Quasi-Geostrophic Theory on the Sphere

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January 2010

QG Theory on the Sphere

- Kuo, H. L., 1959: Finite amplitude three-dimensional harmonic waves on the spherical earth. J. Meteor., 16, 524-534.
- Charney, J. G., and M. E. Stern, 1962: On the stability of internal baroclinic jets in a rotating atmosphere. J. Atmos. Sci., 19, 159-172.
- Schubert, W. H., R. K. Taft, and L. G. Silvers, 2009: Shallow water quasi-geostrophic theory on the sphere. JAMES, 1, No. 2, 1-17.
- Verkley, W. T. M., 2009: A balanced approximation of the one-layer shallow-water equations on a sphere. J. Atmos. Sci, 66, 1735-1748.

The Importance of QG Theory

"I personally regard the successful reduction of the dynamic equations to a single prognostic equation by means of the geostrophic relationship as the greatest single achievement of twentieth-century dynamic meteorology."

Edward Lorenz (2006)
(Annu. Rev. Earth Planet. Sci., Vol. 34, 37-45)

QG Theory Inspires & Fascinates

"I was particularly inspired by the concept of quasi-geostrophy and then fascinated by the fact that even highly simplified dynamical models such as the quasi-geostrophic barotropic model have some relevance to extremely complicated day-to-day weather changes."

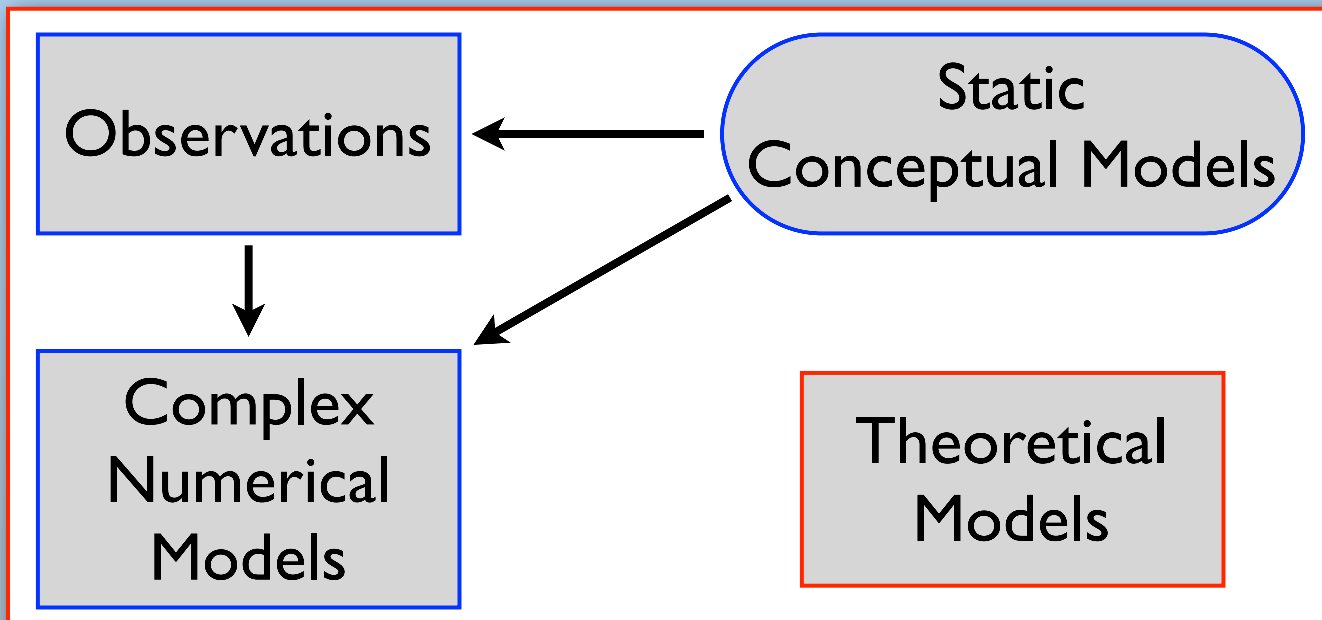
Akio Arakawa (2000)
"General Circulation Model Development"

The Gap between Simulation and Understanding in Climate Modeling

"I have returned repeatedly to Phillips' (1956) original GCM, the two-layer QG model of a statistically steady baroclinically unstable jet on a beta plane -- the E. Coli of climate models."

