1-Hybrid isentropic-sigma vertical coordinate and governing equations in the free atmosphere

This section describes the equations in the free atmosphere of the model. We first discuss the generalized vertical coordinate and describe the θ-σ hybrid vertical coordinate selected for the model. Then we introduce the vertical mass flux equation unique for such a generalized vertical coordinate and briefly discuss the upper and lower boundary conditions. We next describe the vertical grid and vertical discretization in the free atmosphere. The equations governing the dry dynamics closely follow Konor and Arakawa (1997).

1.1 Vertical coordinate

The vertical coordinate is defined by

$$\zeta \equiv \mathrm{F}(\theta,\sigma) \tag{1}$$

and

$$\sigma \equiv G(p,p_B)\,. \tag{2}$$

Since ζ is the vertical coordinate, it is required that $\mathrm{F}(\theta,\sigma)$ be a monotonic function of height. Here we select F as a monotonically increasing function of height and, therefore, the most straightforward choice for G is the same. Now we make the following choice

$$F(\theta,\sigma) \equiv f(\sigma) + g(\sigma)\theta \tag{3}$$

and

$$g(\sigma) \equiv g_o\left(1 - e^{-\alpha\sigma}\right), \tag{4}$$

where

$$g_o \equiv 1\big/\left(1 - e^{-\alpha\sigma_T}\right) \tag{5}$$

Equations (4) and (5) yield $g(\sigma_T) = 1$ where $\sigma_T = G(p_T, p_B)$ at the top of the atmosphere p_T = constant and $g(\sigma_B) = 0$ at the bottom of the free atmosphere (or at the PBL-top). Note that the function $g(\sigma)$ smoothly and exponentially approaches to 1 with height. The rate of approach is controlled by the constant α, which is currently 10.

Requiring $\partial \mathrm{F}/\partial\sigma > 0$ guaranties the monotonousness of ζ with height, which requires $df/d\sigma + \theta\, dg/d\sigma + g\partial\theta/\partial\sigma > 0$ from (3). Since $g > 0$ and $\mathrm{d}g/\mathrm{d}\sigma > 0$ between σ_T and σ_B, we can satisfy the requirement by replacing θ and $\mathrm{d}\theta/\mathrm{d}\sigma$ with their unreachably small values $\theta_{\min}$ and $\left(\mathrm{d}\theta/\mathrm{d}\sigma\right)_{min}$, respectively. Then we write

$$\frac{\mathrm{d}f}{\mathrm{d}\sigma}+\theta_{min}\frac{\mathrm{d}g}{\mathrm{d}\sigma}+g\left(\frac{\partial g}{\partial\sigma}\right)_{min}=0\,. \tag{6}$$

In (6), $\frac{\mathrm{d}g}{\mathrm{d}\sigma}=\alpha g_{\mathrm{o}}\mathrm{e}^{-\alpha\sigma}=\alpha(\,g_{\mathrm{o}}-g\,)$ and $\frac{\mathrm{d}f}{\mathrm{d}\sigma}$, where f is the unknown. By vertically integrating (6) with respect to σ, we obtain

$$f(\sigma)=-\left(\frac{\partial\theta}{\partial\sigma}\right)_{min}\left[g_{\mathrm{o}}(\sigma-\sigma_{\mathrm{T}})-\frac{1}{\alpha}(g-1)\right]-(\theta)_{min}\,(g-1)\,, \tag{8}$$

where we used $f(\sigma_{\mathrm{T}})=0$ and $\mathrm{g}(\sigma_{\mathrm{T}})=1$. Using (3), (4) and (8) in (1), we obtain an expression for the vertical coordinate as

$$\zeta\equiv\mathrm{F}(\theta,\sigma)=(\theta)_{min}+\left[\theta-(\theta)_{min}\right]g-\left(\frac{\partial\theta}{\partial\sigma}\right)_{min}\left[g_{\mathrm{o}}(\sigma-\sigma_{\mathrm{T}})-\frac{1}{\alpha}(g-1)\right]. \tag{9}$$

Definition for $\sigma\equiv\mathrm{G}(p,p_B)$**:**

So far no specific definition is needed for $\sigma\equiv\mathrm{G}(\mathrm{p},\mathrm{p_B})$. Here we discuss a couple of choices for $\sigma\equiv\mathrm{G}(\mathrm{p},\mathrm{p_B})$. An obvious choice for $\mathrm{G}(\mathrm{p},\mathrm{p_B})$ is a linearly decreasing function of p, which can be written as

$$\mathrm{G}(\mathrm{p},\mathrm{p_B})\equiv\frac{\mathrm{p_B}-\mathrm{p}}{\mathrm{p_B}-\mathrm{p_T}}\,. \tag{10}$$

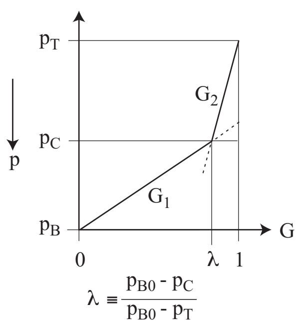
Since this expression is very simple and straightforward, we use it in the development stage of the model. With (10), the partial derivatives become

$$\left.\begin{aligned}\left(\frac{\partial\mathrm{G}}{\partial\mathrm{p}}\right)_{\mathrm{p_B}}&=-\frac{1}{\mathrm{p_B}-\mathrm{p_T}}\\ \left(\frac{\partial\mathrm{G}}{\partial\mathrm{p_B}}\right)_{\mathrm{p}}&=\frac{\mathrm{p}-\mathrm{p_T}}{(\mathrm{p_B}-\mathrm{p_T})^2}\end{aligned}\right\}\,. \tag{11}$$

However, when (10) is used, dependency of ζ on $\mathrm{p_B}$ does not vanish completely in the middle and upper model atmosphere, which is not desirable in a GCM with a variable PBL since it may cause excessive vertical dissipation of moisture and tracers. In the next subsection, we discuss a different expression for $\mathrm{G}(\mathrm{p},\mathrm{p_B})$, which limits the dependency of ζ on $\mathrm{p_B}$ near the PBL top and eliminates it entirely in the middle and upper model atmosphere.

A new definition for $\mathrm{G}(p,p_B)$:

To define G, we first define



$$G_1(p, p_B) \equiv \left(\frac{p_{B0} - p_C}{p_{B0} - p_T}\right)\frac{p_B - p}{p_B - p_C} \qquad \text{for } p_B \geq p \geq p_C, \quad (12)$$

where p_C is a constant pressure, maybe chosen slightly smaller then a possible minimum of p_B, $p_C = p_S$ - 200mb is a reasonable selection, and p_{B0} is a standard constant value of p_B. Then

$$G_2(p) \equiv \frac{p_{B0} - p}{p_{B0} - p_T} \qquad \text{for } p_C \geq p \geq p_T \,. \quad (13)$$

Now for simplicity, we define

$$\tilde{\sigma} \equiv \frac{p_B - p}{p_B - p_C}\,. \qquad (14)$$

By using (14) in (12) and (13), we respectively write

$$G_1(\tilde{\sigma}) \equiv \frac{p_{B0} - p_C}{p_{B0} - p_T}\tilde{\sigma} \quad \text{for} \quad 0 \leq \tilde{\sigma} \leq 1 \qquad (15)$$

and

$$G_2(\tilde{\sigma}, p_B) \equiv \frac{p_{B0} - p_B}{p_{B0} - p_T} + \frac{p_B - p_C}{p_{B0} - p_T}\tilde{\sigma} \quad \text{for } 1 \leq \tilde{\sigma} \leq \frac{p_B - p_T}{p_B - p_C} \qquad (16)$$

Now we require that

$$\frac{\partial G}{\partial \tilde{\sigma}} = a\frac{\partial G_1}{\partial \tilde{\sigma}} + b\frac{\partial G_2}{\partial \tilde{\sigma}}\,, \qquad (17)$$

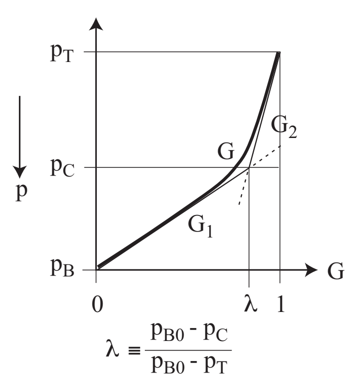
with $a + b = 1$. We select $a \equiv \frac{1}{2}\left\{1 - \tanh\left[\beta\left(\tilde{\sigma} - \tilde{\sigma}_C\right)\right]\right\}$ and it results in $b \equiv \frac{1}{2}\left\{1 + \tanh\left[\beta\left(\tilde{\sigma} - \tilde{\sigma}_C\right)\right]\right\}$. With this choice, we satisfy the following

$$\frac{\partial G}{\partial \tilde{\sigma}} \to \frac{\partial G_1}{\partial \tilde{\sigma}} \quad \text{as} \quad \tilde{\sigma} \to 0\,, \qquad (18)$$

$$\frac{\partial G}{\partial \tilde{\sigma}} \to \frac{\partial G_2}{\partial \tilde{\sigma}} \quad \text{as} \quad \tilde{\sigma} \to \frac{p_B - p_T}{p_B - p_C} \qquad (19)$$

and

$$\frac{\partial G}{\partial \tilde{\sigma}} = \frac{1}{2}\left[\frac{\partial G_1}{\partial \tilde{\sigma}} + \frac{\partial G_2}{\partial \tilde{\sigma}}\right] \quad \text{for} \quad \tilde{\sigma} = \tilde{\sigma}_C\,. \qquad (20)$$



We can satisfy (17) by

$$G(p,p_B) \equiv \frac{1}{2}\,\frac{p_{B0}-p_C}{p_{B0}-p_T}\left[\frac{p_B-p}{p_B-p_C}-\frac{1}{\beta}\,\ell nCosh\left(\beta\frac{p_C-p}{p_B-p_C}\right)\right] + \frac{1}{2}\,\frac{p_B-p_C}{p_{B0}-p_T}\left[\frac{p_B-p}{p_B-p_C}+\frac{1}{\beta}\,\ell nCosh\left(\beta\frac{p_C-p}{p_B-p_C}\right)\right]+C\,, \tag{21}$$

where we choose

$$C \equiv -\frac{1}{2\beta}\,\ell nCosh(-\beta)\left(\frac{p_B-p_{B0}}{p_{B0}-p_T}\right) \tag{22}$$

to satisfy $G=0$ for $p=p_B$. Equations (21) and (22) yield

$$\left(\frac{\partial G}{\partial p}\right)_{p_B} \equiv \frac{1}{2}\,\frac{p_{B0}-p_C}{p_{B0}-p_T}\left[\frac{-1}{p_B-p_C}+\frac{1}{p_B-p_C}\tanh\left(\beta\frac{p_C-p}{p_B-p_C}\right)\right] + \frac{1}{2}\,\frac{p_B-p_C}{p_{B0}-p_T}\left[\frac{-1}{p_B-p_C}-\frac{1}{p_B-p_C}\tanh\left(\beta\frac{p_C-p}{p_B-p_C}\right)\right] \tag{23}$$

and

$$\left(\frac{\partial G}{\partial p_B}\right)_{p} \equiv \frac{1}{2}\,\frac{p_{B0}-p_C}{p_{B0}-p_T}\,\frac{p_C-p}{\left(p_B-p_C\right)^2}\left[-1+\tanh\left(\beta\frac{p_C-p}{p_B-p_C}\right)\right] + \frac{1}{2}\,\frac{1}{p_{B0}-p_T}\left[1+\frac{1}{\alpha}\ell nCosh\left(\beta\frac{p_C-p}{p_B-p_C}\right)-\frac{p_C-p}{p_B-p_C}\tanh\left(\beta\frac{p_C-p}{p_B-p_C}\right)\right] - \frac{1}{2\beta\left(p_{B0}-p_T\right)}\ell nCosh(-\beta) \tag{24}$$

