Structure Preserving Discretizations of the Shallow Water Equations

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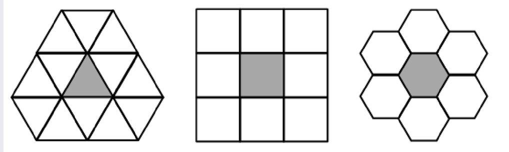
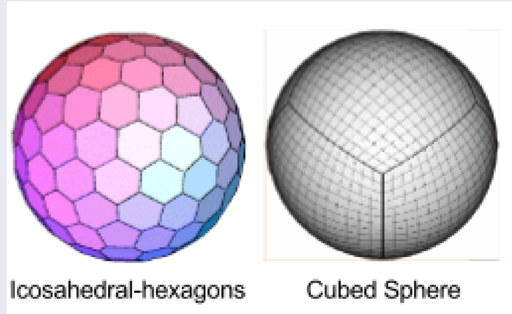
Introduction

1. Shallow water equations are adiabatic, inviscid → Hamiltonian
2. Would like discretizations to inherit (some) of the Hamiltonian structure → conservation laws
3. Wide variety of grids under consideration, schemes should be flexible



Planar Grids



Spherical Grids

4. Generalized Hamiltonian/DEC approach offers this flexibility (builds on work of Salmon, Cotter, Thuburn, Ringler, Dubos, many others)

Non-Canonical Hamiltonian Dynamics

1. Start with non-canonical Hamiltonian dynamics (essentially all invsicid, adiabatic fluid systems are of this form)

$$\frac{\partial \vec{x}}{\partial t} = \mathbb{J}\frac{\delta \mathcal{H}}{\delta \vec{x}}$$

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

$$\frac{d\mathcal{F}}{dt} = -\left(\frac{\delta \mathcal{H}}{\delta \vec{x}}, \mathbb{J}\frac{\delta \mathcal{F}}{\delta \vec{x}}\right)$$

- $\mathbb{J}$ - symplectic operator
- $\mathcal{H}$ - Hamiltonian functional
- $\mathcal{Z}$ - Casimir functional
- $\mathcal{F}$ - arbitrary functional
- $(,)$ - inner product
- $\vec{x}$ - set of variables

Conclusion

1. General discrete framework can conserve **mass, potential vorticity, total energy and potential enstrophy** on **general, non-orthogonal polygonal meshes**
2. Framework cleanly splits **topological** ($\bar{D}_1$, D_2, etc.) and **metrical** ($\mathbf{I}$, $\mathbf{H}$, etc.) aspects; can change one component without changing others
3. Getting total energy and potential enstrophy conservation together is tricky for vector-invariant formulation (having only one is "easy")
4. Hamiltonian/DEC framework also has useful mimetic properties (linear stability, no spurious vorticity production, etc.)

Vector-Invariant Continuous

1. Variables

$$\vec{x} = (h, \vec{u})$$
$$hq = \zeta$$

2. Symplectic Operator

$$\mathbb{J} = \begin{pmatrix} 0 & \vec{\nabla}\cdot \\ \vec{\nabla} & q\hat{k}\times \end{pmatrix}$$

3. Hamiltonian

$$\mathcal{H} = \frac{1}{2}(gh, h) + \frac{1}{2}(h\vec{u}, \vec{u})$$
$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi \\ h\vec{u} \end{pmatrix}$$

4. Potential Enstrophy (Casimir)

$$\mathcal{Z}_{\mathcal{C}} = \frac{1}{2}(\zeta, q)$$
$$\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = \begin{pmatrix} -\frac{q^2}{2} \\ \vec{\nabla}^T q \end{pmatrix}$$

Vorticity-Divergence Continuous

1. Variables

$$\vec{x} = (h, \zeta, \delta)$$

2. Symplectic Operator

$$\mathbb{J} = \begin{pmatrix} 0 & 0 & \vec{\nabla}^2 \\ 0 & -J(q,\bullet) & \vec{\nabla}\cdot(q\vec{\nabla}\bullet) \\ -\vec{\nabla}^2 & -\vec{\nabla}\cdot(q\vec{\nabla}\bullet) & -J(q,\bullet) \end{pmatrix}$$

3. Hamiltonian and Helmholtz Decomposition

$$\mathcal{H} = \frac{1}{2}(gh, h) + \frac{1}{2}(\vec{\nabla}\chi, \frac{\vec{\nabla}\chi}{h}) + \frac{1}{2}(\vec{\nabla}\psi, \frac{\vec{\nabla}\psi}{h}) + (\vec{\nabla}^T\chi, \vec{\nabla}\psi)$$
$$\zeta = \vec{\nabla}\cdot\left(h^{-1}\vec{\nabla}\psi\right) + J(h^{-1}, \chi)$$
$$\delta = \vec{\nabla}\cdot\left(h^{-1}\vec{\nabla}\chi\right) + J(h^{-1}, \psi)$$
$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi \\ -\psi \\ -\chi \end{pmatrix}$$

4. Potential Enstrophy (Casimir)

$$\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = \begin{pmatrix} -\frac{q^2}{2} \\ q \\ 0 \end{pmatrix}$$