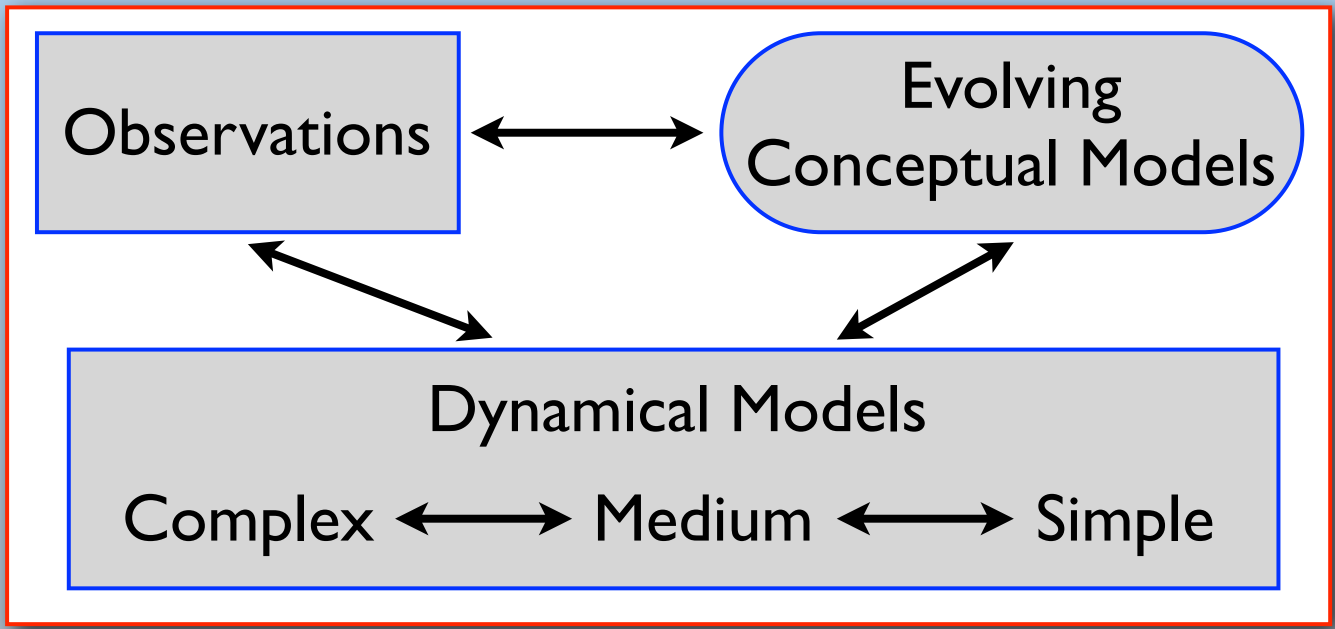
Isaac Held, BAMS, 2005

An Unhealthy Situation in Meteorological Research



Theoretical work proceeds with little contact with observations or numerical models and contributes little to conceptual models (Hoskins, QJRMS, 1983)

The Optimum Situation in Meteorological Research



Observations and models of all complexities are used to produce evolving conceptual models

Shallow Water Model

- Shallow water primitive equations on the sphere:

$$\frac{Du}{Dt} - \left(2\Omega \sin\phi + \frac{u\tan\phi}{a}\right) v + g\frac{\partial h}{a\cos\phi\partial\lambda} = 0$$

$$\frac{Dv}{Dt} + \left(2\Omega \sin\phi + \frac{u\tan\phi}{a}\right) u + g\frac{\partial h}{a\partial\phi} = 0$$

$$\frac{Dh}{Dt} + (\bar{h} + h)\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}\right) = 0$$

- Material Derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{a\cos\phi\partial\lambda} + v\frac{\partial}{a\partial\phi}$$

Potential Vorticity (PV)

- PV equation: $$\frac{DP}{Dt} = 0$$ (PV is conserved)

