UNIFICATION OF DYNAMICAL EQUATIONS:
UNIFIED SYSTEM OF EQUATIONS

(Arakawa and Konor, 2009, *MWR*, **137**)

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OUTLINE:

- Introduction to the unified system of equations.
- Time integrations of the unified system of equations.
- Preliminary results obtained by the unified system based on the direct momentum prediction (square grid).
- Comparison of these results to those obtained by the anelastic system based on the momentum and vorticity predictions

THREE GROUPS OF ATMOSPHERE MODELS

LES MODELS	CLOUD SCALE MODELS	GLOBAL SCALE MODELS
Anelastic and Boussinesq systems	Anelastic, pseudo-incompressible and fully-compressible systems	Quasi-hydrostatic and fully-compressible* systems
100 m	1 km	100 km

Typical grid resolution

*Very few models exist

UNIFIED SYSTEM OF EQUATIONS

- Through a noninvasive approximation, vertically propagating sound waves are eliminated without modifying the dynamically important modes. (See Arakawa and Konor, 2009.)
- Construction of a global numerical model is easy.

UNIFIED SYSTEM OF EQUATIONS

State Equation

$$p = \rho RT \qquad p_{qs} = \rho_{qs} RT_{qs}$$

Continuity Equation

$$\frac{\partial \rho_{qs}}{\partial t} + \nabla_H \cdot \left(\rho_{qs} \mathbf{V}_H\right) + \frac{\partial}{\partial z}\left(\rho_{qs} w\right) = 0$$

$$\frac{\partial p_{qs}}{\partial z} = -g\rho_{qs}$$

Thermodynamic Equation

$$\frac{D\ell n\theta}{Dt} = \frac{Q}{c_p T}$$

Vertical Momentum Equation

$$\frac{Dw}{Dt} = -c_p \theta \frac{\partial(\delta\pi)}{\partial z} + \mathbf{F}_w$$

$$0 = -c_p \theta \frac{\partial \pi_{qs}}{\partial z} - g$$

Horizontal Momentum Equation

$$\frac{D\mathbf{V}_H}{Dt} = -f\mathbf{k} \times \mathbf{V}_H - c_p \theta \nabla_H \left(\pi_{qs} + \delta\pi\right) + \mathbf{F}_H$$

Some Definitions

$$\pi_{qs}\theta = T_{qs} \qquad \pi_{qs} \equiv \left(p_{qs}/p_{00}\right)^{\kappa} \qquad \pi \equiv \pi_{qs} + \delta\pi \qquad \rho_{qs} = \frac{p_{00}\pi_{qs}^{(1-\kappa)/\kappa}}{R\theta}$$

DETERMINATION OF QUASI-HYDROSTATIC VARIABLES

Two different versions of the hydrostatic equation

$$p_{qs} = \left(p_{qs}\right)_T - g\int_{z_T}^{z} \rho_{qs}\, dz \qquad \pi_{qs} = \left(\pi_{qs}\right)_T - \frac{g}{c_p}\int_{z_T}^{z} \frac{1}{\theta}\, dz$$

Two different versions of the hydrostatic equation must satisfy the state equation

Surface and upper boundary tendency equations for π_{qs}

$$\frac{\partial}{\partial t}\left(\pi_{qs}\right)_S = \frac{\kappa g}{\left(p_{qs}/\pi_{qs}\right)_T - \left(p_{qs}/\pi_{qs}\right)_S}\left[\int_{z_T}^{z_S} \frac{\partial \rho_{qs}}{\partial t} dz + \frac{1}{R}\left(\frac{p_{qs}}{\pi_{qs}}\right)_T \int_{z_T}^{z_S} \frac{1}{\theta^2}\frac{\partial \theta}{\partial t} dz\right]$$

$$\frac{\partial}{\partial t}\left(\pi_{qs}\right)_T = \frac{\kappa g}{\left(p_{qs}/\pi_{qs}\right)_T - \left(p_{qs}/\pi_{qs}\right)_S}\left[\int_{z_T}^{z_S} \frac{\partial \rho_{qs}}{\partial t} dz + \frac{1}{R}\left(\frac{p_{qs}}{\pi_{qs}}\right)_S \int_{z_T}^{z_S} \frac{1}{\theta^2}\frac{\partial \theta}{\partial t} dz\right]$$