Vector-Invariant Discrete (C-Grid)

1. Discrete Variables

$$\vec{x} = (m_i, u_e)$$
$$\zeta_v = \bar{D}_2 u_e$$
$$m_v q_v = \zeta_v + f = \eta_v \text{ where}$$
$$m_v = \mathrm{R} m_i$$

2. Symplectic Operator

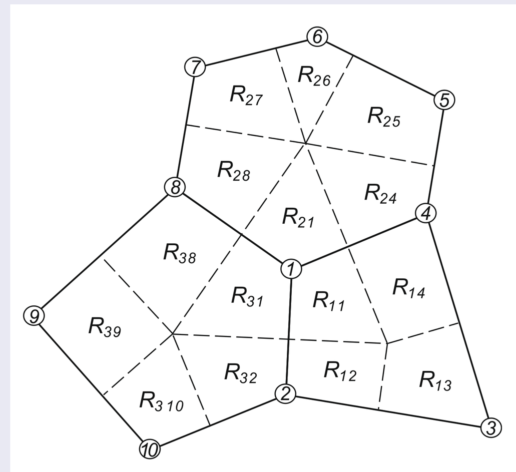
$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & Q \end{pmatrix}$$

3. Hamiltonian

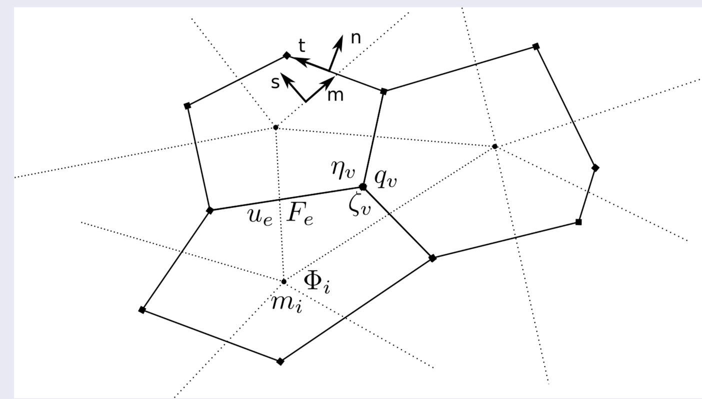
$$\mathcal{H} = \frac{1}{2}g(m_i, m_i)_{\mathrm{I}} + \frac{1}{2}(F_e, u_e)_{\mathrm{H}}$$
$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathrm{I}\Phi_i \\ \mathrm{H}F_e \end{pmatrix}$$

4. Potential Enstrophy (Casimir)

$$\mathcal{Z}_{\mathcal{C}} = \frac{1}{2}(\eta_v, \mathrm{J}^{-1}q_v)_{\mathrm{J}}$$
$$\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = \begin{pmatrix} -\mathrm{R}^T\frac{q_v^2}{2} \\ \bar{D}_2{}^T q_v \end{pmatrix} = \begin{pmatrix} -\mathrm{R}^T\frac{q_v^2}{2} \\ D_1 q_v \end{pmatrix}$$



Construction of $\mathbf{R}$ from Thuburn et. al 2009



Non-orthogonal primal-dual C grid

Vorticity-Divergence Discrete (Z-Grid)

1. Discrete Variables

$$\vec{x} = (m_i, \zeta_i, \delta_i)$$
$$m_i q_i = \zeta_i$$

2. Symplectic Operator

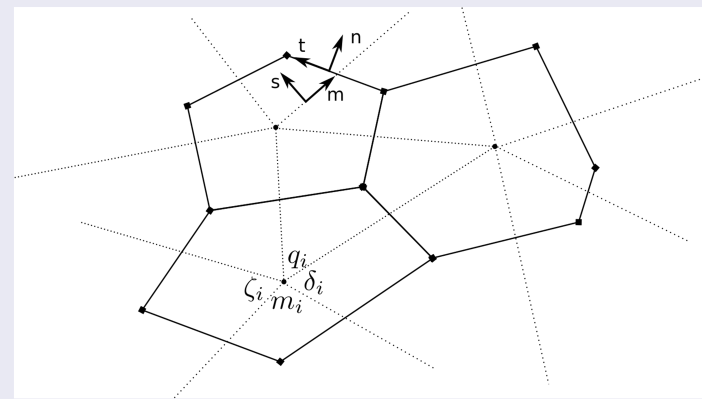
$$\mathbb{J} = \begin{pmatrix} 0 & 0 & \mathrm{L} \\ 0 & -\mathrm{J}_\zeta(q_i,\bullet) & \mathrm{FD}(q_i,\bullet) \\ -\mathrm{L} & -\mathrm{FD}(q_i,\bullet) & -\mathrm{J}_\delta(q_i,\bullet) \end{pmatrix}$$

3. Hamiltonian

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi_i \\ -\psi_i \\ -\chi_i \end{pmatrix}$$

4. Potential Enstrophy (Casimir)

$$\mathcal{Z}_{\mathcal{C}} = \sum_i m_i \frac{q_i^2}{2}$$
$$\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = \begin{pmatrix} -\frac{q_i^2}{2} \\ q_i \\ 0 \end{pmatrix}$$



Z grid Distribution of Variables