- PV: $$P = \frac{\bar{h}}{\bar{h} + h}\left(2\Omega \sin\phi + \frac{\partial v}{a \cos\phi \partial\lambda} - \frac{\partial(u \cos\phi)}{a \cos\phi \partial\phi}\right)$$
$$= 2\Omega\mu + \left(\frac{\bar{h}}{\bar{h} + h}\right)\nabla^2\psi - 2\Omega\mu\left(\frac{h}{\bar{h} + h}\right)$$

where: $$(u_\psi, v_\psi) = \left(-\frac{\partial\psi}{a\partial\phi}, \frac{\partial\psi}{a \cos\phi \partial\lambda}\right)$$ $$\mu = \sin\phi$$

Step 1: Approximate PV

- Assume: $h \ll \bar{h}$ (fluid depth deviations are small)
- Then approximate PV:

$$
\begin{aligned}
P &= 2\Omega\mu + \left(\frac{\bar{h}}{\bar{h}+h}\right)\nabla^2\psi - 2\Omega\mu\left(\frac{h}{\bar{h}+h}\right) \\
&\approx 2\Omega\mu + \nabla^2\psi - \frac{2\Omega\mu}{\bar{h}}h \\
&= 2\Omega\mu + q
\end{aligned}
$$

- q is the potential vorticity anomaly

Step 2: Balance Condition

- Follow the arguments of Kuo (1959) and Charney and Stern (1962)
- Assume the following linear balance condition between the mass field (h) and the nondivergent wind field (ψ):

$$\nabla \cdot (2\Omega\mu\nabla\psi) = g\nabla^2 h$$

- Assume $2\Omega\mu$ is slowly varying:

$$\nabla^2(gh - 2\Omega\mu\psi) = 0$$

- Local linear balance condition:

$$gh = 2\Omega\mu\psi$$

QG Theory on the Sphere

- Prediction of q (now the QG PV anomaly):

$$\frac{\partial q}{\partial t} + \frac{1}{a^2}\frac{\partial(\psi, q)}{\partial(\lambda, \mu)} + \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda} = 0$$

- Invertibility relation for the diagnosis of ψ:

$$\nabla^2\psi - \frac{\epsilon\mu^2}{a^2}\psi = q$$

- Lamb's parameter:

$$\epsilon = \frac{4\Omega^2 a^2}{g\bar{h}} = \left(\frac{a}{(g\bar{h})^{\frac{1}{2}}/(2\Omega)}\right)^2$$

Conservation Principles

- Total energy:

$$\frac{d\mathcal{E}}{dt} = 0$$

where

$$\mathcal{E} = \frac{1}{8\pi}\int_{-1}^{1}\int_{0}^{2\pi}\left(\nabla\psi\cdot\nabla\psi + \frac{\epsilon\mu^2}{a^2}\psi^2\right)d\lambda\, d\mu$$

$$= -\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\tfrac{1}{2}\psi q\, d\lambda d\mu$$

- Potential enstrophy:

$$\frac{d\mathcal{Z}}{dt} = 0$$

where

$$\mathcal{Z} = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\tfrac{1}{2}q^2\, d\lambda d\mu$$

Spheroidal Harmonics

- Separable: $$\mathcal{S}_{mn}(\epsilon;\lambda,\mu) = S_{mn}(\epsilon;\mu)e^{im\lambda}$$
- Satisfy: $$\nabla^2\mathcal{S}_{mn} - \frac{\epsilon\mu^2}{a^2}\mathcal{S}_{mn} = -\frac{\alpha_{mn}(\epsilon)}{a^2}\mathcal{S}_{mn}$$
- Meridional structure:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dS_{mn}}{d\mu}\right] + \left(\alpha_{mn} - \epsilon\mu^2 - \frac{m^2}{1-\mu^2}\right)S_{mn} = 0$$

- Special case: $\epsilon = 0$
  - Get the associated Legendre equation
  - Spheroidal harmonics reduce to spherical harmonics

Spheroidal Harmonic Transform Pair

- Spheroidal harmonic expansion of q:

$$q(\lambda, \mu, t) = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} q_{mn}(t)\, \mathcal{S}_{mn}(\epsilon; \lambda, \mu)$$

- Spheroidal harmonic expansion coefficients:

$$q_{mn}(t) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} q(\lambda, \mu, t)\, \mathcal{S}^{*}_{mn}(\epsilon; \lambda, \mu)\, d\lambda\, d\mu$$