1.2- Generalized vertical mass flux equation

Since ζ is the vertical coordinate of the model, we require

$$0=\left(\frac{\partial}{\partial t}\right)_{\zeta}\mathrm{F}(\theta,\sigma)\,. \tag{30}$$

Then using (1) and (2) in (30), we obtain

$$0=\left(\frac{\partial \mathrm{F}}{\partial \theta}\right)_{\sigma}\left(\frac{\partial \theta}{\partial t}\right)_{\zeta}+\left(\frac{\partial \mathrm{F}}{\partial \sigma}\right)_{\theta}\left(\frac{\partial \sigma}{\partial t}\right)_{\zeta}\,. \tag{31}$$

In (31), $\left(\partial\sigma/\partial t\right)_\zeta$ can be obtained from (2) as

$$\left(\frac{\partial\sigma}{\partial t}\right)_\zeta = \left(\frac{\partial G}{\partial p}\right)_{p_B}\left(\frac{\partial p}{\partial t}\right)_\zeta + \left(\frac{\partial G}{\partial p_B}\right)_p\left(\frac{\partial p_B}{\partial t}\right). \tag{32}$$

Using (32), equation (31) can be rewritten as

$$0 = \left(\frac{\partial F}{\partial\theta}\right)_\sigma\left(\frac{\partial\theta}{\partial t}\right)_\zeta + \left(\frac{\partial F}{\partial\sigma}\right)_\theta\left(\frac{\partial G}{\partial p}\right)_{p_B}\left(\frac{\partial p}{\partial t}\right)_\zeta + \left(\frac{\partial F}{\partial\sigma}\right)_\theta\left(\frac{\partial G}{\partial p_B}\right)_p\left(\frac{\partial p_B}{\partial t}\right). \tag{33}$$

The thermodynamic equation for the system is

$$\left(\frac{\partial}{\partial t}\right)_\zeta \theta = -\mathbf{v}\cdot\nabla_\zeta\theta - \dot{\zeta}\frac{\partial\theta}{\partial\zeta} + \frac{Q}{\Pi}. \tag{34}$$

The pressure tendency and surface pressure tendency equations for the system are

$$\left(\frac{\partial p}{\partial t}\right)_\zeta = \int_{\zeta=\zeta_T}^{\zeta}(\mathrm{m}\mathbf{v})\,d\zeta + \left(\mathrm{m}\dot{\zeta}\right) \tag{35}$$

and

$$\frac{\partial p_B}{\partial t} = \int_{\zeta=\zeta_T}^{\zeta_B}(\mathrm{m}\mathbf{v})\,d\zeta + \left(\mathrm{m}\dot{\zeta}\right)_B, \tag{36}$$

respectively. By using (34), (35) and (36) in (33), the equation that determines the generalized vertical mass flux can be obtained as

$$\left\{\left(\frac{\partial F}{\partial\theta}\right)_\sigma\frac{\partial\theta}{\partial\zeta} - \mathrm{m}\left(\frac{\partial F}{\partial\sigma}\right)_\theta\left(\frac{\partial G}{\partial p}\right)_{p_B}\right\}\dot{\zeta} = \left(\frac{\partial F}{\partial\theta}\right)_\sigma\left\{-\mathbf{v}\cdot\nabla_\zeta\theta + \frac{Q}{\Pi}\right\}$$
$$+\left(\frac{\partial F}{\partial\sigma}\right)_\theta\left\{\left(\frac{\partial G}{\partial p}\right)_{p_B}\left[\int_{\zeta=\zeta_T}^{\zeta}(\mathrm{m}\mathbf{v})\,d\zeta\right] + \left(\frac{\partial G}{\partial p_B}\right)_p\left[\int_{\zeta=\zeta_T}^{\zeta_B}(\mathrm{m}\mathbf{v})\,d\zeta + \left(\mathrm{m}\dot{\zeta}\right)_B\right]\right\}. \tag{37}$$

The vertically discrete version of the generalized vertical mass flux equation (37) will be discussed later in this text.

1.3 Upper and Lower boundary conditions

The upper boundary (upper most interface, ζ_T) is an isentropic surface ($\zeta_T \equiv \theta_T = \text{constant}$). We assume that $p_T = \text{constant}$ and $\left(\mathrm{m}\dot{\zeta}\right)_T = 0$ {i.e.

$\dot{\theta}_T = \left(\frac{Q}{\Pi}\right)_T = 0\}$. The lower boundary (lowest interface, ζ_B), which coincides with the PBL top, is a sigma type, $\zeta_B = f(\sigma_B) = \text{constant}$. The vertical mass flux $\left(m\dot{\zeta}\right)_B$ is primarily determined from PBL top entrainment/detrainment and cumulus mass flux from PBL into cumulus clouds.

1.4 Vertical grid in the free atmosphere

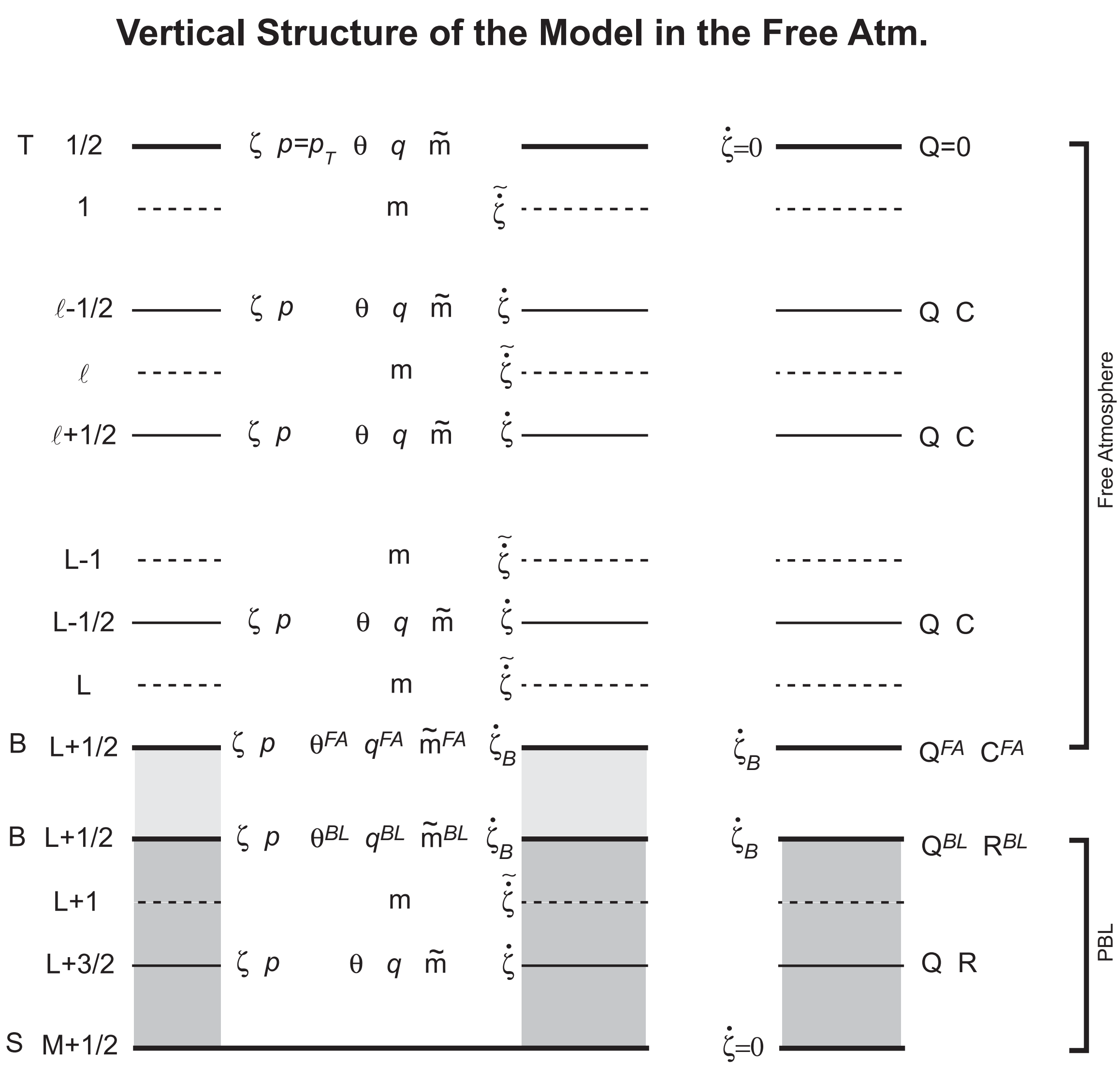


Fig. 1. Vertical grid used in the discretization.

1.5- Vertically discrete equations in the free atmosphere

1.5.a. Mass continuity equation

The vertically discrete version of the mass continuity equation applied to the model layers within the free atmosphere is given by

$$\frac{\partial \mathrm{m}_\ell}{\partial t} + \nabla\cdot(\mathrm{m}\mathbf{v})_\ell + \frac{1}{(\delta\zeta)_\ell}\left[\left(\mathrm{m}\dot{\zeta}\right)_{\ell+1/2} - \left(\mathrm{m}\dot{\zeta}\right)_{\ell-1/2}\right] = 0 \quad \text{for } \ell = 1,2,..,L, \tag{38}$$

where

$$\left.\begin{aligned} \mathrm{m}_\ell &\equiv -(\delta p)_\ell/(\delta\zeta)_\ell \\ (\delta p)_\ell &\equiv p_{\ell+1/2} - p_{\ell-1/2} \\ (\delta\zeta)_\ell &\equiv \zeta_{\ell+1/2} - \zeta_{\ell-1/2} \end{aligned}\right\} \text{ for } \ell = 1,2,...,L, \tag{39}$$

and $\left(\mathrm{m}\dot{\zeta}\right)_{\ell+1/2}$ is the vertical mass flux carried at the interfaces of the model layers. We assume that, at the top of the atmosphere, $\left(\mathrm{m}\dot{\zeta}\right)_{1/2} = 0$. The vertical mass flux at the PBL-top, $\left(\mathrm{m}\dot{\zeta}\right)_{L+1/2}$, where $L+1/2$ and B are interchangeable, is determined by entrainment/detrainment parameterization. The vertically discrete pressure tendency equations can be obtained by vertically summing (38) with (39) as

$$\left.\begin{aligned} \frac{\partial p_{\ell+1/2}}{\partial t} &= \sum_{k=1}^{\ell} \nabla\cdot(\mathrm{m}_k \mathbf{v}_k)(\delta\zeta)_k + \left(\mathrm{m}\dot{\zeta}\right)_{\ell+1/2} \qquad \text{for } \ell = 1,2,...,L \qquad (40.\mathrm{a}) \\ \frac{\partial p_B}{\partial t} &= \sum_{k=1}^{L} \nabla\cdot(\mathrm{m}_k \mathbf{v}_k)(\delta\zeta)_k + \left(\mathrm{m}\dot{\zeta}\right)_B, \qquad (40.\mathrm{b}) \end{aligned}\right\}$$

where we assumed that $(\partial p_T/\partial t) = 0$, where subscript $1/2$ and T are interchangeable.