where

$$\int_{z_T}^{z_S} \frac{\partial \rho_{qs}}{\partial t} dz = -\nabla_H \cdot \int_{z_T}^{z_S} \rho_{qs} \mathbf{V}_H\, dz$$

$$\left(\rho_{qs} w\right)_S = \left(\rho_{qs} w\right)_T = 0$$

Quasi-hydrostatic density

$$\rho_{qs} = \frac{p_{00}\pi_{qs}^{(1-\kappa)/\kappa}}{R\theta}$$

DETERMINATION OF NONHYDROSTATIC VARIABLES

Horizontal and vertical components of mass weighted-momentum equation

$$\frac{\partial}{\partial t}\left(\rho_{qs}\mathbf{V}_H\right) = \mathbf{G}_3 - \left(\rho_{qs}c_p\theta\right)\nabla_H\left(\delta\pi\right) \qquad \frac{\partial}{\partial t}\left(\rho_{qs}w\right) = \left(G_4\right) - \left(\rho_{qs}c_p\theta\right)\frac{\partial\left(\delta\pi\right)}{\partial z}$$

Continuity equation

$$\frac{\partial\rho_{qs}}{\partial t} + \nabla_H\cdot\left(\rho_{qs}\mathbf{V}_H\right) + \frac{\partial}{\partial z}\left(\rho_{qs}w\right) = 0$$

3D elliptic equation for $\delta\pi$

$$\nabla_H\cdot\left[\left(\rho_{qs}c_p\theta\right)\nabla_H\left(\delta\pi\right)\right] + \frac{\partial}{\partial z}\left[\left(\rho_{qs}c_p\theta\right)\frac{\partial\left(\delta\pi\right)}{\partial z}\right] = \nabla_H\cdot\mathbf{G}_3 + \frac{\partial}{\partial z}G_4 + \frac{\partial^2\rho_{qs}}{\partial t^2}$$

Other nonhydrostatic variables

$$\delta p \approx \frac{p_{qs}}{\kappa\pi_{qs}}\delta\pi \qquad \frac{\delta T}{T_{qs}} \approx \frac{\delta\pi}{\pi_{qs}} \qquad \frac{\delta\rho}{\rho_{qs}} \approx \left(1-\kappa\right)\frac{\delta p}{p_{qs}}$$

INTEGRATION PROCEDURES OF THE UNIFIED SYSTEM OF EQUATIONS (I)

Continuity equation

$$\int_{z_T}^{z_S} (\rho_{qs})^{n+1} dz = \int_{z_T}^{z_S} (\rho_{qs})^{n} dz - \nabla_H \cdot \int_{z_T}^{z_S} (\rho_{qs}\mathbf{V}_H)^{n} dz$$

Thermodynamic equation

$$\frac{\theta^{n+1}-\theta^{n}}{\Delta t} = (G_5)^n + \frac{Q^n}{(\pi_{qs})^n}$$

↓

Quasi-hydrostatic variables

$$(\pi_{qs}^{1/\kappa})_T^{n+1} = \left[(\pi_{qs})_T^{n+1} - \frac{g}{c_p}\int_{z_T}^{z_S}\frac{1}{\theta^{n+1}}dz\right]^{1/\kappa} + \frac{g}{p_{00}}\int_{z_T}^{z_S}(\rho_{qs})^{n+1}dz$$

$$(\pi_{qs})^{n+1} = (\pi_{qs})_T^{n+1} - \frac{g}{c_p}\int_{z_T}^{z}\frac{1}{\theta^{n+1}}dz$$

↓

$$(\rho_{qs})^{n+1} = \frac{p_{00}\left(\pi_{qs}^{(1-\kappa)/\kappa}\right)^{n+1}}{R\theta^{n+1}}$$

↓

Nonhydrostatic variables

$$\nabla_H \cdot \left[(\rho_{qs}c_p\theta)^n \nabla_H(\delta\pi)\right] + \frac{\partial}{\partial z}(\rho_{qs}c_p\theta)^n \frac{\partial(\delta\pi)}{\partial z} = \nabla_H \cdot (\mathbf{G}_3)^n + \frac{\partial}{\partial z}(G_4)^n + \frac{(\rho_{qs})^{n+1} - 2(\rho_{qs})^n + (\rho_{qs})^{n-1}}{\Delta t^2}$$