- Invertibility relation in spectral space:

$$\psi_{mn} = -\frac{a^2 q_{mn}}{\alpha_{mn}(\epsilon)}$$

Rossby-Haurwitz Waves

- Linearized PV equation:

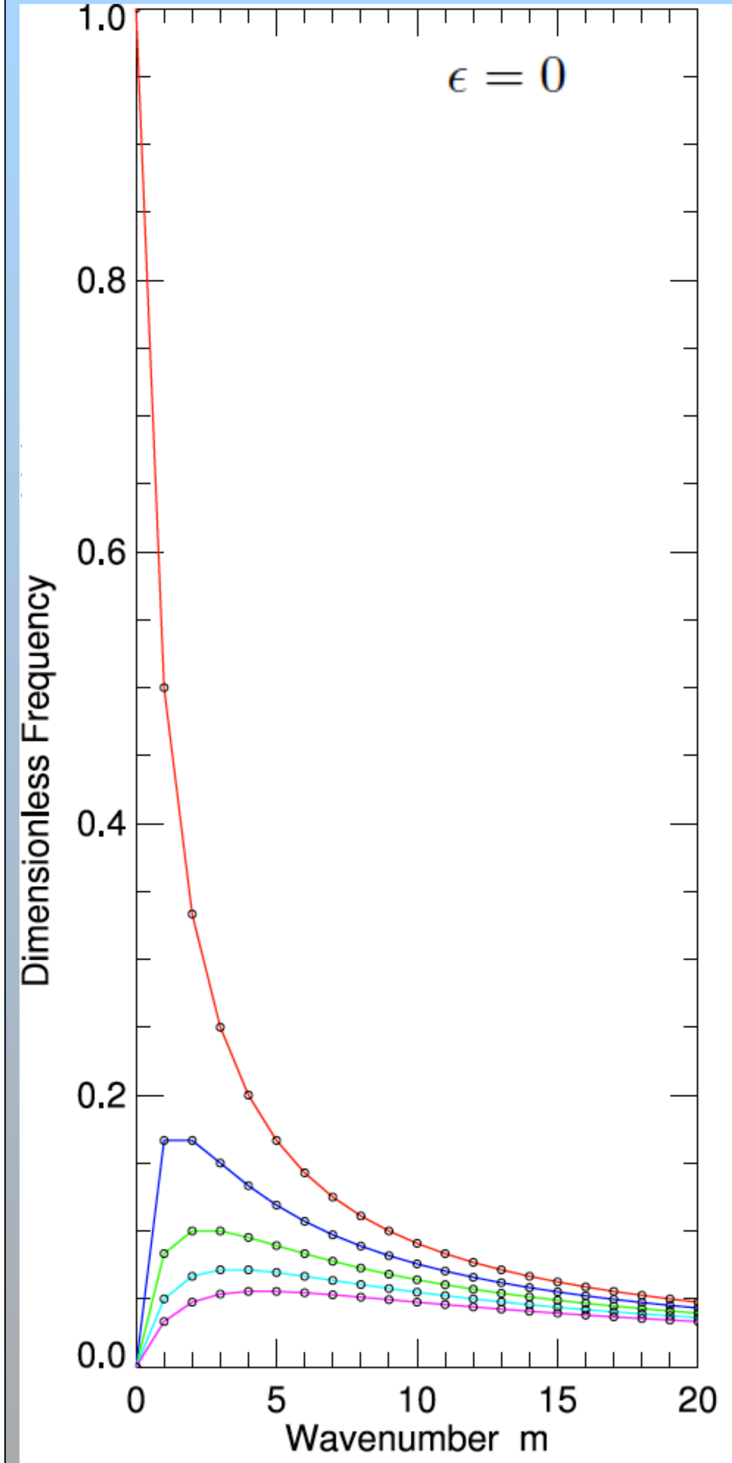
$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{\epsilon\mu^2}{a^2}\psi\right) + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} = 0$$

