Outline

❖ There will be two parts:

1) Development of the 3D poisson equation solver

2) Development of the advection based on grids associated with the icosahedral grid

3D Poisson equation solver

- ❖ Based on Arakawa, Jung and Konor
- ❖ The equation we are interested is this:

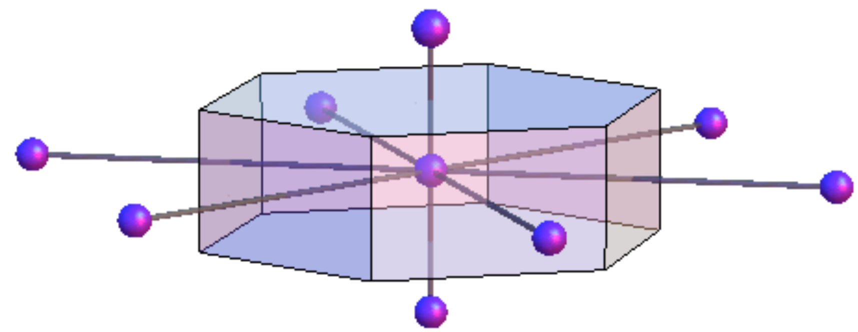
$$\nabla^2 w + \frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial z}(\rho w)\right] = rhs$$

- ❖ Re-write as a parabolic equation:

$$\mu\frac{\partial w}{\partial t} = \nabla^2 w + \frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial z}(\rho w)\right] - rhs$$

The Discrete form of the equations

❖ The discrete form of the equations looks like this:



$$\mu \frac{w_{i,k+1/2}^{(\kappa+1)} - w_{i,k+1/2}^{(\kappa)}}{\delta t} = \frac{1}{A_i} \sum_{i'} \frac{w_{i+i',k+1/2}^{(\kappa)} - w_{i,k+1/2}^{(\kappa+1)}}{L_{i;i+i'}} l_{i;i+i'}$$

$$+ \frac{1}{\delta z_{k+1/2}} \left[\frac{1}{\rho_{k+1} \delta z_{k+1}} \left(\rho_{k+3/2} w_{k+3/2}^{(\kappa+1)} - \rho_{k+1/2} w_{k+1/2}^{(\kappa+1)} \right) - \frac{1}{\rho_k \delta z_k} \left(\rho_{k+1/2} w_{k+1/2}^{(\kappa+1)} - \rho_{k-1/2} w_{k-1/2}^{(\kappa+1)} \right) \right] - rhs_{i,k+1/2}$$

❖ Re-arrange to form an implicit tridiagonal system in the vertical:

$$\frac{\rho_{k-1/2}}{\delta z_{k+1/2} \rho_k \delta z_k} w_{k-1/2}^{(\kappa+1)} - \left[\frac{\mu}{\delta t} + \frac{1}{A_i} \sum_{i'} \frac{l_{i;i+i'}}{L_{i;i+i'}} + \frac{\rho_{k+1/2}}{\delta z_{k+1/2}} \left(\frac{1}{\rho_{k+1} \delta z_{k+1}} + \frac{1}{\rho_k \delta z_k} \right) \right] w_{i,k+1/2}^{(\kappa+1)} + \frac{\rho_{k+3/2}}{\delta z_{k+1/2} \rho_{k+1} \delta z_{k+1}} w_{k+3/2}^{(\kappa+1)}$$

$$= rhs_{i,k+1/2} - \frac{\mu}{\delta t} w_{i,k+1/2}^{(\kappa)} - \frac{1}{A_i} \sum_{i'} \frac{l_{i;i+i'}}{L_{i;i+i'}} w_{i+i',k+1/2}^{(\kappa)}$$