↓

Momentum equation

$$\frac{(\rho_{qs}\mathbf{V}_H)^{n+1} - (\rho_{qs}\mathbf{V}_H)^n}{\Delta t} = (\mathbf{G}_3)^n - (\rho_{qs}c_p\theta)^n \nabla_H(\delta\pi)$$

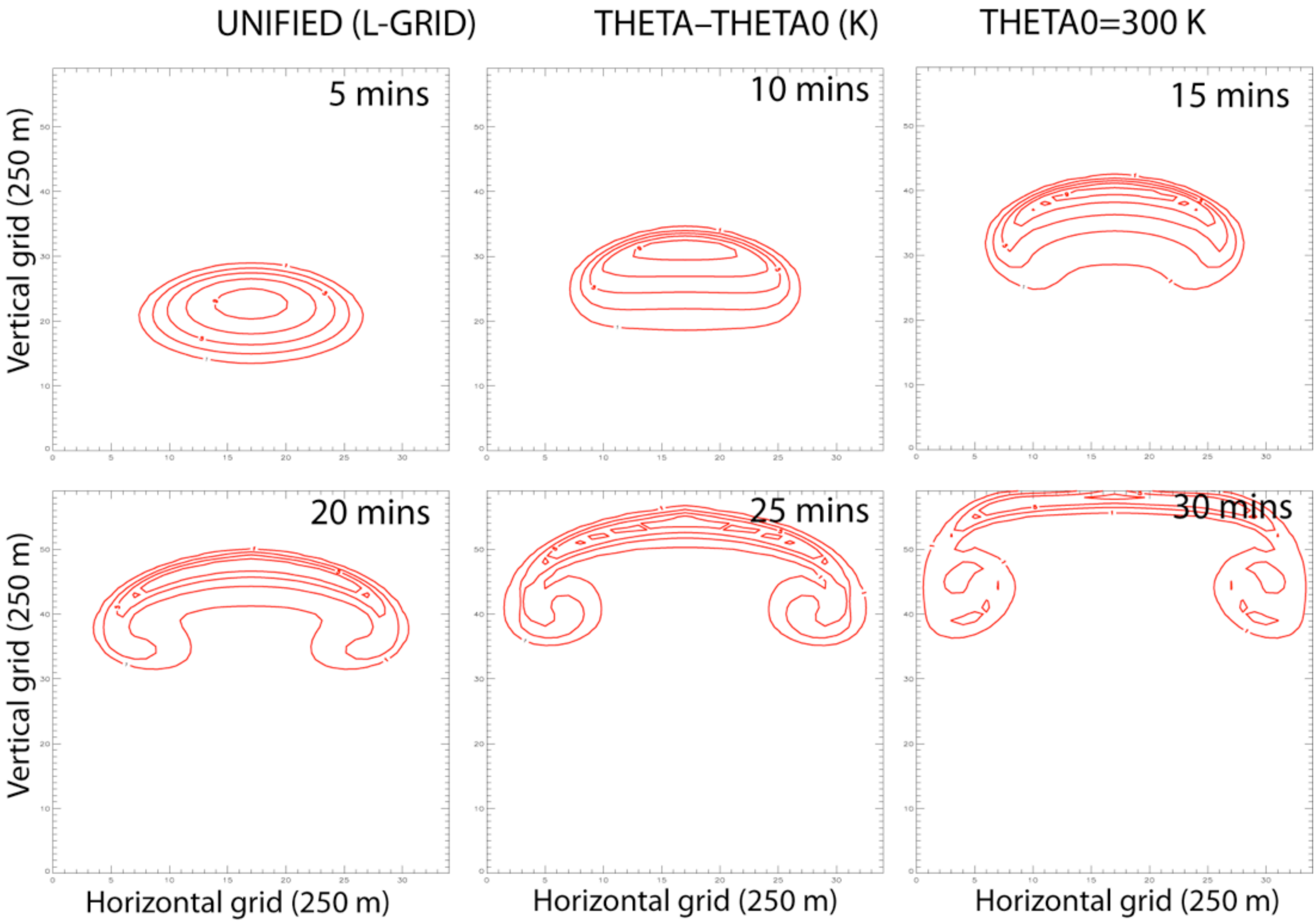
↓

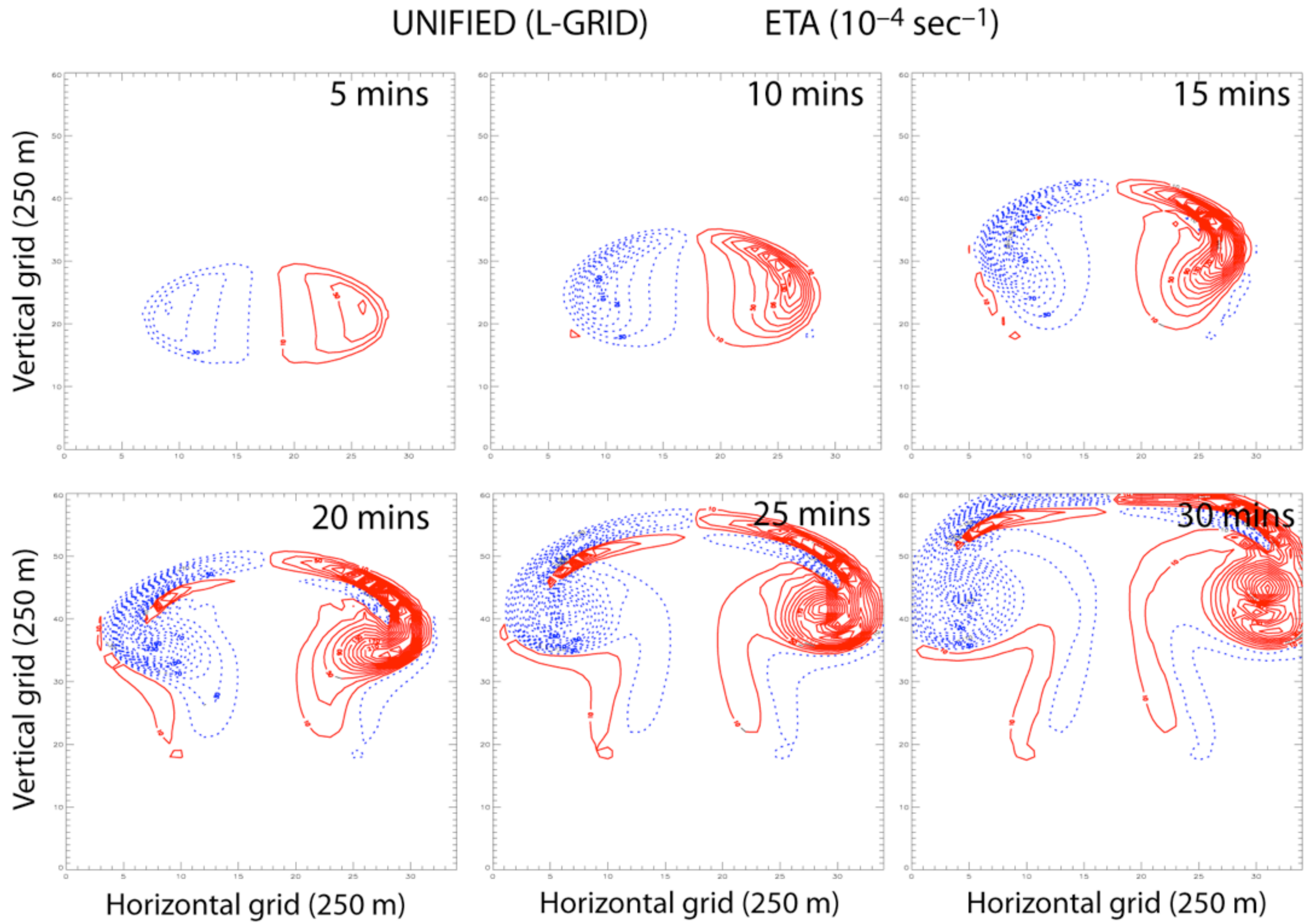
Continuity equation

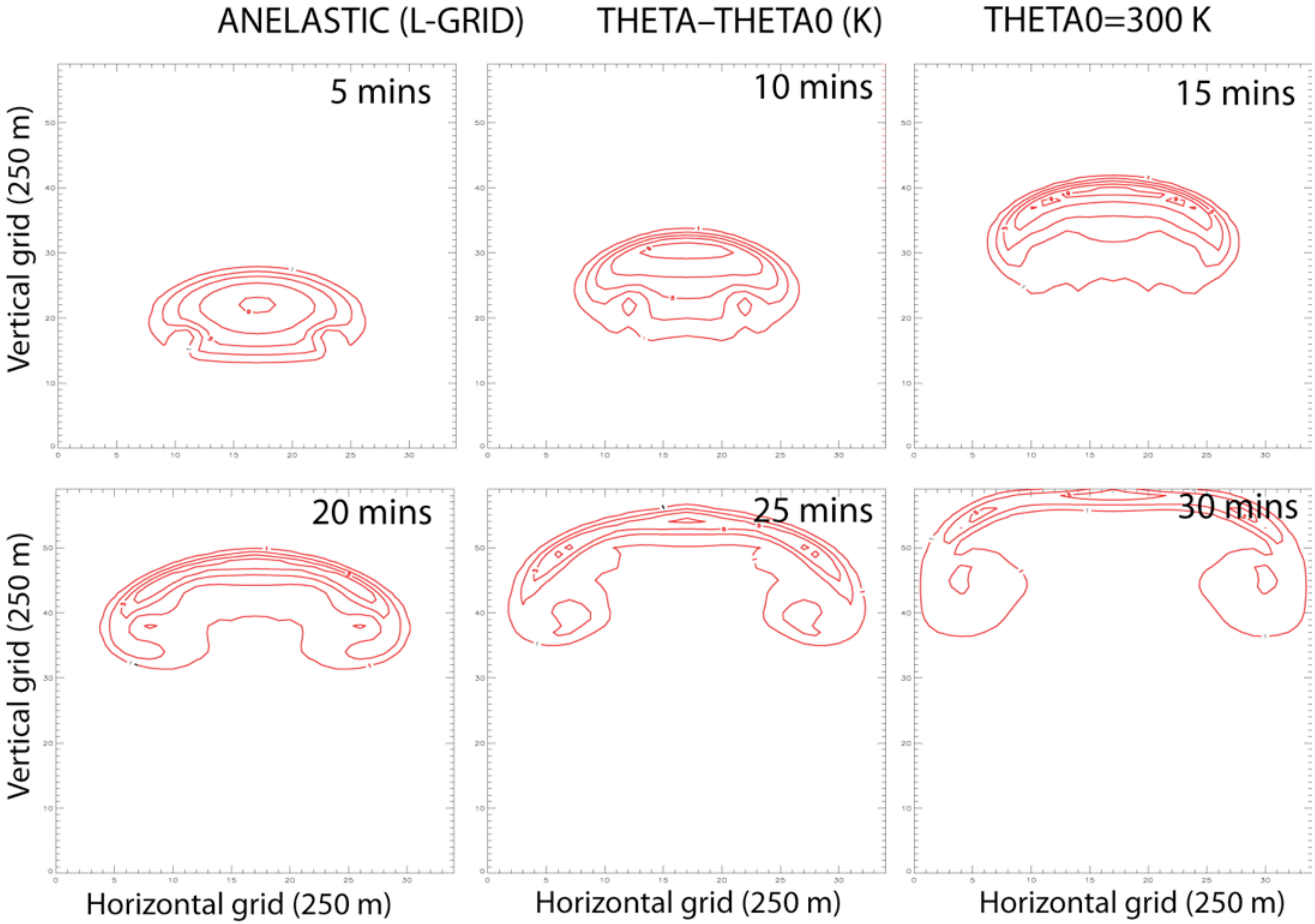
$$(\rho_{qs}w)^{n+1} = \int_{z_T}^{z} \nabla_H \cdot (\rho_{qs}\mathbf{V}_H)^{n+1} dz - \int_{z_T}^{z} \frac{(\rho_{qs})^{n+1} - (\rho_{qs})^n}{\Delta t} dz$$