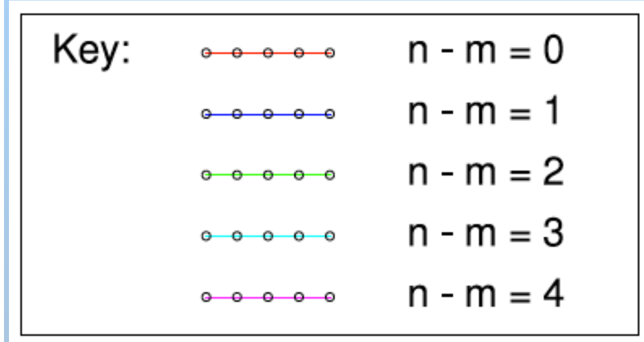
- Dispersion relation:

$$\nu_{mn}(\epsilon) = \frac{2\Omega m}{\alpha_{mn}(\epsilon)}$$

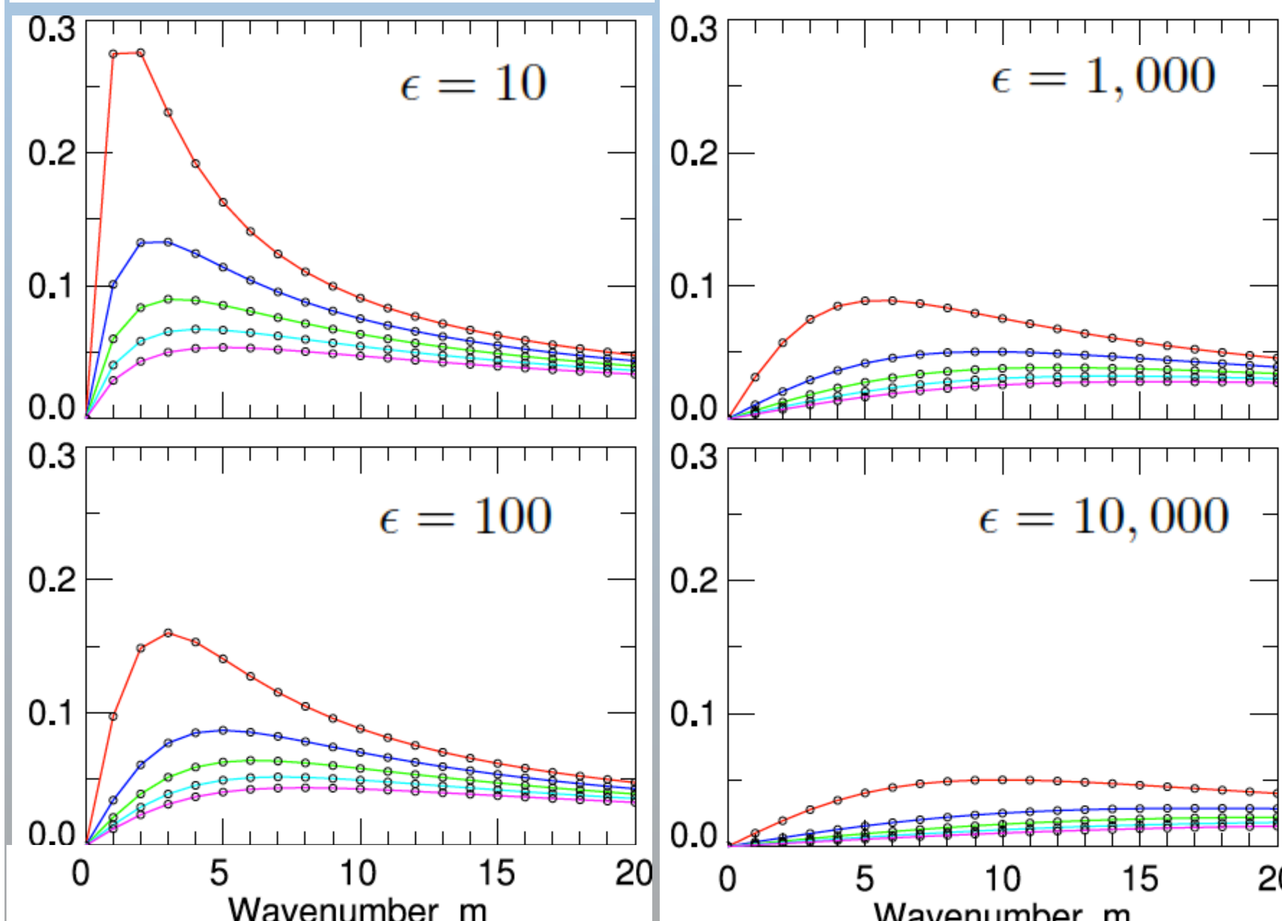
- This is a very accurate approximation to the shallow water PE results of Longuet-Higgins (1968)