3D Poisson equation solver

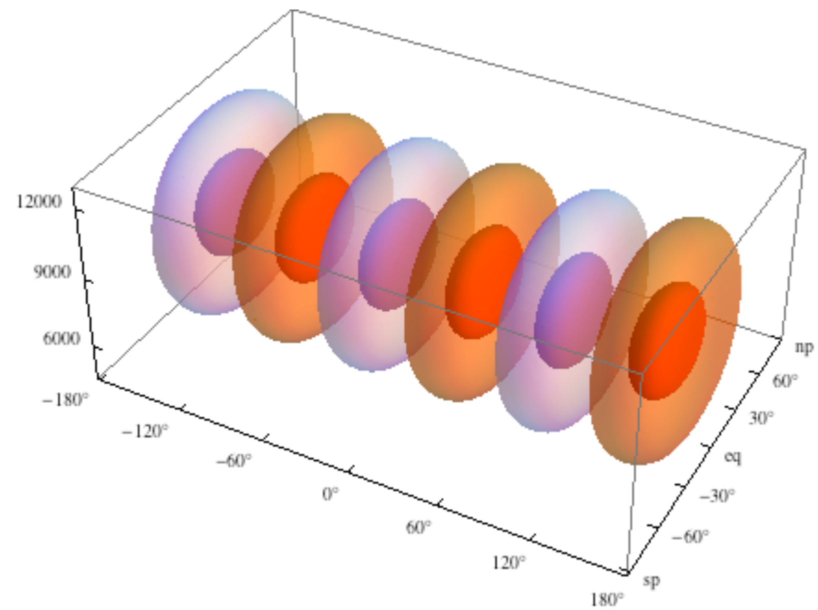
❖ A simple analytic test function:

$$w(\lambda,\varphi,z) = w_1(\lambda,\varphi)\, w_2(z)$$

where

$$w_1(\lambda,\varphi) = \sin 3\lambda \cos^4 \varphi$$

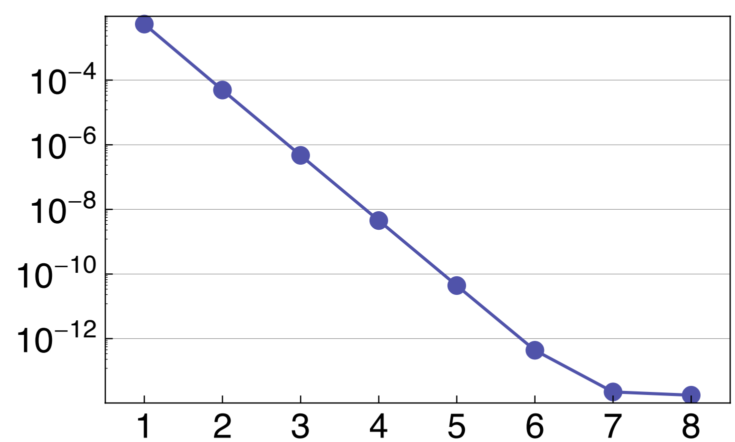
$$w_2(z) = \sin^8\left(\pi \frac{z}{z_{\max}}\right)$$



❖ 40962 cells -- 40 vertical layers

❖ Note that this problem is very tightly coupled in the vertical.

❖ The inf-norm error looks like this:

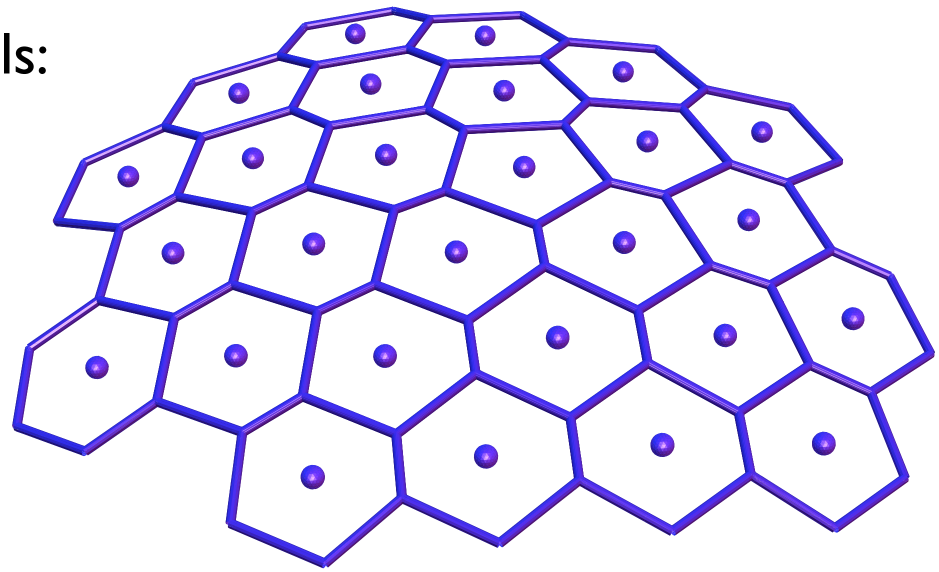


Advection based on cell corners and edges

❖ Recall that advection of quantities defined at cell centers uses hexagons and pentagons as control volumes.