1.5.b. Thermodynamic equation

The vertically discrete version of the thermodynamic equation within the free atmosphere applied to the interfaces of the model layers are given by

$$\left.\begin{aligned}
&\frac{\partial\theta_{1/2}}{\partial t}=0 && \text{(41.a)}\\
&\frac{\partial\theta_{\ell+1/2}}{\partial t}+(\mathbf{v}\cdot\nabla\theta)_{\ell+1/2}+\left(\frac{\partial\theta}{\partial\zeta}\right)_{\ell+1/2}\dot{\zeta}_{\ell+1/2}=(Q/\Pi)_{\ell+1/2}\quad \text{for}\ \ \ell=1,2,\ldots,L-1 && \text{(41.b)}\\
&\frac{\partial\theta_{L+1/2}}{\partial t}+\mathbf{v}_L\cdot\nabla\theta_{L+1/2}+\left(\frac{\partial\theta}{\partial\zeta}\dot{\zeta}\right)_{L+1/2}=(Q/\Pi)_{L+1/2} && \text{(41.c)}
\end{aligned}\right\}.$$

Equation (41.a) is a result of choosing the upper boundary placed on an isentropic surface. In (41.b) and (41.c),

$$(\mathbf{v}\cdot\nabla\theta)_{\ell+1/2}\equiv\left[\frac{\Pi_{\ell+1}\,\mathrm{m}_{\ell+1}(\delta\zeta)_{\ell+1}\mathbf{v}_{\ell+1}+\Pi_\ell\,\mathrm{m}_\ell(\delta\zeta)_\ell\mathbf{v}_\ell}{\Pi_{\ell+1}\,\mathrm{m}_{\ell+1}(\delta\zeta)_{\ell+1}+\Pi_\ell\,\mathrm{m}_\ell(\delta\zeta)_\ell}\right]\cdot\nabla\theta_{\ell+1/2}\quad\text{for}\ \ \ell=1,2,\ldots,L-1,\quad \text{(41.d)}$$

$$\left(\frac{\partial\theta}{\partial\zeta}\right)_{\ell+1/2}\equiv\frac{\mathrm{m}_{\ell+1/2}\Pi_{\ell+1/2}(\theta_{\ell+1}-\theta_\ell)}{\frac{1}{2}\left[\Pi_{\ell+1}\mathrm{m}_{\ell+1}(\delta\zeta)_{\ell+1}+\Pi_\ell\mathrm{m}_\ell(\delta\zeta)_\ell\right]}\qquad\text{for}\quad \ell=1,2,\ldots,L-1,\qquad \text{(41.e)}$$

where

$$\left.\begin{aligned}
&\Pi_\ell\equiv\frac{\Pi_{\ell+1/2}p_{\ell+1/2}-\Pi_{\ell-1/2}p_{\ell-1/2}}{(\kappa+1)\left(p_{\ell+1/2}-p_{\ell-1/2}\right)}\\
&\theta_\ell\equiv\frac{1}{2}\left(\theta_{\ell+1/2}+\theta_{\ell-1/2}\right)
\end{aligned}\right\}\ \text{for}\ \ \ell=1,2,\ldots,L\,,\qquad \text{(41.f)}$$

$$\left(\frac{\partial\theta}{\partial\zeta}\dot{\zeta}\right)_{L+1/2}\equiv\frac{1}{\mathrm{m}_{L+1/2}(\delta\zeta)^{FA}_{L+1/2}}\left[\left(\hat{\theta}_{L+1/2}-\theta_{L+1/2}\right)\left(\mathrm{m}\dot{\zeta}\right)_{L+1/2}+\left(\theta_{L+1/2}-\theta_L\right)\left(\mathrm{m}\dot{\zeta}\right)_L\right],\ \text{(41.g)}$$

$$\left.\begin{aligned}
&\hat{\theta}_{L+1/2}\equiv\theta_{L+1/2}\quad \mathit{if}\quad \left(\mathrm{m}\dot{\zeta}\right)_{L+1/2}<0\\
&\hat{\theta}_{L+1/2}\equiv\theta^{BL}_{L+1/2}\quad \mathit{if}\quad \left(\mathrm{m}\dot{\zeta}\right)_{L+1/2}>0
\end{aligned}\right\}.\qquad \text{(41.h)}$$

In (41.h), superscript *BL* denotes values from the upper most level of the PBL, which is also indexed by $L+1/2$. In the equations above, we use $\Pi_{\ell+1/2}\equiv c_p\left(p_{\ell+1/2}/p_\mathrm{o}\right)^\kappa$ and, if it is necessary, $p_\ell\equiv p_o\left(\Pi_\ell/c_p\right)^{1/\kappa}$.

When we consider condensation process only, heating Q can be written as

$$Q_{\ell+1/2}=LC_{\ell+1/2}\,. \qquad (42)$$

1.5.c. Vertical mass flux equation for temporally- and vertically-discrete system

Let us define $\Delta\theta$, Δp and Δp_B as the deviation of respected variables due to horizontal advection and physical processes at models grid points. Note that a significant portion of Δp_B is due to PBL-top entrainment. Then we define

$$(\Delta F) = F\left(\theta + \Delta\theta, p + \Delta p, p_B + \Delta p_B\right) - F\left(\theta, p, p_B\right). \tag{43}$$

Since F remains unchanged on ζ surfaces, (ΔF) must be compensated by the vertical advection, therefore,

$$(\Delta F) + (\delta t)\left(\frac{\partial F}{\partial \sigma}\right)_\theta \left(\frac{\partial G}{\partial p}\right)_{p_B} \mathrm{m}\dot{\zeta} - (\delta t)\left(\frac{\partial F}{\partial \theta}\right)_\sigma \left(\frac{\partial \theta}{\partial \zeta}\right)\dot{\zeta} = 0, \tag{44}$$

where (δt) is the time step. From (44), we obtain the vertical mass flux equation for the time-discrete case

$$\mathrm{m}\dot{\zeta} = \frac{(\Delta F)}{(\delta t)\left(\frac{\partial F}{\partial \theta}\right)_\sigma \frac{1}{\mathrm{m}}\left(\frac{\partial \theta}{\partial \zeta}\right) - (\delta t)\left(\frac{\partial F}{\partial \sigma}\right)_\theta \left(\frac{\partial G}{\partial p}\right)_{p_B}}. \tag{45}$$

The solution of (45) requires iteration. During the iteration, θ and p change following $\left(\partial\theta/\partial t\right) = -\left(\partial\theta/\partial\zeta\right)\dot{\zeta}$ and $\left(\partial p/\partial t\right) = \mathrm{m}\dot{\zeta}$, respectively. $\left(\partial\theta/\partial\zeta\right)$, which is calculated from $\theta + \Delta\theta$ and $p + \Delta p$, and the PBL-top pressure ($= p_B + \Delta p_B$) remain unchanged Iteration continues until calculated $\mathrm{m}\dot{\zeta}$ becomes virtually zero. Then the vertical mass flux is determined from $\mathrm{m}\dot{\zeta} = \left(p_{final} - p_{initial}\right)/(\delta t)$, where p_{final} and $p_{initial}$ $(\equiv p + \Delta p)$ are the pressure at the end and at the beginning of the iteration, respectively. In the vertically discrete system, the vertical mass flux equation (45) is applied at the model interfaces. $\left(\partial\theta/\partial\zeta\right)$ must be consistent with the vertical finite difference term in the discrete thermodynamic equation given by (41e) and 41g).

1.5.d. Moisture equation

At the upper most interface of the model, the equation that predicts the mass-weighted water vapor mixing ratio is written as

$$\frac{\partial (q\mathrm{m})_{1/2}}{\partial t} = -\nabla \cdot (q\mathrm{m}\mathbf{v})_{1/2} - \frac{1}{(\delta\zeta)_{1/2}}\left(q\mathrm{m}\dot{\zeta}\right)_1 - (\mathrm{m}C)_{1/2}, \tag{46}$$

where q is the water vapor mixing ration and

$$(q\mathrm{m})_{1/2} \equiv q_{1/2}\mathrm{m}_1 \,, \tag{47a}$$

$$\mathrm{m}_{1/2} \equiv \mathrm{m}_1 \,, \tag{47b}$$

$$(\delta\zeta)_{1/2} \equiv (\delta\zeta)_1 \,, \tag{47c}$$

$$(q\mathrm{m}\mathbf{v})_{1/2} \equiv q_{1/2}\mathrm{m}_1\mathbf{v}_1 \,, \tag{47d}$$

$$\left(q\mathrm{m}\dot{\zeta}\right)_1 \equiv \frac{1}{2} q_{3/2}\left(\mathrm{m}\dot{\zeta}\right)_{3/2} \,. \tag{47e}$$

Within the free atmosphere, the equation that predicts the mass-weighted water vapor mixing ratio is written as

$$\frac{\partial (q\mathrm{m})_{\ell+1/2}}{\partial t} = -\nabla\cdot(q\mathrm{m}\mathbf{v})_{\ell+1/2} - \frac{1}{(\delta\zeta)_{\ell+1/2}}\left[\left(q\mathrm{m}\dot{\zeta}\right)_{\ell+1} - \left(q\mathrm{m}\dot{\zeta}\right)_{\ell}\right] - (\mathrm{m}C)_{\ell+1/2} \,, \tag{48}$$

where

$$(q\mathrm{m})_{\ell+1/2} \equiv q_{\ell+1/2}\mathrm{m}_{\ell+1/2} \,, \tag{49a}$$

$$\mathrm{m}_{\ell+1/2} \equiv \frac{1}{2(\delta\zeta)_{\ell+1/2}}\left[(\delta\zeta)_{\ell+1}\mathrm{m}_{\ell+1} + (\delta\zeta)_{\ell}\mathrm{m}_{\ell}\right], \tag{49b}$$

$$(\delta\zeta)_{\ell+1/2} \equiv \frac{1}{2}\left[(\delta\zeta)_{\ell+1} + (\delta\zeta)_{\ell}\right], \tag{49c}$$

$$(q\mathrm{m}\mathbf{v})_{\ell+1/2} \equiv \frac{q_{\ell+1/2}}{2(\delta\zeta)_{\ell+1/2}}\left[(\delta\zeta)_{\ell+1}\mathrm{m}_{\ell+1}\mathbf{v}_{\ell+1} + (\delta\zeta)_{\ell}\mathrm{m}_{\ell}\mathbf{v}_{\ell}\right], \tag{49d}$$

Definitions of $\left(q\mathrm{m}\dot{\zeta}\right)_\ell$ will be discussed in more detail later in this text. For convenience in the early versions of the model, we used $\left(q\mathrm{m}\dot{\zeta}\right)_\ell \equiv \frac{1}{2}\left[q_{\ell+1/2}\left(\mathrm{m}\dot{\zeta}\right)_{\ell+1/2} + q_{\ell-1/2}\left(\mathrm{m}\dot{\zeta}\right)_{\ell-1/2}\right]$. With this definition the vertical moisture advection scheme becomes a second-order centered finite differencing when $\mathrm{m}\dot{\zeta}$ is uniform and $\delta\zeta$ is constant. Such a centered scheme may produce large dispersion errors and it cannot accurately represent the PBL-Free Atmosphere mass exchange process.

