Implicit-Explicit Multistep Methods for Fast-Wave Slow-Wave Problems

Peter Blossey & Dale Durran

10 August 2011

Fast waves require small timesteps, unless you ...

- choose governing equations that prohibit fast waves (anelasitc, unified framework, etc.)
- split fast waves from rest of physics, so that they alone are integrated on small timesteps. (As in WRF)
- treat them implicitly. (ECMWF, Met Office, ...)

First and third options require the solution of a global (usually linear) system at each timestep.

In this talk, we are focused on the third option.

Fast waves: sound (and sometimes gravity) waves.
Slow waves: advection (and sometimes gravity) waves.

Splitting Up the PDE

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u}) + \mathbf{L}\mathbf{u},$$

where

- $\mathbf{u}$ is the state variable,
- $\mathbf{L}$ is the matrix associated with a linear operator modeling processes with short timescales (sound and possibly gravity waves)
- $\mathbf{f}(\mathbf{u})$ is everything else

IMEX Multistep Approximation

$$\sum_{k=-M}^{1} \alpha_k \mathbf{q}^{n+k} = \Delta t \left[\sum_{k=-M}^{0} \beta_k \mathbf{f}(\mathbf{q}^{n+k}) + \sum_{k=-M}^{1} \nu_k \mathbf{L} \mathbf{q}^{n+k} \right].$$

- $\mathbf{q}^n$ approximates $\mathbf{u}(n\Delta t)$
- (α_k, β_k) define the explicit method
- (α_k, ν_k) define the implicit scheme

Common in Atmospheric Models

Implicit: Trapezoidal over $2\Delta t$, backward weighted if $\theta > 0.5$

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^{n-1}}{2\Delta t} = \theta \mathbf{L}\mathbf{q}^{n+1} + (1 - \theta)\mathbf{L}\mathbf{q}^{n-1}$$

Explicit: Asselin-filtered leapfrog

$$\begin{aligned} \frac{\mathbf{q}^{n+1} - \tilde{\mathbf{q}}^{n-1}}{2\Delta t} &= \mathbf{f}(\mathbf{q}^n) \\ \tilde{\mathbf{q}}^n &= \mathbf{q}^n + \gamma \left(\tilde{\mathbf{q}}^{n-1} - 2\mathbf{q}^n + \mathbf{q}^{n+1} \right), \end{aligned}$$

where typically $0.05 \leq \gamma \leq 0.2$

Weaknesses

Asselin-filtered leapfrog-trapezoidal is

- first-order
- off-centering the trapezoidal method is not very scale selective
- leapfrog limits choice of spatial differences
- physical parameterizations treated with forward Euler over $2\Delta t$

Desired Properties of Alternate Schemes

Implicit schemes:

- *A*-stability (numerical approximations to

$$\frac{du}{dt} = \eta u$$

  satisfy $|A| \equiv |q^{n+1}/q^n| \leq 1$ if $\Re\{\eta\} \leq 0$)
  (But *A*-stable scheme can be no better than 2nd-order.)
- Damps high frequencies ($|A|$ becomes small as $\Im\{\eta\}\Delta t \rightarrow \pm\infty$).

Desired Properties of Alternate Schemes

Explicit schemes:

- Good stability for purely oscillatory phenomena (i.e. larger CFL limit).
- Able to handle both oscillatory and damping phenomena.

A family of implicit-explicit Adams methods

Building on Frank et al. 1997, consider a family of implicit Adams schemes (A-stable for $c \geq 0$):

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} = \frac{1}{2}\left[\mathbf{Lq}^{n+1} + \mathbf{Lq}^n\right] + \frac{c}{2}\left[\mathbf{Lq}^{n+1} - 2\mathbf{Lq}^n + \mathbf{Lq}^{n-1}\right]$$

Trapezoidal is $c = 0$.

Similarly, consider a family of three-step explicit Adams schemes:

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} = \frac{3}{2}\mathbf{f}(\mathbf{q}^n) - \frac{1}{2}\mathbf{f}\left(\mathbf{q}^{n-1}\right) + \frac{b}{2}\left[\mathbf{f}\left(\mathbf{q}^n\right) - 2\mathbf{f}\left(\mathbf{q}^{n-1}\right) + \mathbf{f}\left(\mathbf{q}^{n-2}\right)\right]$$

$b = 0$ gives 2nd order Adams-Bashforth, $b = 5/6$ gives AB3.

We seek combinations of b and c which yield good stability for fast wave-slow wave problems.