❖ Suppose the grid contains *N* cells:

- 12 pentagons
- *N*-12 hexagons



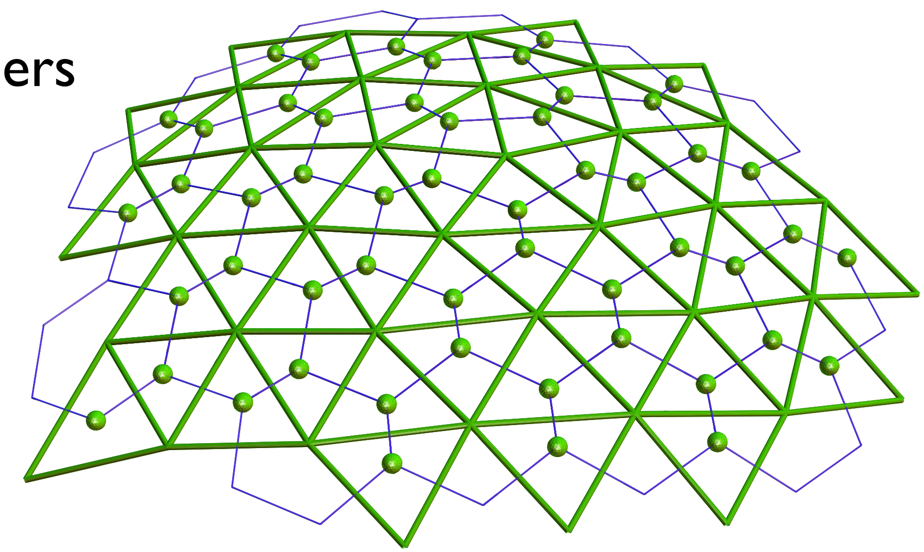
❖ In the VVM advected quantities will also be defined at cell corners and edges.

Control volumes for the corner grid

❖ This is the dual of the hexagon/pentagon grid.

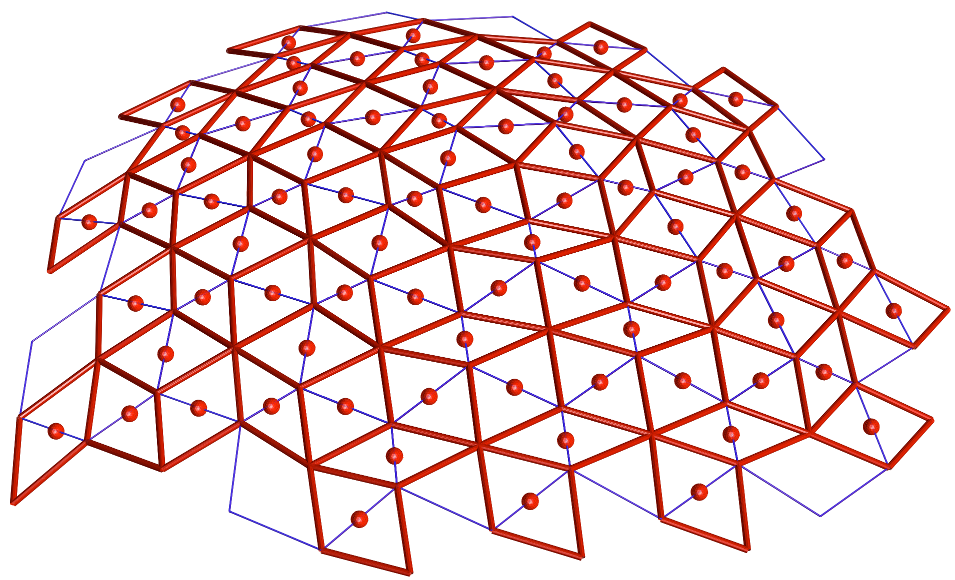
❖ Suppose the grid contains *N* cells.