At the lower most interface of the free atmosphere, the equation that predicts the mass-weighted water vapor mixing ratio is written as

$$\frac{\partial(\mathrm{qm})^{\mathrm{FA}}_{\mathrm{L+1/2}}}{\partial \mathrm{t}} = -\nabla\cdot(\mathrm{qm}\mathbf{v})^{\mathrm{FA}}_{\mathrm{L+1/2}} - \frac{1}{(\delta\zeta)^{\mathrm{FA}}_{\mathrm{L+1/2}}}\left\{\hat{\mathrm{q}}_{\mathrm{L+1/2}}\left[\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{L+1/2}} - \mathrm{gM_B}\right] + \mathrm{q}^{\mathrm{BL}}_{\mathrm{L+1/2}}\mathrm{gM_B} - \left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{L}}\right\}$$
$$-(\mathrm{mC})_{\mathrm{L+1/2}}, \quad (50)$$

where

$$\left(q\mathrm{m}\right)^{FA}_{L+1/2} \equiv q^{FA}_{L+1/2}\mathrm{m}_L, \quad (51a)$$
$$\mathrm{m}^{FA}_{L+1/2} \equiv \mathrm{m}_L, \quad (51b)$$
$$\left(\delta\zeta\right)^{FA}_{L+1/2} \equiv \frac{1}{2}\left(\delta\zeta\right)_L, \quad (51c)$$
$$\left(q\mathrm{m}\mathbf{v}\right)^{FA}_{L+1/2} \equiv q^{FA}_{L+1/2}\mathrm{m}_L\mathbf{v}_L. \quad (51d)$$

In (50), we used upstream treatment for exchange associated with $\mathrm{M_B}$. The mass-weighted vertical mass flux at the lowest layer of the model is given by $\left(q\mathrm{m}\dot{\zeta}\right)_L \equiv \frac{1}{2}\left[\hat{q}_{L+1/2}\left(\mathrm{m}\dot{\zeta}\right)_{L+1/2} + q_{L-1/2}\left(\mathrm{m}\dot{\zeta}\right)_{L-1/2}\right]$, where

$$\left.\begin{array}{ll}\hat{\mathrm{q}}_{\mathrm{L+1/2}} \equiv \mathrm{q}^{\mathrm{FA}}_{\mathrm{L+1/2}} & \text{if} \quad \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{L+1/2}} - \mathrm{gM_B} < 0 \\ \hat{\mathrm{q}}_{\mathrm{L+1/2}} \equiv \mathrm{r}^{\mathrm{BL}}_{\mathrm{L+1/2}} & \text{if} \quad \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{L+1/2}} - \mathrm{gM_B} > 0\end{array}\right\}. \quad (52)$$

Following the argument above, the expression for $\left(q\mathrm{m}\dot{\zeta}\right)_L$ is modified to better represent the PBL-Free Atmosphere mass exchange process. This will be discussed later in this text.

1.5.e. Vertical moisture fluxes

To calculate the vertical moisture fluxes at the layers, we will rewrite (48) by omitting horizontal advection and diabatic terms as

$$\frac{\partial(\mathrm{m}q)_{\ell+1/2}}{\partial t} = -\frac{1}{(\delta\zeta)_{\ell+1/2}}\left[\left(q\mathrm{m}\dot{\zeta}\right)_{\ell+1} - \left(q\mathrm{m}\dot{\zeta}\right)_{\ell}\right] \quad (53)$$

Following Hsu and Arakawa's (1990) positive-definite scheme, we define the vertical mass fluxes if $\left(\mathrm{m}\dot{\zeta}\right)$ is upward [$\left(\mathrm{m}\dot{\zeta}\right)^+ \equiv \left(\mathrm{m}\dot{\zeta}\right)$ and $\left(\mathrm{m}\dot{\zeta}\right)^- \equiv 0$] by

$$\left(q\mathrm{m}\dot{\zeta}\right)_\ell \equiv \left(\mathrm{m}\dot{\zeta}\right)^+_\ell \frac{q_{\ell+1/2} + q_{\ell-1/2}}{2} - \alpha\left[\left(\mathrm{m}\dot{\zeta}\right)^+_\ell \beta^+_\ell\left(q_{\ell-1/2} - q_{\ell+1/2}\right) - \widehat{\left(\mathrm{m}\dot{\zeta}\right)}^+_\ell \hat{\beta}^+_\ell\left(q_{\ell+1/2} - q_{\ell+3/2}\right)\right]$$
$$\text{for} \quad \ell = 1,2,\ldots,L-1 \quad (54)$$

and

$$\left(q\mathrm{m}\dot{\zeta}\right)_L \equiv \left(\mathrm{m}\dot{\zeta}\right)_L^+ \hat{q}_{L+1/2} \, . \tag{55}$$

We define the vertical mass fluxes if $\left(\mathrm{m}\dot{\zeta}\right)$ is downward [$\left(\mathrm{m}\dot{\zeta}\right)^- \equiv \left(\mathrm{m}\dot{\zeta}\right)$ and $\left(\mathrm{m}\dot{\zeta}\right)^+ \equiv 0$] by

$$\left(q\mathrm{m}\dot{\zeta}\right)_\ell \equiv \left(\mathrm{m}\dot{\zeta}\right)_\ell^- \frac{q_{\ell+1/2} + q_{\ell-1/2}}{2} - \alpha\left[\left(\mathrm{m}\dot{\zeta}\right)_\ell^- \beta_\ell^- \left(q_{\ell+1/2} - q_{\ell-1/2}\right) - \widehat{\left(\mathrm{m}\dot{\zeta}\right)}_\ell^- \hat{\beta}_\ell^- \left(q_{\ell-1/2} - q_{\ell-3/2}\right)\right]$$
$$\text{for} \quad \ell = 2,3,\ldots,L \tag{56}$$

and

$$\left(q\mathrm{m}\dot{\zeta}\right)_1 \equiv \left(\mathrm{m}\dot{\zeta}\right)_1^- q_{1/2} \, .$$

In (54) and (56),

$$\alpha = 2/9 \, , \tag{57a}$$

$$\beta_\ell^\pm \equiv 1 + \left(\frac{1-2\alpha}{2\alpha}\right)\gamma_\ell^\pm \tag{57b}$$

$$\hat{\beta}_\ell^\pm \equiv 1 - \hat{\gamma}_\ell^\pm \, , \tag{57c}$$

$$\gamma_\ell^+ = \hat{\gamma}_\ell^+ \equiv \left[\frac{P_{\ell+1/2}^2}{P_{\ell+1/2}^2 + q_{\ell+1/2} q_{\ell-1/2}}\right]^2 \quad \text{for} \quad \ell = 1,2,\ldots,L-1 \, , \tag{57d}$$

$$\gamma_\ell^- = \hat{\gamma}_\ell^- \equiv \left[\frac{P_{\ell-1/2}^2}{P_{\ell-1/2}^2 + q_{\ell+1/2} q_{\ell-1/2}}\right]^2 \quad \text{for} \quad \ell = 2,3,\ldots,L \, , \tag{57e}$$

$$P_{\ell+1/2} \equiv \left|q_{\ell+3/2} - 2q_{\ell+1/2} + q_{\ell-1/2}\right| + \varepsilon \, , \tag{57f}$$

$$\widehat{\left(\mathrm{m}\dot{\zeta}\right)}_\ell^+ \equiv \sqrt{\left(\mathrm{m}\dot{\zeta}\right)_\ell^+ \left(\mathrm{m}\dot{\zeta}\right)_{\ell+1}^+} \, , \tag{57g}$$

and

$$\widehat{(\mathrm{m}\dot{\zeta})}^{-}_{\ell} \equiv -\sqrt{(\mathrm{m}\dot{\zeta})^{-}_{\ell}(\mathrm{m}\dot{\zeta})^{-}_{\ell-1}} \,. \qquad (57h)$$

In (57f), ε is an infinitesimally small positive constant.

1.5.f. Correction step of moisture prediction

In the time discrete case, the solution of the vertical advection scheme discussed above is subject to over and under shooting errors. To minimize these errors, we employed a procedure, which is based on the principle that the advection process cannot generate new maximums and minimums in the field. Prior to the vertical advection, we locally determine the upper and lower bounds of advected values by

$$\left.\begin{array}{l} \left.\begin{array}{l} (q_{\max})_{1/2} = Max\{q_{1/2}, q_{3/2}\} \\ (q_{\min})_{1/2} = Min\{q_{1/2}, q_{3/2}\} \end{array}\right\} \quad \text{for } (\mathrm{m}\dot{\zeta})_{1} > 0 \\ (q_{\max})_{1/2} = (q_{\min})_{1/2} = q_{1/2} \qquad \text{for } (\mathrm{m}\dot{\zeta})_{1} < 0 \end{array}\right\}, \qquad (58a)$$

$$\left.\begin{array}{l} (q_{\max})_{\ell+1/2} = Max\{q_{\ell+1/2}, q_{\ell+3/2}\} \text{ for } (\mathrm{m}\dot{\zeta})_{\ell+1} > 0 \\ (q_{\max})_{\ell+1/2} = Max\{q_{\ell+1/2}, q_{\ell-1/2}\} \text{ for } (\mathrm{m}\dot{\zeta})_{\ell} < 0 \end{array}\right\} \text{for } \ell = 1, 2, \cdots, \mathrm{L}-1, \qquad (58b)$$

$$\left.\begin{array}{l} (q_{\min})_{\ell+1/2} = Min\{q_{\ell+1/2}, q_{\ell+3/2}\} \text{ for } (\mathrm{m}\dot{\zeta})_{\ell+1} > 0 \\ (q_{\min})_{\ell+1/2} = Min\{q_{\ell+1/2}, q_{\ell-1/2}\} \text{ for } (\mathrm{m}\dot{\zeta})_{\ell} < 0 \end{array}\right\} \text{for } \ell = 1, 2, \cdots, \mathrm{L}-1. \qquad (58c)$$

In (58b) and (58c), $\mathrm{q}_{\mathrm{L}+1/2} \equiv \mathrm{q}^{\mathrm{FA}}_{\mathrm{L}+1/2}$. At the lowest interface of the free atmosphere, the estimated max and min values are

$$\left.\begin{array}{ll} (q_{\max})^{FA}_{L+1/2} = Max\{q^{FA}_{L+1/2}, q^{BL}_{L+1/2}\} & \text{for } (\mathrm{m}\dot{\zeta})_{B} > 0 \\ (q_{\max})^{FA}_{L+1/2} = Max\{q^{FA}_{L+1/2}, q_{L-1/2}\} & \text{for } (\mathrm{m}\dot{\zeta})_{B} < 0 \end{array}\right\}, \qquad (58d)$$

$$\left.\begin{array}{ll} (q_{\min})^{FA}_{L+1/2} = Max\{q^{FA}_{L+1/2}, q^{BL}_{L+1/2}\} & \text{for } (\mathrm{m}\dot{\zeta})_{B} > 0 \\ (q_{\min})^{FA}_{L+1/2} = Max\{q^{FA}_{L+1/2}, q_{L-1/2}\} & \text{for } (\mathrm{m}\dot{\zeta})_{B} < 0 \end{array}\right\}. \qquad (58e)$$

In (58a-e), $q_{\ell+1/2}$ (for $\ell = 0,1,2,\cdots,L$) are pre-advection values. Then, after the advection, we apply corrections. Now let us first express $q^*_{\ell+1/2}$ for $\ell = 0,1,2,\cdots,L$ as post-advection values. Then the correction process can be summarized as follows

---

If $\left(\mathrm{m}\dot{\zeta}\right)_{L+1/2} > 0$,

If $q^{*FA}_{L+1/2} < \left(q_{min}\right)^{FA}_{L+1/2}$ $\begin{cases} q^{FA}_{L+1/2} = \left(q_{min}\right)^{FA}_{L+1/2} \\ q^{*BL}_{L+1/2} = q^{BL}_{\ell+1/2} - \left[\left(q_{min}\right)^{FA}_{L+1/2} - q^{*FA}_{L+1/2}\right]\mathrm{m}^{FA}_{L+1/2}(\delta\zeta)^{FA}_{L+1/2} \Big/ \mathrm{m}(\delta\zeta)^{BL}_{L+1/2} \end{cases}$

If $\left(\mathrm{m}\dot{\zeta}\right)_{L} > 0$:

If $q^{*FA}_{L+1/2} > \left(q_{\max}\right)^{FA}_{L+1/2}$ $\begin{cases} q_{L-1/2} = q^*_{L-1/2} + \left[q^{*FA}_{L+1/2} - \left(q_{\min}\right)^{FA}_{L+1/2}\right]\mathrm{m}^{FA}_{L+1/2}(\delta\zeta)^{FA}_{L+1/2} \Big/ \mathrm{m}_{L-1/2}(\delta\zeta)_{L-1/2} \\ q^{FA}_{L+1/2} = \left(q_{\max}\right)^{FA}_{L+1/2} \end{cases}$

For $\ell = L-1, L-2, \cdots, 1$ and if $\left(\mathrm{m}\dot{\zeta}\right)_{\ell} > 0$,

If $q^*_{\ell+1/2} > \left(q_{\max}\right)_{\ell+1/2}$ $\begin{cases} q_{\ell-1/2} = q^*_{\ell-1/2} + \left[q^*_{\ell+1/2} - \left(q_{\min}\right)_{\ell+1/2}\right]\mathrm{m}_{\ell+1/2}(\delta\zeta)_{\ell+1/2} \Big/ \mathrm{m}_{\ell-1/2}(\delta\zeta)_{\ell-1/2} \\ q_{\ell+1/2} = \left(q_{\max}\right)_{\ell+1/2} \end{cases}$

where $q_{L+1/2} \equiv q^{FA}_{L+1/2}$.