Dispersion Diagrams





$$\nu_{mn}(\epsilon) = \frac{2\Omega m}{\alpha_{mn}(\epsilon)}$$



Frequency Errors in Early NWP

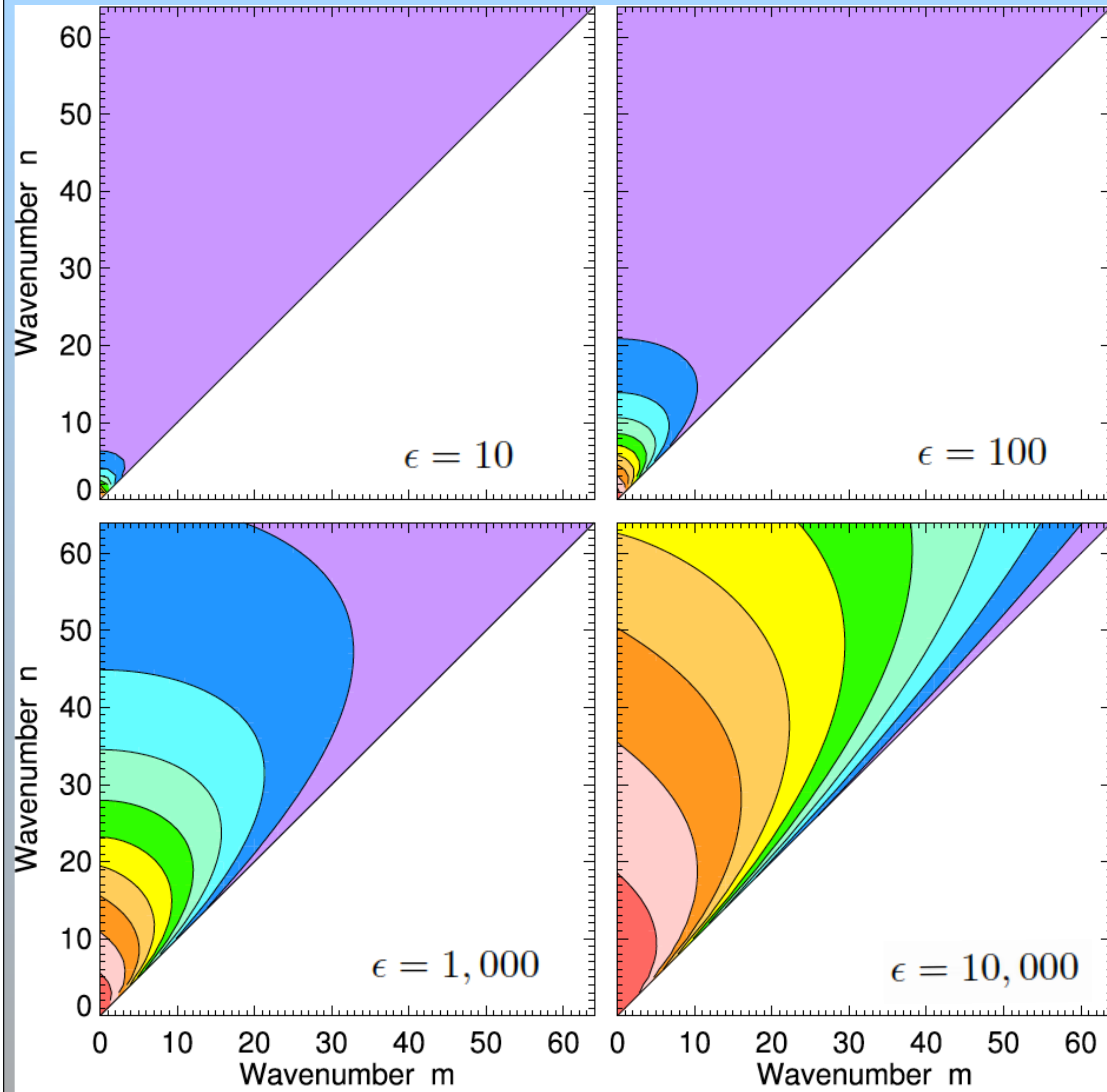
- When early barotropic forecast models became nearly hemispheric, erroneous westward propagation of ultralong waves was noticed
- Cressman (1958) suggested an empirical "divergence" correction
- Spherical QG theory provides a more precise theoretical interpretation of this problem