It is easy to show 2 (*N*-2) corners



Control volumes for the edge grid

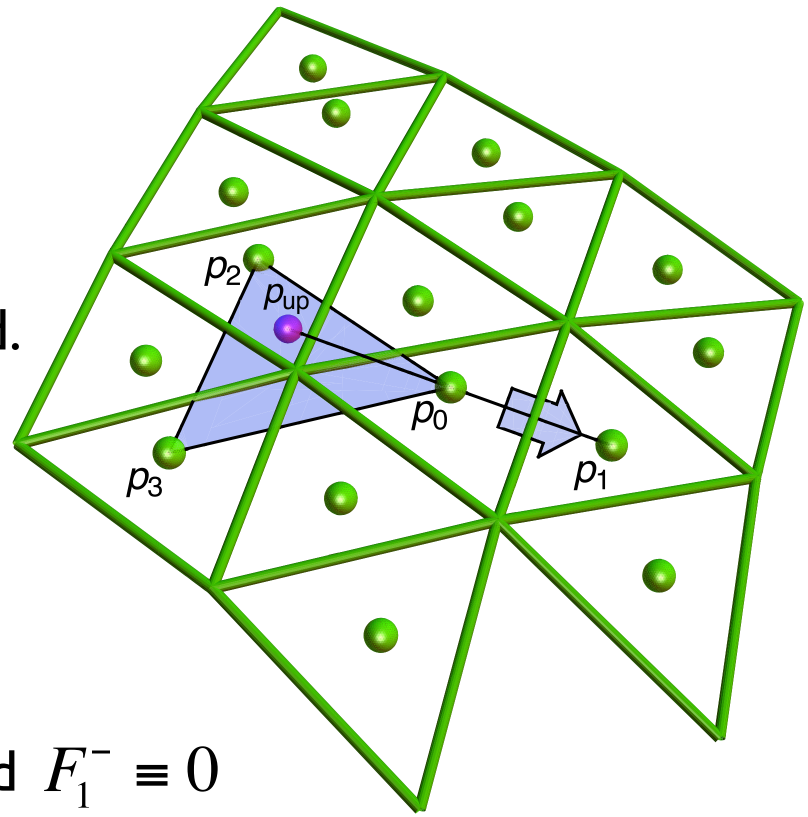
- ❖ Each grid point is associated with an edge of the icosahedral grid.
- ❖ Suppose the grid contains *N* cells.
  It is easy to show 3 (*N*-2) edges.
- ❖ This is more difficult than the corner grid and more important to do well.



The 3rd-order (upstream-biased) advection

❖ Based on Hsu and Arakawa (1990)

$$\frac{\partial m}{\partial t} + \frac{1}{A}\sum_{i=1} F_i = 0$$



❖ The incoming and outgoing fluxes depend on the direction of the wind.

$$F_i = F_i^+ + F_i^-$$

❖ Suppose the wind directed from p_0 toward p_1, then, for example

$$F_1^+ \equiv F_1^+\left(m_{up}, m_0, m_1, \mathbf{v}_{up}, \mathbf{v}_1\right) \text{ and } F_1^- \equiv 0$$

where F_1^+ depends on the curvature of the upstream field.

❖ The scheme can be positive-definite or not.

Pure Advection Test

❖ Williamson *et al.* (1992)

$$h(\lambda,\varphi) = \begin{cases} (h_0/2)\left(1+\cos(\pi r/R)\right) & r < R \\ 0 & r \geq R \end{cases}$$

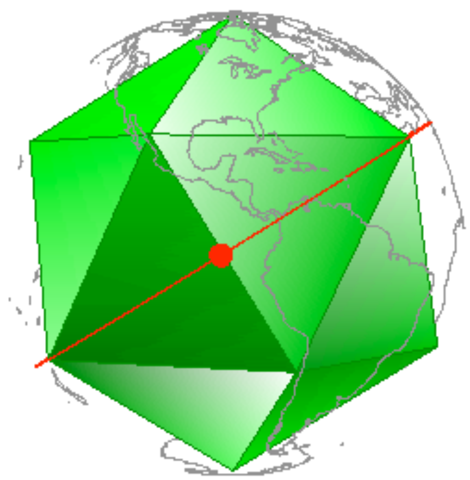
where h_0 = 1000 m and R = a/3

❖ The wind is prescribed:

$$u = u_0\left(\cos\varphi\cos\alpha + \sin\varphi\cos\lambda\sin\alpha\right)$$

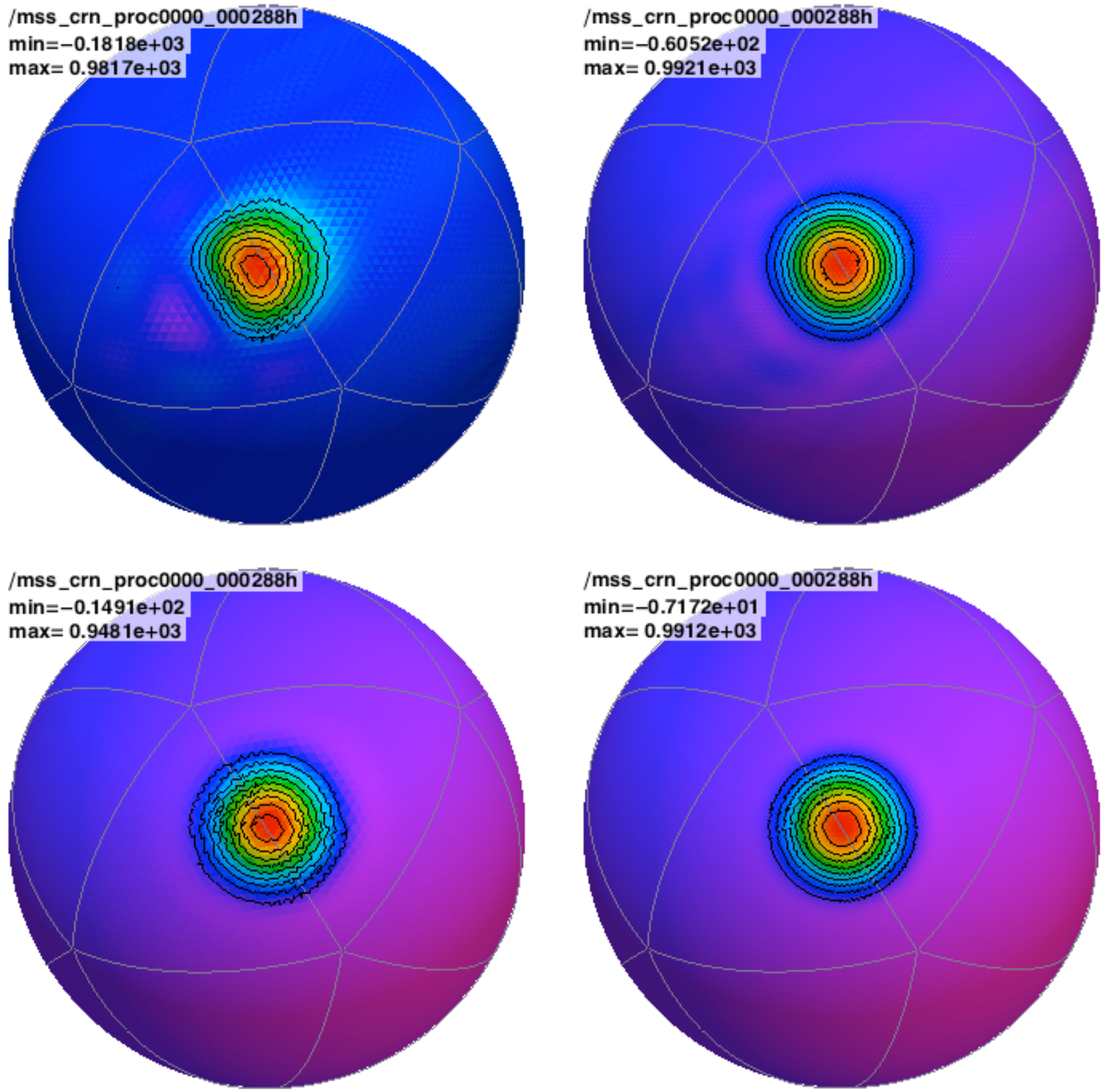
$$v = -u_0\sin\lambda\sin\alpha$$

❖ The value of å is selected to advect along a great circle path over four pentagons.