For $\ell = 1,2,\cdots,L$ and if $\left(\mathrm{m}\dot{\zeta}\right)_{\ell} > 0$,

If $q^*_{\ell-1/2} < \left(q_{\min}\right)_{\ell-1/2}$ $\begin{cases} q_{\ell-1/2} = \left(q_{\min}\right)_{\ell-1/2} \\ q_{\ell+1/2} = q^*_{\ell+1/2} - \left[\left(q_{\min}\right)_{\ell-1/2} - q^*_{\ell-1/2}\right]\mathrm{m}_{\ell-1/2}(\delta\zeta)_{\ell-1/2} \Big/ \mathrm{m}_{\ell+1/2}(\delta\zeta)_{\ell+1/2} \end{cases}$

for $\ell = 1,2,\cdots,L$

where $q_{L+1/2} \equiv q^{FA}_{L+1/2}$.

For $\ell = 1,2,\cdots,L$ and if $\left(\mathrm{m}\dot{\zeta}\right)_{\ell} < 0$,

If $q^*_{\ell-1/2} > \left(q_{\max}\right)_{\ell-1/2}$ $\begin{cases} q_{\ell-1/2} = \left(q_{\max}\right)_{\ell-1/2} \\ q_{\ell+1/2} = q^*_{\ell+1/2} + \left[q^*_{\ell-1/2} - \left(q_{\max}\right)_{\ell-1/2}\right]\mathrm{m}_{\ell-1/2}(\delta\zeta)_{\ell-1/2} \Big/ \mathrm{m}_{\ell+1/2}(\delta\zeta)_{\ell+1/2} \end{cases}$

where $q_{L+1/2} \equiv q^{FA}_{L+1/2}$.

For $\ell = 1,2,\cdots,L$ and if $\left(\mathrm{m}\dot{\zeta}\right)_\ell < 0$,

$$\text{If } q^*_{\ell+1/2} < \left(q_{\min}\right)_{\ell+1/2} \quad \begin{cases} q_{\ell-1/2} = q^*_{\ell-1/2} - \left[\left(q_{\min}\right)_{\ell+1/2} - q^*_{\ell+1/2}\right]\mathrm{m}_{\ell+1/2}\left(\delta\zeta\right)_{\ell+1/2} \big/ \mathrm{m}_{\ell-1/2}\left(\delta\zeta\right)_{\ell-1/2} \\ q_{\ell+1/2} = \left(q_{\min}\right)_{\ell+1/2} \end{cases}$$

where $q_{L+1/2} \equiv q^{FA}_{L+1/2}$.

During the correction process, no modification is made to the values if $\left(q_{\min}\right)_{\ell+1/2} \le q^*_{\ell+1/2} \le \left(q_{\max}\right)_{\ell+1/2}$ {i.e. $q_{\ell+1/2} = q^*_{\ell+1/2}$}.

1.5.g. Momentum equation

The vertically discrete momentum equation applied to the model layers is given by

$$\frac{\partial \mathbf{v}_\ell}{\partial t} + \mathbf{v}_\ell \cdot \nabla \mathbf{v}_\ell + \left(\dot{\zeta}\frac{\partial \mathbf{v}}{\partial \zeta}\right)_\ell = -\left(\nabla_p \Phi\right)_\ell - f\,\mathbf{k}\times\mathbf{v}_\ell \quad \text{for } \ell = 2,3,...,L\,, \tag{59}$$

where $\mathbf{v}$ is the horizontal velocity, f is the Coriolis parameter, $\mathbf{k}$ is the unit vertical vector. The vertical advection of momentum is defined by

$$\left.\begin{aligned}
&\left(\dot{\zeta}\frac{\partial \mathbf{v}}{\partial \zeta}\right)_1 \equiv \frac{1}{2\mathrm{m}_1\left(\delta\zeta\right)_1}\left(\mathbf{v}_2 - \mathbf{v}_1\right)\left(\mathrm{m}\dot{\zeta}\right)_{3/2} && \text{(60a)} \\
&\left(\dot{\zeta}\frac{\partial \mathbf{v}}{\partial \zeta}\right)_\ell \equiv \frac{1}{2\mathrm{m}_\ell\left(\delta\zeta\right)_\ell}\left[\left(\mathbf{v}_{\ell+1} - \mathbf{v}_\ell\right)\left(\mathrm{m}\dot{\zeta}\right)_{\ell+1/2} + \left(\mathbf{v}_\ell - \mathbf{v}_{\ell-1}\right)\left(\mathrm{m}\dot{\zeta}\right)_{\ell-1/2}\right] && \text{(60b)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ell = 2,3,...,L-1 && \text{(60c)} \\
&\left(\dot{\zeta}\frac{\partial \mathbf{v}}{\partial \zeta}\right)_L \equiv \frac{1}{\mathrm{m}_L\left(\delta\zeta\right)_L}\left[\left(\mathbf{v}_{L+1/2} - \mathbf{v}_L\right)\left(\mathrm{m}\dot{\zeta}\right)_B + \frac{1}{2}\left(\mathbf{v}_L - \mathbf{v}_{L-1}\right)\left(\mathrm{m}\dot{\zeta}\right)_{L-1/2}\right] && \text{(60d)}
\end{aligned}\right\},$$

where $\mathbf{v}_{L+1/2} = \mathbf{v}_{B^+} \equiv f^+\left(\mathbf{v}_L, \mathbf{v}_{L-1}\right)$ for $\left(\mathrm{m}\dot{\zeta}\right)_B < 0$, which is an extrapolation from above, and $\mathbf{v}_{L+1/2} = \mathbf{v}_{B^-} \equiv f^-\left(\mathbf{v}_{L+2}, \mathbf{v}_{L+1}\right)$ for $\left(\mathrm{m}\dot{\zeta}\right)_B > 0$, which is an extrapolation from PBL. Currently, we are using $\mathbf{v}_{B^+} \equiv \mathbf{v}_L$ and $\mathbf{v}_{B^-} \equiv \mathbf{v}_{L+1}$.

1.5.h. Horizontal pressure gradient force

The first term on the right hand side of (59) is the pressure gradient force given by

$$-\left(\nabla_p \Phi\right)_\ell = -\nabla M_\ell + \Pi_\ell \nabla \theta_\ell , \tag{61}$$

where the Montgomery potential is given by $M_\ell \equiv \Pi_\ell \theta_\ell + \Phi_\ell$.

1.5.i. Hydrostatic equation

The vertically discrete hydrostatic equation is given by

$$\Phi_L = \Phi_{L+1/2} + \left(\Pi_{L+1/2} - \Pi_L\right) \theta^{FA}_{L+1/2} \tag{62a}$$

and

$$\Phi_{\ell-1} = \Phi_\ell + \left(\Pi_\ell - \Pi_{\ell-1/2}\right)\theta_\ell + \left(\Pi_{\ell-1/2} - \Pi_{\ell-1}\right)\theta_{\ell-1} \quad \text{for} \quad \ell = L, L-1, ..., 1 . \tag{62b}$$

In (62a),$\Phi_B \equiv \Phi_{L+1/2}$ is obtained by vertically integrating hydrostatic equation within the PBL starting from the Surface. $\theta^{FA}_{L+1/2}$ is the predicted potential temperature at the lowest interface of the free atmosphere, which is also expressed as θ_{B^+} and $\Pi_B \equiv \Pi_{L+1/2}$ is the PBL-top Exner function. The influence of the moisture on the geopotential height can be included in (62a) and (62b) by replacing the potential temperature by the virtual potential temperature given by $\theta_v \equiv \theta(1 + 0.608q)$. The geopotential height at the interfaces can be calculated from

$$\Phi_{\ell-1/2} = \Phi_\ell + \left(\Pi_\ell - \Pi_{\ell-1/2}\right)\theta_\ell \quad \text{for} \quad \ell = L, L-1, ..., 1 . \tag{63}$$

or

$$\Phi_{\ell+1/2} = \Phi_\ell - \left(\Pi_{\ell+1/2} - \Pi_\ell\right)\theta_\ell \quad \text{for} \quad \ell = L, L-1, ..., 1 . \tag{64}$$

2-Vertical discretization in the PBL

This section describes the vertical discretization in the PBL. The PBL consists of multiple layers between the free atmosphere of the model and Earth's surface. A shared coordinate surface, referred as PBL-top, separates the free atmosphere from the PBL. The height of the PBL-top is predicted through a mass budget equation for the PBL. The mass budget of the PBL is primarily controlled by the PBL-top entrainment and horizontal mass convergence within the PBL. Konor and Arakawa (2000) discuss the rational behind the multi-layer PBL approach. The vertical discretization within the PBL follows Arakawa and Konor (1996), which describe a sigma vertical coordinate model with a Charney-Phillips type vertical grid.