Model Problem 1: Oscillations Forced at Two-Frequencies

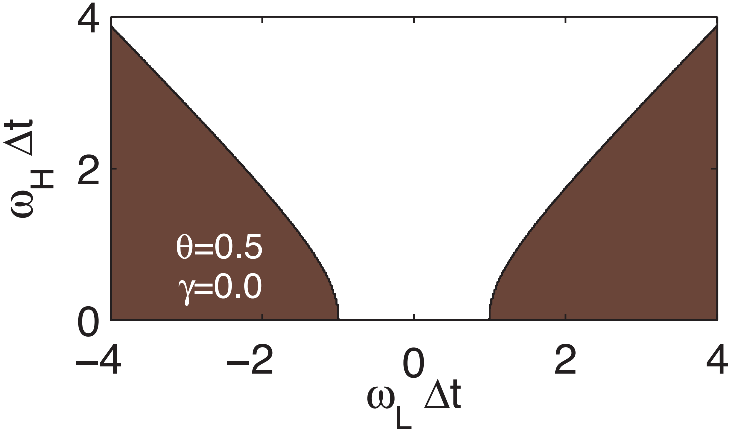
$$\frac{\partial q}{\partial t} = i\omega_L q + i\omega_H q$$

- ω_H is the high-frequency forcing.
- ω_L is the low-frequency forcing.

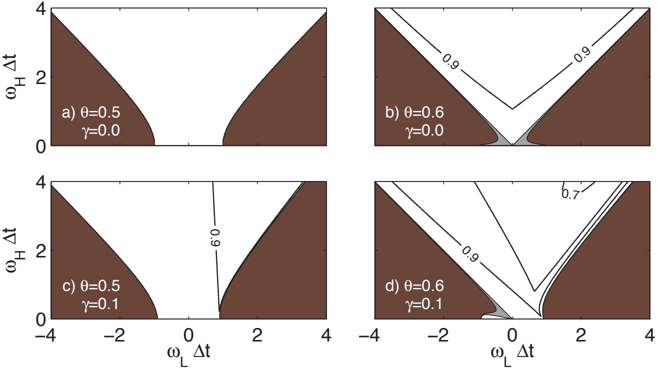
Leapfrog-trapezoidal amplification factor

No Asselin, no off-centering

$|A| \leq 1$ throughout white region

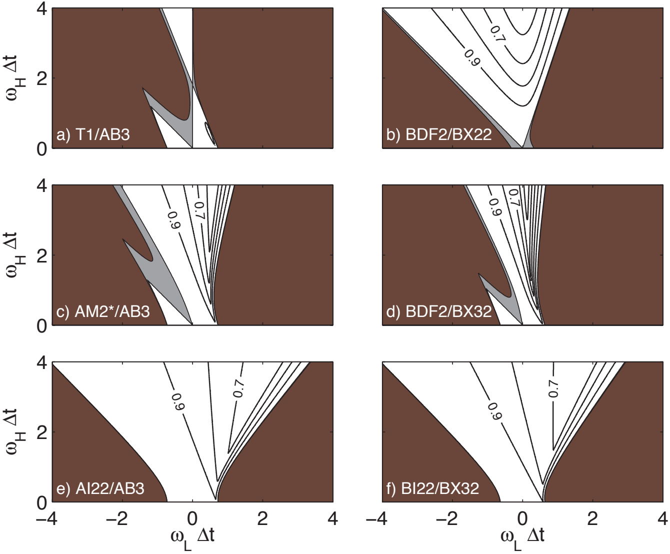


Influence of Asselin filtering and off-centering

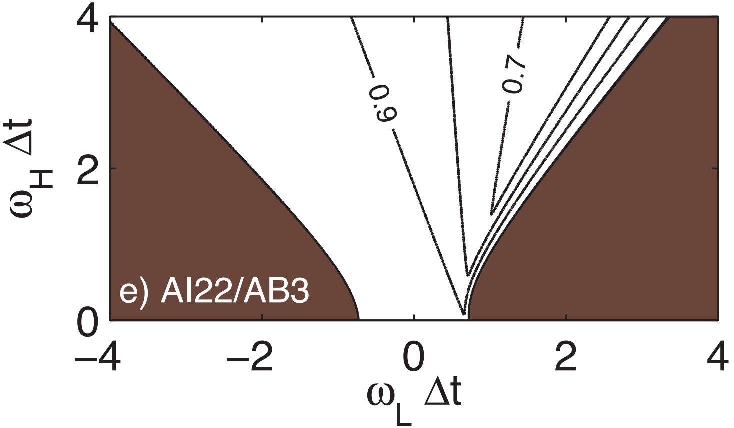


Off-centering the trapezoidal spoils stability in limit $\omega_{\mathrm{H}}\Delta t \rightarrow 0$.

Amplification factors for other schemes



AI22/AB3 stability region



- Here, $b = 5/6$ and $c = 3/2$.
- A whole family of implicit-explicit Adams methods with good stability properties exists for $c = 3b - 1$.
- A similar family of BDF schemes also exists.