Frequency Correction Factor

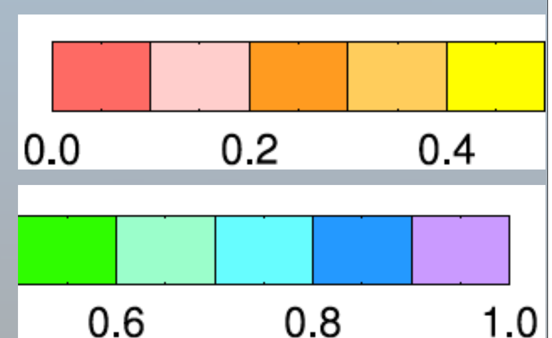
$$\nu_{mn}(\epsilon) = \left(\frac{n(n+1)}{\alpha_{mn}(\epsilon)}\right)\left(\frac{2\Omega m}{n(n+1)}\right)$$

- The spherical QG Rossby-Haurwitz wave frequency is the product of the frequency correction factor and the nondivergent Rossby-Haurwitz wave frequency
- Frequency correction factor is the ratio of the spherical harmonic eigenvalue to the spheroidal harmonic eigenvalue

QG Frequency Correction Factor



$$\left(\frac{n(n+1)}{\alpha_{mn}(\epsilon)}\right)$$



Waves and Turbulence

- Recall: $$\frac{\partial q}{\partial t} + \frac{1}{a^2}\frac{\partial(\psi, q)}{\partial(\lambda, \mu)} + \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda} = 0$$
  - Contains nonlinear advection of q
  - Contains linear term associated with R-H waves
- Dynamics is wavelike if: $$\frac{2\Omega m}{\alpha_{mn}(\epsilon)} \gg \frac{[\alpha_{mn}(\epsilon)]^{\frac{1}{2}}\, V_{\mathrm{rms}}}{a}$$
- Dynamics is turbulent if: $$\frac{2\Omega m}{\alpha_{mn}(\epsilon)} \ll \frac{[\alpha_{mn}(\epsilon)]^{\frac{1}{2}}\, V_{\mathrm{rms}}}{a}$$