2.1-Vertical coordinate in the PBL

A sigma type vertical coordinate is used in the PBL portion of model, which is given by

$$\zeta \equiv \zeta_B \left[1 - (1-\gamma)\left(\frac{p - p_B}{p_S - p_B} \right) \right] \tag{65}$$

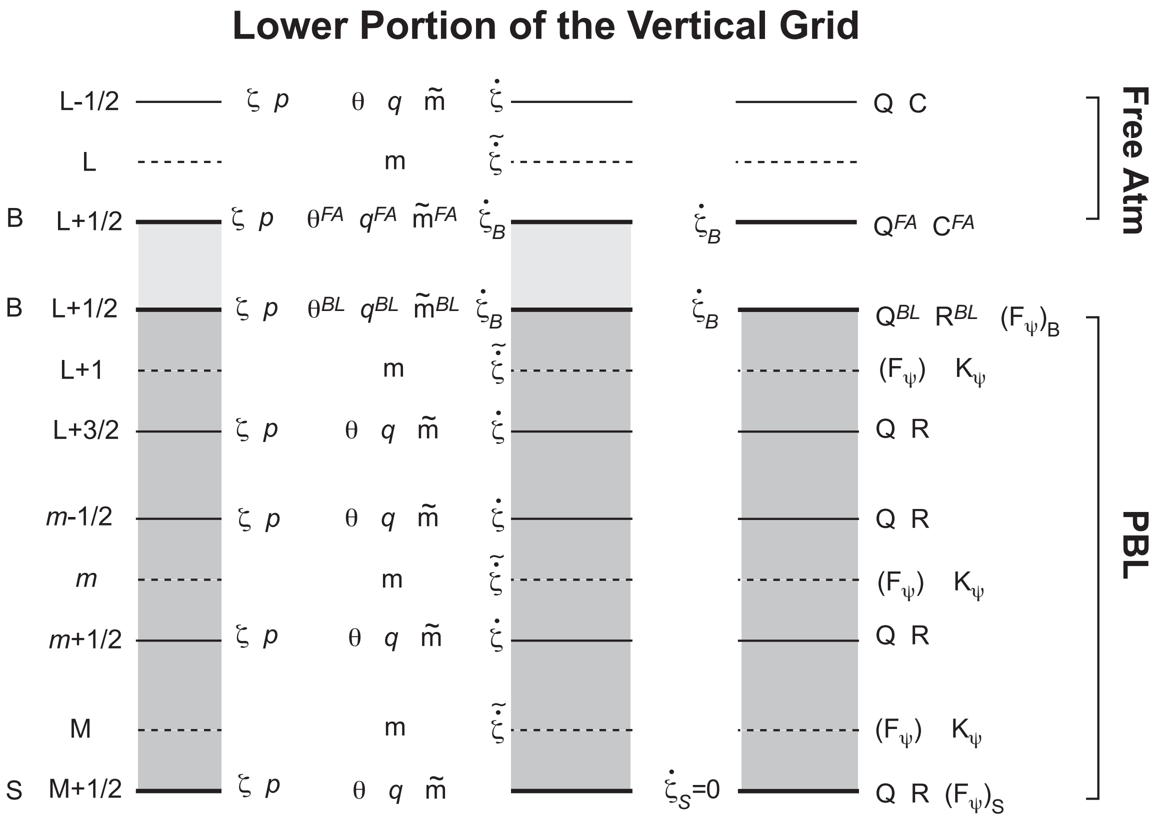
with $\zeta_S = \gamma \zeta_B$ and $0 < \gamma < 1$. The constant PBL mass is written by

$$\mathrm{m} \equiv -\frac{\partial p}{\partial \zeta} = \frac{p_S - p_B}{\zeta_B - \zeta_S} \tag{66}$$

and $p = p_B + \mathrm{m}(\zeta_B - \zeta)$.

In the model, we first prescribe $\theta(y,p)$ field between $p_S(y)$ and p_T to be used in the determination of the ζ coordinate. Then we define $p_B(y)$ to represent the initial PBL-top. We then prescribe the function $g(\sigma)$ and obtain the corresponding $f(\sigma)$ as discussed in subsection 1.a. This gives us ζ_B and, with a prescribed γ, ζ_S. Form (65) and (66), we can obtain $\zeta = \zeta(p)$ and m, respectively, for the PBL.

2.2. Vertical grid in the PBL



2.3. Vertically discrete equations in the PBL

2.3.a. Continuity and Vertical mass flux Equations

The continuity equation within the PBL is given by

$$\frac{\partial \mathrm{m}}{\partial t} = \frac{1}{(\delta\zeta)_{PBL}}\left[-\sum_{k=L+1}^{M}\nabla\cdot(\mathrm{m}\mathbf{v})_k(\delta\zeta)_k + \left(\mathrm{m}\dot{\zeta}\right)_B\right]. \tag{67}$$

Here we make following definitions

$$\left.\begin{array}{l} (\delta p)_m \equiv p_{m+1/2} - p_{m-1/2} \\ (\delta p)_{PBL} \equiv p_S - p_B \\ (\delta\zeta)_m \equiv \zeta_{m+1/2} - \zeta_{m-1/2} \\ (\delta\zeta)_{PBL} \equiv \zeta_S - \zeta_B \end{array}\right\}, \tag{68}$$

where we used m as vertical index. From (67) and (68), we can obtain pressure tendency equations within the PBL as

$$\left.\begin{aligned}\frac{\partial p_{m+1/2}}{\partial t}&=\frac{\partial p_B}{\partial t}+\sum_{k=L+1}^{m}\nabla\cdot(m\mathbf{v})_k(\delta\zeta)_k+\left(m\dot{\zeta}\right)_{m+1/2}-\left(m\dot{\zeta}\right)_B\\ \frac{\partial p_S}{\partial t}&=\frac{\partial p_B}{\partial t}+\sum_{k=L+1}^{M}\nabla\cdot(m\mathbf{v})_k(\delta\zeta)_k-\left(m\dot{\zeta}\right)_B\end{aligned}\right\}. \tag{69}$$

The PBL-top mass flux is determined from the entrainment/detrainment and cumulus mass flux by

$$\left(m\dot{\zeta}\right)_B=-g\left(E-M_B\right), \tag{70}$$

where negative E corresponds to detrainment. A limiting procedure is applied to $\left(m\dot{\zeta}\right)_B$ [or $\left(m\dot{\zeta}\right)_{L+1/2}$] to prevent the PBL to get very deep or shallow. It is assumed that the limiting procedure modifies E while M_B remains unchanged. The equation that determines the vertical mass flux within the PBL is written as

$$\left(m\dot{\zeta}\right)_{m+1/2}=\frac{\left(\zeta_S-\zeta_{m+1/2}\right)}{\left(\zeta_S-\zeta_B\right)}\left(m\dot{\zeta}\right)_B+\frac{\left(\zeta_{m+1/2}-\zeta_B\right)}{\left(\zeta_S-\zeta_B\right)}\sum_{k=L+1}^{M}\nabla\cdot(m\mathbf{v})_k(\delta\zeta)_k-\sum_{k=L+1}^{m}\nabla\cdot(m\mathbf{v})_k(\delta\zeta)_k. \tag{71}$$

2.3.b. Thermodynamic Equation

At the lowest interface of the free-atmosphere (*FA*), the vertically discrete thermodynamic equation is written by

$$\left[m_{L+1/2}\frac{\partial\theta_{L+1/2}}{\partial t}+(m\mathbf{v})_{L+1/2}\cdot\nabla\theta_{L+1/2}\right]^{FA}+\left(m\dot{\zeta}\frac{\partial\theta}{\partial\zeta}\right)^{FA}_{L+1/2}=\frac{(mQ)^{FA}_{L+1/2}}{\Pi_{L+1/2}}, \tag{72}$$

where

$$\left(m\dot{\zeta}\frac{\partial\theta}{\partial\zeta}\right)^{FA}_{L+1/2}\equiv\frac{1}{(\delta\zeta)^{FA}_{L+1/2}}\left\{\left(\hat{\theta}_{L+1/2}-\theta^{FA}_{L+1/2}\right)\left[\left(m\dot{\zeta}\right)_{L+1/2}-gM_B\right]+\left(\theta^{BL}_{L+1/2}-\theta^{FA}_{L+1/2}\right)gM_B+\left(\theta^{FA}_{L+1/2}-\theta_L\right)\left(m\dot{\zeta}\right)_L\right\} \tag{73}$$

and

$$\left.\begin{aligned}\hat{\theta}_{L+1/2}&\equiv\theta^{FA}_{L+1/2}\quad\text{if}\quad\left(m\dot{\zeta}\right)_{L+1/2}-gM_B<0\\ \hat{\theta}_{L+1/2}&\equiv\theta^{BL}_{L+1/2}\quad\text{if}\quad\left(m\dot{\zeta}\right)_{L+1/2}-gM_B>0\end{aligned}\right\}. \tag{74}$$

In (73) and (74), we used upstream treatment for the exchange associated with M_B. On the other hand, at the Upper most interface of the PBL (*BL*), the vertically discrete thermodynamic equation is written by

$$\left[m\frac{\partial\theta_{L+1/2}}{\partial t}+(m\mathbf{v})_{L+1/2}\cdot\nabla\theta_{L+1/2}\right]^{BL}+\left(m\dot{\zeta}\frac{\partial\theta}{\partial\zeta}\right)^{BL}_{L+1/2}$$
$$=\frac{(mQ)^{BL}_{L+1/2}}{\Pi_{L+1/2}}-\frac{g}{(\delta\zeta)^{BL}_{L+1/2}}\left[(F_\theta)_{L+1}-(F_\theta)_{L+1/2}\right]+(G_\theta)_{L+1/2}, \qquad (75)$$

where

$$\left(m\dot{\zeta}\frac{\partial\theta}{\partial\zeta}\right)^{BL}_{L+1/2}\equiv\frac{1}{(\delta\zeta)^{BL}_{L+1/2}}\left\{\left(\theta_{L+1}-\theta^{BL}_{L+1/2}\right)\left(m\dot{\zeta}\right)_{L+1}+\left(\theta^{BL}_{L+1/2}-\hat{\theta}_{L+1/2}\right)\left[\left(m\dot{\zeta}\right)_{L+1/2}-gM_B\right]\right\}. \qquad (76)$$

$$\left.\begin{aligned}
&(m\mathbf{v})^{BL}_{L+1/2}\equiv m\mathbf{v}^{BL}_{L+1/2}\\
&\mathbf{v}^{BL}_{L+1/2}\equiv\mathbf{v}_{L+1}\\
&(\delta\zeta)^{BL}_{L+1/2}\equiv\frac{1}{2}(\delta\zeta)_{L+1}\\
&\left(m\dot{\zeta}\right)_{L+1}\equiv\frac{1}{2}\left[\left(m\dot{\zeta}\right)_{L+3/2}+\left(m\dot{\zeta}\right)_{L+1/2}\right]
\end{aligned}\right\}. \qquad (77)$$

In (76), we used upstream treatment for the exchange associated M_B. Note that F_θ and G_θ in (75) represent turbulent fluxes and other PBL processes in the PBL. They should not be mistaken by the functions used in the definition of the vertical coordinate in subsection 1.a and 1.b.