Model Problem 2: Compressible Boussinesq System

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) u + \underbrace{\frac{\partial P}{\partial x}}_{\mathrm{s}} = 0, \quad (1)$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) w + \underbrace{\frac{\partial P}{\partial z}}_{\mathrm{s}} = \underbrace{b}_{\mathrm{b}}, \quad (2)$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) b + \underbrace{N^2 w}_{\mathrm{b}} = 0, \quad (3)$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) P + \underbrace{c_{\mathrm{s}}^2\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)}_{\mathrm{s}} = 0. \quad (4)$$

More stability analysis of implicit-explicit schemes for linearized form of this system in paper.

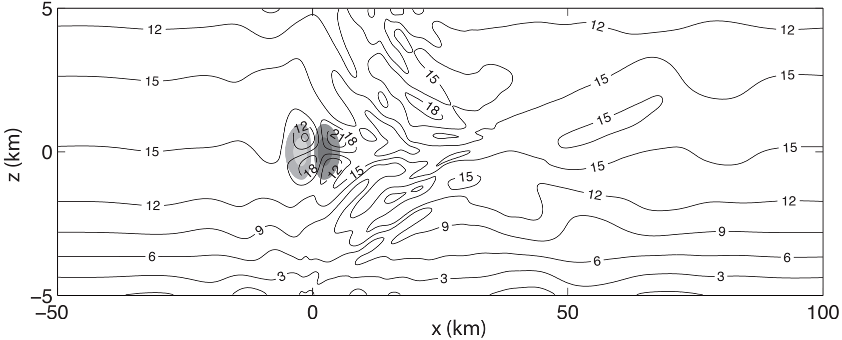
Numerical simulations

Fixed spatial discretization, explore convergence in time to compressible solution computed with very small Δt.

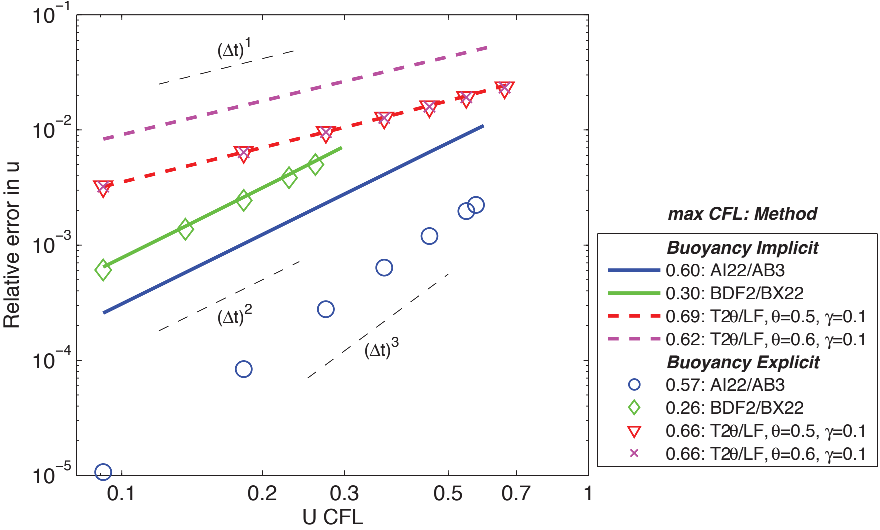
- Compressible Boussinesq system, $c_s = 350$ m s^{-1}
- Mean shear flow, $5 \leq U(z) \leq 15$
- Constant mean static stability, $N = .01$ s^{-1}
- Periodic lateral BC, rigid top and bottom
- Buoyancy waves generated by compact nondivergent forcing

Time-converged solution

u contours at 3000 s; shading shows streamlines of forcing field

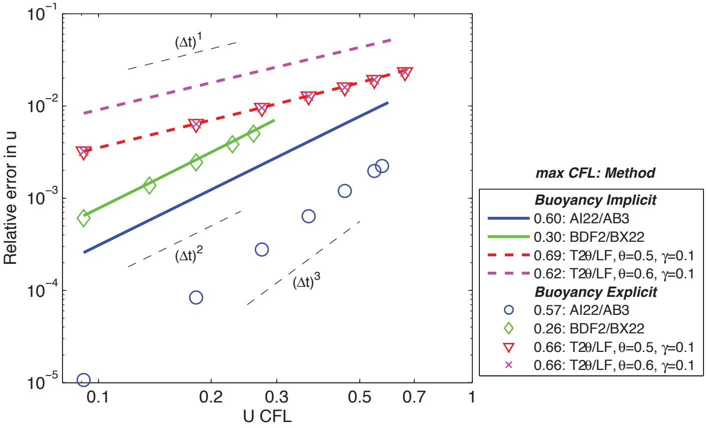


Empirical convergence rates



Explicit buoyancy can improve accuracy

- forward biased T2θ/LF ($\theta = 0.6$)
- AI22/AB3



Improvement at almost no CPU cost

14% reduction in maximum Δt for AI22/AB3 relative to Asselin-filtered leapfrog scheme