Anisotropic Rhines Barrier

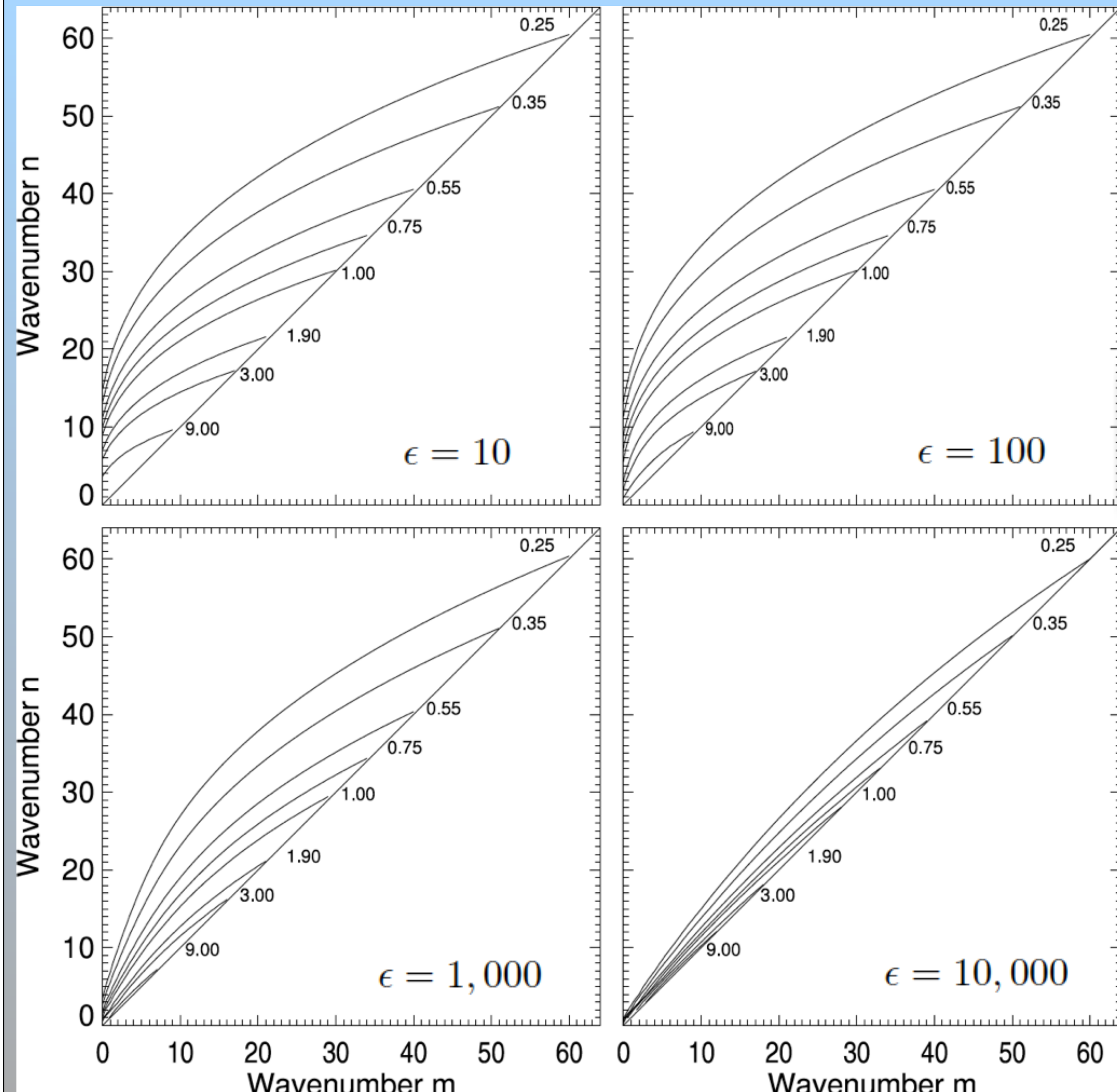
- Defined by equating the two time scales:

$$\frac{m}{\left[\alpha_{mn}(\epsilon)\right]^{\frac{3}{2}}} = \frac{V_{\mathrm{rms}}}{2\Omega a}$$

- For a given right hand side and given Lamb's parameter, this defines a curve in the spheroidal harmonic (m, n) wavenumber plane

- This barrier can be viewed as a surface in the three-dimensional (m, n, ϵ) wavenumber space

Rhines Barrier Diagrams



$$\frac{m}{[\alpha_{mn}(\epsilon)]^{\frac{3}{2}}} = \frac{V_{\rm rms}}{2\Omega a}$$

Value shown by each curve is the value of $V_{\rm rms}$ in m/s

Two Approaches (on the Sphere)

$$q = \nabla^2 \psi - \frac{2\Omega\mu}{\bar{h}} h$$

(A) Leave the h term in its present form and approximate the vorticity by its geostrophic value (i.e., in terms of h)

- Initialize with h observations only
- Bad approach on the sphere

(B) Leave the vorticity term in its present form and approximate the h term by ψ

- Initialize with wind observations only
- Good approach on the sphere

Two Approaches (Phillips)

$$q = \nabla^2 \psi - \frac{2\Omega\mu}{\bar{h}} h$$

(A) Leave the h term in its present form, approximate the vorticity by its geostrophic value (i.e., in terms of h), and advect the potential vorticity by the geostrophic wind

- Results in forecast errors (Phillips 1958)

(B) Leave the vorticity term in its present form, approximate the h term by ψ, and advect the potential vorticity using the nondivergent wind

- Results in better forecasts

Ingredients of Global QG

- To obtain an invertibility principle, leave $\nabla^2\psi$ unchanged and approximate gh by $2\Omega\mu\psi$, rather than leaving gh unchanged and approximating $\nabla^2\psi$ in terms of gh
- Flow partitioning is between nondivergent and irrotational components rather than between geostrophic and ageostrophic components
- Utilize spheroidal harmonics to understand Rossby waves and energy/potential enstrophy cascades