Within the PBL, the vertically discrete thermodynamic equation is written by

$$\left[m\frac{\partial\theta_{m+1/2}}{\partial t}+(m\mathbf{v})_{m+1/2}\cdot\nabla\theta_{m+1/2}\right]+\left(m\dot{\zeta}\right)_{m+1/2}\left(\frac{\partial\theta}{\partial\zeta}\right)_{m+1/2}$$
$$=\frac{(mQ)_{m+1/2}}{\Pi_{m+1/2}}-\frac{g}{(\delta\zeta)_{m+1/2}}\left[(F_\theta)_{m+1}-(F_\theta)_{m}\right]+(G_\theta)_{m+1/2} \quad \text{for} \quad m=L+1, L+2,\cdots, M-1, \qquad (78)$$

where

$$\left(m\dot{\zeta}\right)_{m+1/2}\left(\frac{\partial\theta}{\partial\zeta}\right)_{m+1/2} \equiv \frac{1}{(\delta\zeta)_{m+1/2}}\left[\frac{\Pi_{m+3/2}\theta_{m+3/2}-\Pi_{m-1/2}\theta_{m-1/2}}{2\Pi_{m+1/2}}-\left(p_{m+3/2}-p_{m-1/2}\right)\frac{\kappa\Pi_{m+1/2}}{2p_{m+1/2}}\theta_{m+1/2}\right]\left(m\dot{\zeta}\right)_{m+1/2}$$

for $m = L+1, L+2, \cdots, M-1$, (79)

$$\left.\begin{aligned}(m\mathbf{v})_{m+1/2} &\equiv \frac{1}{2(\delta\zeta)_{m+1/2}}\left[(m\mathbf{v})_{m+1}(\delta\zeta)_{m+1}+(m\mathbf{v})_{m}(\delta\zeta)_{m}\right]\\ (\delta\zeta)_{m+1/2} &\equiv \frac{1}{2}\left[(\delta\zeta)_{m+1}+(\delta\zeta)_{m}\right]\end{aligned}\right\}\text{ for } m = L+1, L+2, \cdots, M-1, \quad (80)$$

At the lowest interface of the PBL, the vertically discrete thermodynamic equation is written by

$$m\frac{\partial\theta_{M+1/2}}{\partial t}+(m\mathbf{v})_{M+1/2}\cdot\nabla\theta_{M+1/2} = \frac{(mQ)_{M+1/2}}{\Pi_{M+1/2}}-\frac{g}{(\delta\zeta)_{M+1/2}}\left[\left(F_\theta\right)_{M+1/2}-\left(F_\theta\right)_{M}\right]+\left(G_\theta\right)_{M+1/2}, \quad (81)$$

where

$$\left.\begin{aligned}(m\mathbf{v})_{M+1/2} &\equiv m\mathbf{v}_{M+1/2}\\ \mathbf{v}_{M+1/2} &\equiv \mathbf{v}_{M}\\ (\delta\sigma)_{M+1/2} &\equiv \frac{1}{2}(\delta\sigma)_{M}\end{aligned}\right\}. \quad (82)$$

<u>2.3.c. Moisture Equation</u>

The moisture equation applied to the upper most interface of the PBL (BL) can be given by

$$\left[\frac{\partial(mq)_{L+1/2}}{\partial t}+\nabla\cdot(qm\mathbf{v})_{L+1/2}\right]^{BL}+\frac{1}{(\delta\zeta)^{BL}_{L+1/2}}\left\{\left(qm\dot{\zeta}\right)_{L+1}-\hat{q}_{L+1/2}\left[\left(m\dot{\zeta}\right)_{L+1/2}-gM_B\right]+q^{BL}_{L+1/2}gM_B\right\}$$

$$=-(mC)_{L+1/2}-\frac{g}{(\delta\zeta)^{BL}_{L+1/2}}\left[\left(F_q\right)_{L+1}-\left(F_q\right)_{L+1/2}\right]+\left(G_q\right)_{L+1/2}, \quad (83)$$

where

$$(qm\mathbf{v})^{BL}_{L+1/2} \equiv q_{L+1/2}m\mathbf{v}^{BL}_{L+1/2}, \quad (84a)$$

$$\left.\begin{aligned}\left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{L+1}} &\equiv \mathrm{q}_{\mathrm{L+3/2}}\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{L+1}} \quad \text{if } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{L+1}} > 0\\ \left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{L+1}} &\equiv \mathrm{q}^{\mathrm{BL}}_{\mathrm{L+1/2}}\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{L+1}} \quad \text{if } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{L+1}} < 0\end{aligned}\right\} \tag{84b}$$

and

$$\left.\begin{aligned}\hat{\mathrm{q}}_{\mathrm{L+1/2}} &\equiv \mathrm{q}^{\mathrm{FA}}_{\mathrm{L+1/2}} \qquad \text{if} \quad \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{L+1/2}} - \mathrm{gM_B} < 0\\ \hat{\mathrm{q}}_{\mathrm{L+1/2}} &\equiv \mathrm{q}^{\mathrm{BL}}_{\mathrm{L+1/2}} \qquad \text{if} \quad \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{L+1/2}} - \mathrm{gM_B} > 0\end{aligned}\right\}. \tag{84c}$$

Note that $\mathrm{F_q}$ and $\mathrm{G_q}$ in (83) represent turbulent fluxes and additional PBL processes in the PBL. They are different from the functions used in the definition of the vertical coordinate in subsection 1.a and 1.b. The predicted $\mathrm{q}^{\mathrm{BL}}_{\mathrm{L+1/2}}$ is further corrected using a procedure similar to the one described for the free atmosphere. We will discuss this procedure later in this text.

The moisture equation applied to the interfaces within the PBL can be given by

$$\frac{\partial(\mathrm{mq})_{\ell+1/2}}{\partial t} + \nabla\cdot\left(\mathrm{qm}\mathbf{v}\right)_{\mathrm{m+1/2}} + \frac{1}{(\delta\zeta)_{\mathrm{m+1/2}}}\left[\left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{m+1}} - \left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{m}}\right]$$
$$= -(\mathrm{mC})_{\mathrm{m+1/2}} - \frac{\mathrm{g}}{(\delta\zeta)_{\mathrm{m+1/2}}}\left[\left(\mathrm{F_q}\right)_{\mathrm{m+1}} - \left(\mathrm{F_q}\right)_{\mathrm{m}}\right] + \left(\mathrm{G_q}\right)_{\mathrm{m+1/2}}, \tag{85}$$

where

$$\left(\mathrm{qm}\mathbf{v}\right)_{\mathrm{m+1/2}} \equiv \frac{\mathrm{q_{m+1/2}}}{(\delta\zeta)_{\mathrm{m+1/2}}}\left[\mathrm{m}\mathbf{v}_{\mathrm{m+1}}(\delta\zeta)_{\mathrm{m+1}} + \mathrm{m}\mathbf{v}_{\mathrm{m}}(\delta\zeta)_{\mathrm{m}}\right] \tag{86a}$$

and

$$\left.\begin{aligned}\left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{m}} &\equiv \mathrm{q}_{\mathrm{m+1/2}}\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{m}} \quad \text{if } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{m}} > 0\\ \left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{m}} &\equiv \mathrm{q}_{\mathrm{m-1/2}}\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{m}} \quad \text{if } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{m}} < 0\end{aligned}\right\}. \tag{86b}$$

In (86b), $\left(\mathrm{m}\dot{\zeta}\right)_{m} \equiv \frac{1}{2}\left\{\left(\mathrm{m}\dot{\zeta}\right)_{m+1/2} + \left(\mathrm{m}\dot{\zeta}\right)_{m-1/2}\right\}$ for $m = \mathrm{L}+1, \mathrm{L}+2, \cdots, \mathrm{M}-1$. The predicted q is further corrected using a procedure similar to the one described for the free atmosphere. We will discuss this procedure later in this text. The moisture equation applied to the lowest interface of the PBL can be given by

$$\frac{\partial(\mathrm{mq})_{\mathrm{M+1/2}}}{\partial t} + \nabla\cdot\left(\mathrm{qm}\mathbf{v}\right)_{\mathrm{M+1/2}} + \frac{1}{(\delta\zeta)_{\mathrm{M+1/2}}}\left[\mathrm{q}_{\mathrm{M+1/2}}\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{M+1/2}} - \left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{M}}\right]$$

$$= -(\mathrm{mC})_{\mathrm{M}+1/2} - \frac{\mathrm{g}}{(\delta\zeta)_{\mathrm{M}+1/2}}\left[\left(\mathrm{F_q}\right)_{\mathrm{M}+1/2} - \left(\mathrm{F_q}\right)_{\mathrm{M}}\right] + \left(\mathrm{G_q}\right)_{\mathrm{M}+1/2} \tag{87}$$

where

$$\left(\mathrm{qm}\mathbf{v}\right)_{\mathrm{M}+1/2} \equiv \mathrm{q}_{\mathrm{M}+1/2}\mathrm{m}\mathbf{v}_{\mathrm{M}} \tag{88a}$$

and

$$\left.\begin{aligned}\left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{M}} &\equiv \mathrm{q}_{\mathrm{M}+1/2}\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{M}} \ \text{ if } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{m}} > 0 \\ \left(\mathrm{qm}\dot{\zeta}\right)_{\mathrm{M}} &\equiv \mathrm{q}_{\mathrm{M}-1/2}\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{M}} \ \text{ if } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{m}} < 0\end{aligned}\right\}. \tag{88b}$$

In (88b), $\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{M}} \equiv \frac{1}{2}\left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{M}-1/2}$. A correction procedure on the predicted $\mathrm{q}_{\mathrm{M}+1/2}$ is discussed below.

2.3.d. Correction step of moisture prediction

The estimated max and min values at the upper most level of the PBL are

$$\left.\begin{aligned}\left(\mathrm{q}_{max}\right)^{\mathrm{BL}}_{\mathrm{L}+1/2} &= \mathrm{Max}\left\{\mathrm{q}_{\mathrm{L}+3/2}, \mathrm{q}^{\mathrm{BL}}_{\mathrm{L}+1/2}\right\} \qquad \text{for } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{B}} > 0 \\ \left(\mathrm{q}_{max}\right)^{\mathrm{BL}}_{\mathrm{L}+1/2} &= \mathrm{Max}\left\{\mathrm{q}^{\mathrm{BL}}_{\mathrm{L}+1/2}, \mathrm{q}^{\mathrm{FA}}_{\mathrm{L}+1/2}\right\} \qquad \text{for } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{B}} < 0\end{aligned}\right\}, \tag{89a}$$

$$\left.\begin{aligned}\left(\mathrm{q}_{min}\right)^{\mathrm{BL}}_{\mathrm{L}+1/2} &= \mathrm{Max}\left\{\mathrm{q}_{\mathrm{L}+3/2}, \mathrm{q}^{\mathrm{FA}}_{\mathrm{L}+1/2}\right\} \qquad \text{for } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{B}} > 0 \\ \left(\mathrm{q}_{min}\right)^{\mathrm{BL}}_{\mathrm{L}+1/2} &= \mathrm{Max}\left\{\mathrm{q}^{\mathrm{FA}}_{\mathrm{L}+1/2}, \mathrm{q}^{\mathrm{BL}}_{\mathrm{L}+1/2}\right\} \qquad \text{for } \left(\mathrm{m}\dot{\zeta}\right)_{\mathrm{B}} < 0\end{aligned}\right\}. \tag{89b}$$

The estimated max and min values at the interfaces within the PBL are

$$\left.\begin{aligned}\left(\mathrm{q}_{max}\right)_{m+1/2} &= \mathrm{Max}\left\{\mathrm{q}_{m+1/2}, \mathrm{q}_{m+3/2}\right\} \ \text{ for } \left(\mathrm{m}\dot{\zeta}\right)_{m+1} > 0 \\ \left(\mathrm{q}_{max}\right)_{m+1/2} &= \mathrm{Max}\left\{\mathrm{q}_{m+1/2}, \mathrm{q}_{m-1/2}\right\} \ \text{ for } \left(\mathrm{m}\dot{\zeta}\right)_{m} < 0\end{aligned}\right\} \text{ for } m = \mathrm{L}+1, \mathrm{L}+2, \cdots, \mathrm{M}-1, \tag{89c}$$

$$\left.\begin{aligned}\left(\mathrm{q}_{min}\right)_{m+1/2} &= \mathrm{Min}\left\{\mathrm{q}_{m+1/2}, \mathrm{q}_{m+3/2}\right\} \ \text{ for } \left(\mathrm{m}\dot{\zeta}\right)_{m+1} > 0 \\ \left(\mathrm{q}_{min}\right)_{m+1/2} &= \mathrm{Min}\left\{\mathrm{q}_{m+1/2}, \mathrm{q}_{m-1/2}\right\} \ \text{ for } \left(\mathrm{m}\dot{\zeta}\right)_{m} < 0\end{aligned}\right\} \text{ for } m = \mathrm{L}+1, \mathrm{L}+2, \cdots, \mathrm{M}-1. \tag{89d}$$

In (89c) and (89d), $q_{L+1/2} \equiv q_{L+1/2}^{BL}$. At the lower boundary, the estimated max and min values are given by

$$\left.\begin{array}{ll} (q_{max})_{M+1/2} = (q_{min})_{M+1/2} = q_{M+1/2} & \text{for } (m\dot{\zeta})_M > 0 \\ \left.\begin{array}{l}(q_{max})_{M+1/2} = \mathrm{Max}\{q_{M+1/2}, q_{M-1/2}\} \\ (q_{min})_{M+1/2} = \mathrm{Min}\{q_{M+1/2}, q_{M-1/2}\}\end{array}\right\} & \text{for } (m\dot{\zeta})_M < 0 \end{array}\right\} \tag{89e}$$

If $(m\dot{\zeta})_M > 0$:

If $q^*_{M+1/2} > (q_{max})_{M+1/2}$ $\begin{cases} q_{M-1/2} = q^*_{M-1/2} + \left[q^*_{M+1/2} - (q_{min})_{M+1/2}\right](\delta\zeta)_{M+1/2}\Big/(\delta\zeta)_{M-1/2} \\ q_{M+1/2} = (q_{max})_{M+1/2} \end{cases}$

For $m = M-1, M-2, \cdots, L+1$ and if $(m\dot{\zeta})_m > 0$,

If $q^*_{m+1/2} > (q_{max})_{m+1/2}$ $\begin{cases} q_{m-1/2} = q^*_{m-1/2} + \left[q^*_{m+1/2} - (q_{min})_{m+1/2}\right](\delta\zeta)_{\ell+1/2}\Big/(\delta\zeta)_{\ell-1/2} \\ q_{m+1/2} = (q_{max})_{m+1/2} \end{cases}$

where $q_{L+1/2} \equiv q^{FA}_{L+1/2}$.