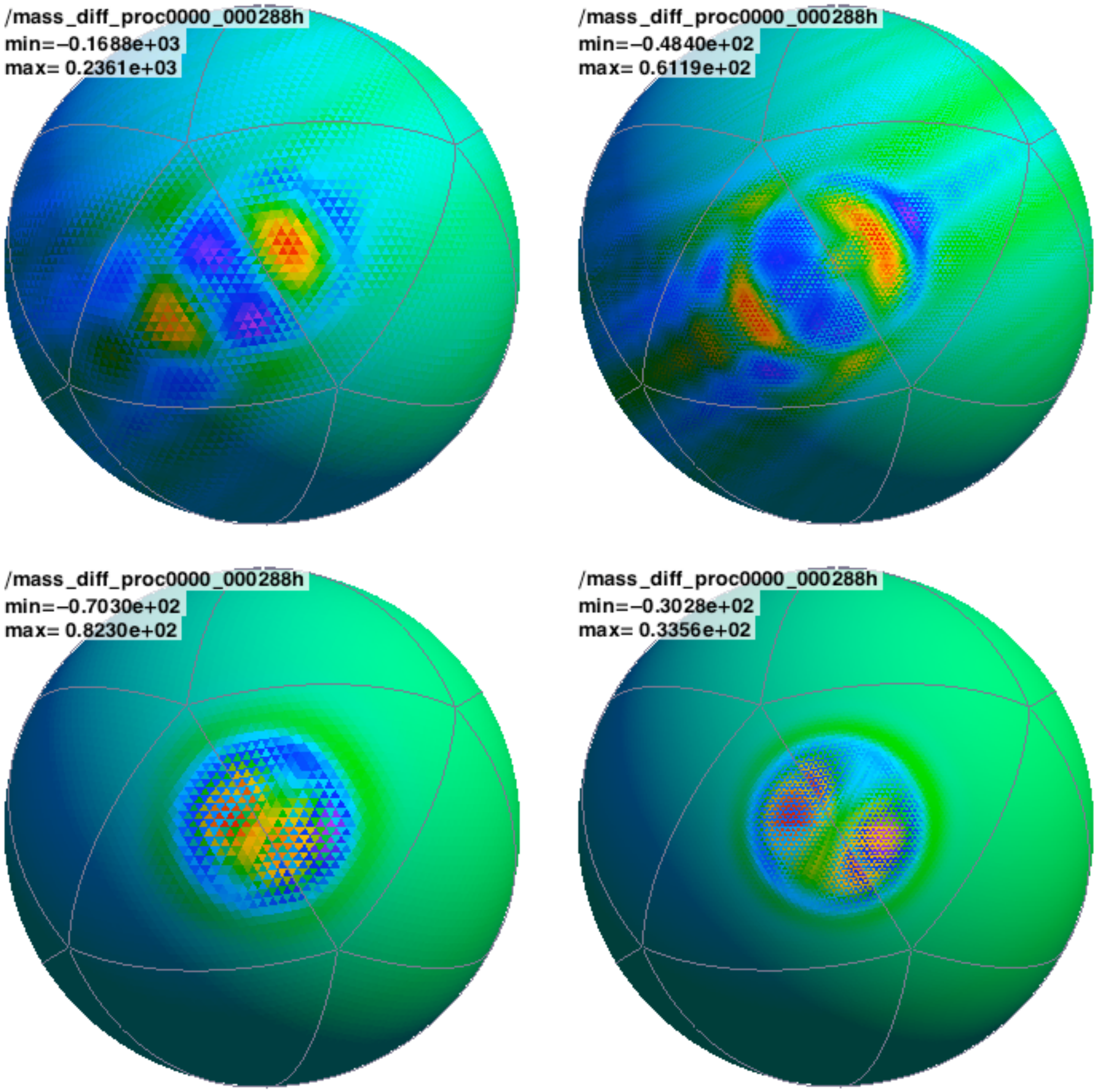
(appx. mass) at 12 days

- Columns: 10242 and 40962
- Rows: Centered in space and upstream biased



(true mass) - (appx. mass) at 12 days

- Columns: 10242 and 40962
- Rows: Centered in space and upstream biased



normalized RMS and inf-norm errors

- Columns: 10242 and 40962
- Rows: Centered in space and upstream biased
- red line: rms error
- blue line: inf-norm error