \tag{102}$$

Since $[\rho_\theta v]\cos\varphi = 0$ at both poles, we conclude that

$$\left[\rho_\theta v\right] = 0 \text{ for all } \theta \text{ and } \varphi \text{ , for steady flow without heating or friction,} \tag{103}$$

from which it follows that

$$\left[\hat{v}\right] = 0 \text{ for all } \theta \text{ and } \varphi \text{ , for steady flow without heating or friction.} \tag{104}$$

This means that for steady flow with no heating the meridional advection term of (101) vanishes, which is quite amazing. Naturally the tendency and vertical advection terms of (101) are also zero in this case. It follows that for steady flow in the absence of heating, the Eliassen-Palm flux is non-divergent:

$$\nabla_\theta \cdot \mathbf{EPF} = 0 \text{ for steady flow without heating or friction.} \tag{105}$$

Another way of saying this is that, for steady flow in the absence of heating, the zonally averaged meridional angular momentum transport is due only to the eddies, and is balanced by the form drag on isentropic surfaces. This beautifully simple result is pretty nearly exact. It is a statement of the Eliassen-Palm theorem.

With the isentropic system, there is no need to define a "residual" circulation, because the true zonally averaged circulation as seen in isentropic coordinates *is* the residual circulation. This circulation vanishes for a steady state (or time average) with no heating, even when friction is present. The time-averaged mean meridional circulation in isentropic coordinates is due entirely to heating.

Potential vorticity fluxes

Consider the momentum equation in isentropic coordinates:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{k} \times (\eta \mathbf{V}) + \nabla_\theta (K + s) + \dot{\theta} \frac{\partial \mathbf{V}}{\partial \theta} + \mathbf{F} = 0 \,. \tag{106}$$

Here s is the Montgomery Stream Function,

$$\eta \equiv \zeta + f \tag{107}$$

is the absolute vorticity, where

$$\zeta \equiv \mathbf{k} \cdot (\nabla_\theta \times \mathbf{V}), \tag{108}$$

and $\mathbf{F}$ is the friction vector. Note that ζ is the vorticity computed by taking derivatives of the horizontal wind along isentropic surfaces.

To derive the vorticity equation, we apply $\mathbf{k} \cdot \nabla_\theta \times$ to (106). Starting from standard vector identities, we can show that

$$\mathbf{k} \cdot \{\nabla \times (\mathbf{k} \times \mathbf{A})\} = \nabla \cdot \mathbf{A} \,, \tag{109}$$

and

$$\mathbf{k}\cdot(\nabla\times\mathbf{A})=-\nabla\cdot(\mathbf{k}\times\mathbf{A}),$$

(110)

where $\mathbf{A}$ is an arbitrary horizontal vector. With these relations, and using the fact that the Coriolis parameter is independent of time, we can show that

$$\frac{\partial\eta}{\partial t}+\nabla_{\theta}\cdot(\mathbf{V}\eta)=\nabla_{\theta}\cdot\left\{\mathbf{k}\times\left(\dot{\theta}\frac{\partial\mathbf{V}}{\partial\theta}+\mathbf{F}\right)\right\}.$$

(111)

Notice that the *vertical* advection term of (106) now appears inside a *horizontal* divergence operator! This comes from the use of (110). We now define

$$q\equiv\frac{\eta}{\rho_{\theta}}$$

(112)

as the Ertel potential vorticity. Here

$$\rho_{\theta}\equiv-\frac{1}{g}\frac{\partial p}{\partial\theta}>0$$

(113)

is the pseudo-density. Then (111) becomes

$$\boxed{\frac{\partial(\rho_{\theta}q)}{\partial t}+\nabla_{\theta}\cdot(\rho_{\theta}\mathbf{V}q)=\nabla_{\theta}\cdot\left\{\mathbf{k}\times\left(\dot{\theta}\frac{\partial\mathbf{V}}{\partial\theta}+\mathbf{F}\right)\right\}}.$$

(114)

This equation was derived and discussed by Haynes and McIntyre (1987). According to (114), the average over the sphere of the mass-weighted potential vorticity on an isentropic surface cannot change, *even in the presence of heating and friction*. This amazing conclusion, that neither vertical advection of momentum nor friction can alter the mass-weighted average PV on an isentropic surface, is called the "*impermeability theorem*." For further discussion, see Bretherton and Schär (1993).

The zonal average of (114) gives

$$\frac{\partial}{\partial t}[\rho_{\theta}q]+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left([\rho_{\theta}vq]\cos\varphi\right)=\frac{-1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\left[\dot{\theta}\frac{\partial u}{\partial\theta}+F_{\lambda}\right]\cos\varphi\right).$$

(115)

Using (88), (93) and (94), we can rewrite (115) as

$$\frac{\partial}{\partial t}[\rho_\theta q]+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\left([\rho_\theta][\hat{v}][q]+\left[(\rho_\theta v)^* q^*\right]\right)\cos\varphi\right]=\frac{-1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\left[\dot{\theta}\frac{\partial u}{\partial\theta}+F_\lambda\right]\cos\varphi\right).$$

(116)

For a steady state, (116) can be written as

$$\frac{\partial}{\partial\varphi}\left\{\left([\rho_\theta][\hat{v}][q]+\left[(\rho_\theta v)^* q^*\right]+\left[\dot{\theta}\frac{\partial u}{\partial\theta}\right]+[F_\lambda]\right)\cos\varphi\right\}=0\ .$$

(117)

The quantity in curly braces is independent of latitude. It is zero at both poles, because of the factor of $\cos\varphi$. Therefore it must be zero at every latitude, i.e.,

$$[\rho_\theta][\hat{v}][q]+\left[(\rho_\theta v)^* q^*\right]+\left[\dot{\theta}\frac{\partial u}{\partial\theta}\right]+[F_\lambda]=0 \text{ for all } \theta \text{ and } \varphi \text{, for steady flow.}$$

(118)

This is equation (3.4) of Haynes and McIntyre (1987). All three terms on the left-hand side vanish at the poles. With no heating or friction, and using (102), this reduces to

$$\left[(\rho_\theta v)^* q^*\right]=0 \text{ for all } \theta \text{ and } \varphi \text{, for steady flow without heating or friction.}$$

(119)

Compare with (16), which was derived using the quasi-geostrophic approximation.

Summary

Waves and other eddies produce important effects on the large-scale circulation of the atmosphere. Important fluxes are associated with a wide variety of waves, including gravity waves, Rossby waves, and Kelvin waves. Momentum fluxes and temperature fluxes can tend to produce mutually counteracting effects, so that the mean zonal flow and temperature may not be altered. The appreciable effects of the eddies on the mean flow are typically associated with developing or decaying eddies, rather than steady, equilibrated eddies.

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