If $(m\dot{\zeta})_{L+1/2} > 0$:

If $q^{*BL}_{L+1/2} > (q_{max})^{BL}_{L+1/2}$ $\begin{cases} q^{FA}_{L+1/2} = q^{*FA}_{L+1/2} + \left[q^{*FA}_{L+1/2} - (q_{min})^{FA}_{L+1/2}\right] m(\delta\zeta)^{BL}_{L+1/2}\Big/ m^{FA}_{L+1/2}(\delta\zeta)^{FA}_{L+1/2} \\ q^{BL}_{L+1/2} = (q_{max})^{BL}_{L+1/2} \end{cases}$

If $(m\dot{\zeta})_{L+1/2} > 0$,

If $q^{*FA}_{L+1/2} < (q_{min})^{FA}_{L+1/2}$ $\begin{cases} q^{FA}_{L+1/2} = (q_{min})^{FA}_{L+1/2} \\ q^{*BL}_{L+1/2} = q^{BL}_{\ell+1/2} - \left[(q_{min})^{FA}_{L+1/2} - q^{*FA}_{L+1/2}\right] m^{FA}_{L+1/2}(\delta\zeta)^{FA}_{L+1/2}\Big/ m(\delta\zeta)^{BL}_{L+1/2} \end{cases}$

For $m = L+1, \cdots, M-2, M-1$ and if $(m\dot{\zeta})_m > 0$, If $q^*_{m+1/2} < \left(q_{min}\right)_{m+1/2}$ $\begin{cases} q_{m+1/2} = q^*_{m+1/2} + \left[q^*_{m-1/2} - \left(q_{min}\right)_{m-1/2}\right](\delta\zeta)_{m-1/2} \Big/ (\delta\zeta)_{m+1/2} \\ q_{m-1/2} = \left(q_{max}\right)_{m-1/2} \end{cases}$

where $q_{L+1/2} \equiv q^{FA}_{L+1/2}$.

For $m = \mathrm{L}+1, \mathrm{L}+2, \cdots, \mathrm{M}-1$ and if $\left(\mathrm{m}\dot{\zeta}\right)_m < 0$,

If $q^*_{m+1/2} < \left(q_{min}\right)_{m+1/2}$ $\begin{cases} q_{m-1/2} = q^*_{m-1/2} - \left[\left(q_{min}\right)_{m+1/2} - q^*_{m+1/2}\right](\delta\zeta)_{m+1/2} \Big/ (\delta\zeta)_{m-1/2} \\ q_{m+1/2} = \left(q_{min}\right)_{m+1/2} \end{cases}$

where $q_{L+1/2} \equiv q^{FA}_{L+1/2}$.

For $m = \mathrm{L}+1, \mathrm{L}+2, \cdots, \mathrm{M}-1$ and if $\left(\mathrm{m}\dot{\zeta}\right)_m < 0$,

If $q^*_{m-1/2} < \left(q_{min}\right)_{m-1/2}$ $\begin{cases} q_{m+1/2} = q^*_{m+1/2} - \left[\left(q_{min}\right)_{m-1/2} - q^*_{m-1/2}\right](\delta\zeta)_{m-1/2} \Big/ (\delta\zeta)_{m+1/2} \\ q_{m-1/2} = \left(q_{min}\right)_{m-1/2} \end{cases}$

2.3.e. Momentum Equation

The vertically discrete momentum equation applied to the layers in the PBL is given by

$$\left.\begin{aligned}
&\frac{\partial \mathbf{v}_{\mathrm{L}+1}}{\partial t} + \mathbf{v}_{\mathrm{L}+1}\cdot\nabla\mathbf{v}_{\mathrm{L}+1} + \left(\dot{\zeta}\frac{\partial \mathbf{v}}{\partial\zeta}\right)_{\mathrm{L}+1} = \\
&\qquad -\left(\nabla_p\Phi\right)_{\mathrm{L}+1} - f\,\mathbf{k}\times\mathbf{v}_{\mathrm{L}+1} - \frac{g}{\mathrm{m}(\delta\zeta)_{\mathrm{L}+1}}\left(\mathbf{F}_\mathrm{v}\right)_{\mathrm{L}+3/2} + \frac{\left(\mathbf{G}_\mathrm{v}\right)_{\mathrm{L}+1}}{\mathrm{m}} \qquad (76a) \\
&\frac{\partial \mathbf{v}_m}{\partial t} + \mathbf{v}_m\cdot\nabla\mathbf{v}_m + \left(\dot{\zeta}\frac{\partial \mathbf{v}}{\partial\zeta}\right)_m = \\
&\qquad -\left(\nabla_p\Phi\right)_m - +f\,\mathbf{k}\times\mathbf{v}_m - \frac{g}{\mathrm{m}(\delta\zeta)_m}\left\{\left(\mathbf{F}_\mathrm{v}\right)_{m+1/2} - \left(\mathbf{F}_\mathrm{v}\right)_{m-1/2}\right\} + \frac{\left(\mathbf{G}_\mathrm{v}\right)_m}{\mathrm{m}} \\
&\qquad\qquad m = \mathrm{L}+2,\ldots,\mathrm{M} \quad (76b)
\end{aligned}\right\},$$

where the vertical momentum advection is

$$\left.\begin{aligned}
\left(\dot{\zeta}\frac{\partial \mathbf{v}}{\partial \zeta}\right)_{L+1} &\equiv \frac{1}{m(\delta\zeta)_{L+1}}\left[\frac{1}{2}\left(\mathbf{v}_{L+2}-\mathbf{v}_{L+1}\right)\left(m\dot{\zeta}\right)_{L+3/2}+\left(\mathbf{v}_{L+1}-\mathbf{v}_{L+1/2}\right)\left(m\dot{\zeta}\right)_{B}\right] && \text{(77a)}\\
\left(\dot{\zeta}\frac{\partial \mathbf{v}}{\partial \zeta}\right)_{m} &\equiv \frac{1}{2m(\delta\zeta)_{m}}\left[\left(\mathbf{v}_{m+1}-\mathbf{v}_{m}\right)\left(m\dot{\zeta}\right)_{m+1/2}+\left(\mathbf{v}_{m}-\mathbf{v}_{m-1}\right)\left(m\dot{\zeta}\right)_{m-1/2}\right] \\
&\qquad \text{for } m = L+1,..,M-1 && \text{(77b)}\\
\left(\dot{\zeta}\frac{\partial \mathbf{v}}{\partial \zeta}\right)_{M} &\equiv \frac{1}{2m(\delta\zeta)_{M}}\left(\mathbf{v}_{M}-\mathbf{v}_{M-1}\right)\left(m\dot{\zeta}\right)_{M-1/2} && \text{(77c)}
\end{aligned}\right\}.$$

In (77a), we formally define $\mathbf{v}_{L+1/2} = \mathbf{v}_{B^+} \equiv f^+(\mathbf{v}_L, \mathbf{v}_{L-1})$ for $\left(m\dot{\zeta}\right)_B < 0$, which must be an extrapolation from above, and $\mathbf{v}_{L+1/2} = \mathbf{v}_{B^-} \equiv f^-(\mathbf{v}_{L+2}, \mathbf{v}_{L+1})$ for $\left(m\dot{\zeta}\right)_B > 0$, which must be an extrapolation from PBL. Currently, we are using $\mathbf{v}_{B^+} \equiv \mathbf{v}_L$ and $\mathbf{v}_{B^-} \equiv \mathbf{v}_{L+1}$.

2.3.f. Horizontal pressure gradient force

On the right hand side (77a) and (77b), the vertically discrete pressure gradient force is given by

$$-\left(\nabla_p \Phi\right) = -\frac{1}{m}\nabla(m\Phi)_m + \frac{1}{\left(p_{m+1/2}-p_{m-1/2}\right)}\left[\Phi_{m+1/2}\nabla p_{m+1/2} - \Phi_{m-1/2}\nabla p_{m-1/2}\right]$$
$$\text{for} \quad m = L+1, L+2, \cdots, M \qquad (78)$$

2.3.g. Hydrostatic equation

The geopotential height within the PBL is determined by vertically summing the hydrostatic equation starting from the surface where $\Phi_{M+1/2} \equiv \Phi_S$ is prescribed. The vertically discrete hydrostatic equation is given by

$$\Phi_M = \Phi_{M+1/2} + \frac{\kappa\left(p_{M+1/2}-p_{M-1/2}\right)}{2p_{M+1/2}}\Pi_{M+1/2}\theta_{M+1/2}, \qquad (79a)$$

$$\Phi_m = \Phi_{m+1} + \frac{\kappa\left(p_{m+3/2}-p_{m-1/2}\right)}{2p_{m+1/2}}\Pi_{m+1/2}\theta_{m+1/2} \quad \text{for} \quad m = M-1, M-2, \cdots, L+1. \quad (79b)$$

At the top of the PBL, the geopotential height is calculated from

$$\Phi_{L+1/2} = \Phi_{L+1} + \frac{\kappa\left(p_{L+3/2} - p_{L+1/2}\right)}{2p_{L+1/2}} \Pi_{L+1/2} \theta^{BL}_{L+1/2}, \tag{79c}$$

where $\Phi_B \equiv \Phi_{L+1/2}$. At the interfaces of layers, the geopotential height can be calculated from

$$\Phi_{m-1/2} = \Phi_m + \frac{\kappa\left(p_{m+1/2} - p_{m-1/2}\right)}{2p_{m-1/2}} \Pi_{m-1/2} \theta_{m-1/2} \quad \text{for} \quad m = M, M-1, \cdots, L+2 \tag{79d}$$

or

$$\Phi_{m+1/2} = \Phi_m - \frac{\kappa\left(p_{m+1/2} - p_{m-1/2}\right)}{2p_{m+1/2}} \Pi_{m+1/2} \theta_{m+1/2} \quad \text{for} \quad m = M-1, M-2, \cdots, L+1 \tag{79e}$$