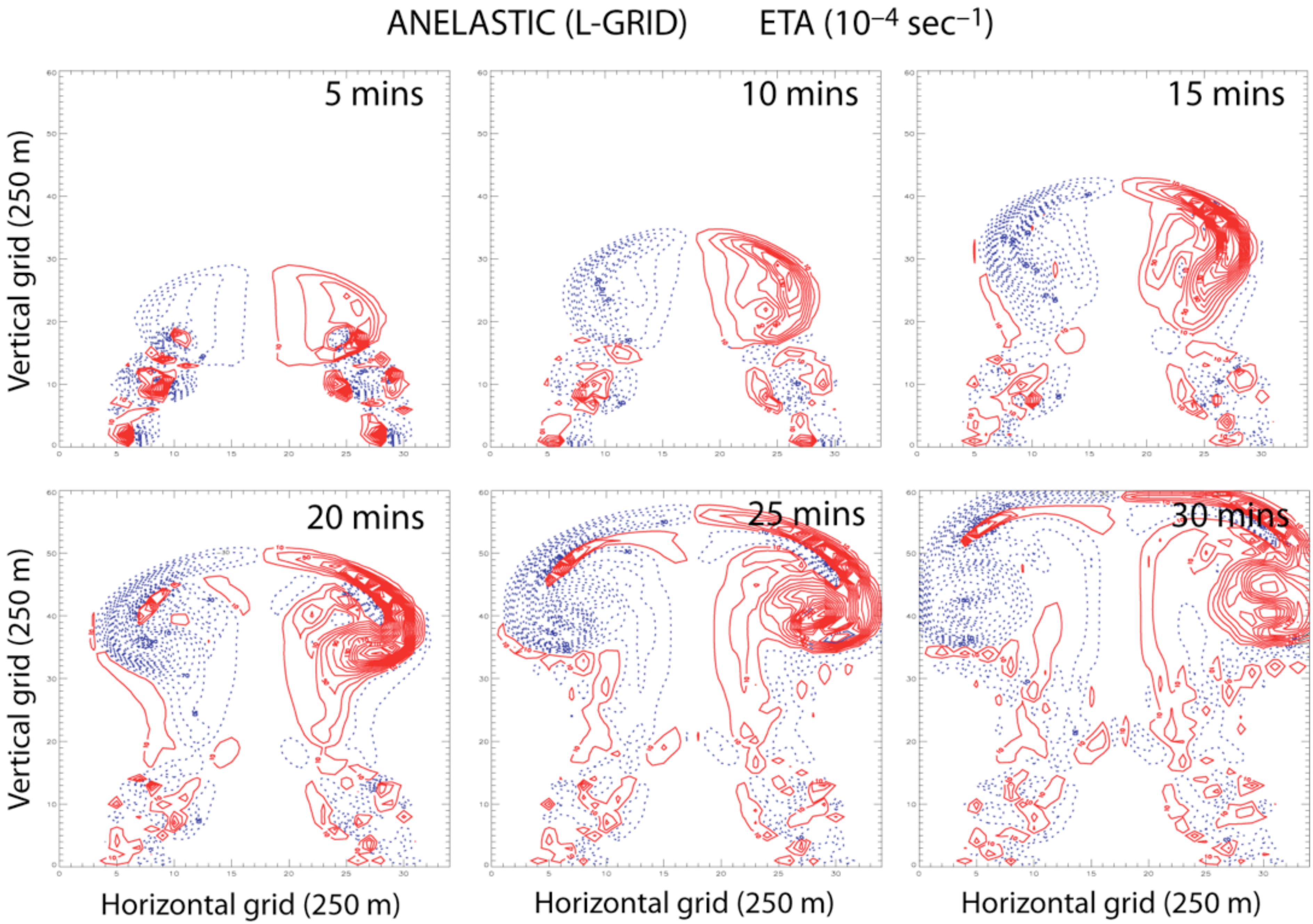
BUOYANT BUBBLE SIMULATIONS
WITH THE UNIFIED, ANELASTIC AND VVM MODELS

- The unified and anelastic models are based on the direct momentum prediction. The momentum equation is discretized following Arakawa and Lamb (1981). Integration procedure (I) is used.
- The thermodynamic equation is discretized on the Lorenz (L) grid. The anelastic model has also the Charney-Phillips (CP) grid version.
- Horizontal and vertical grid distance is 250 m.

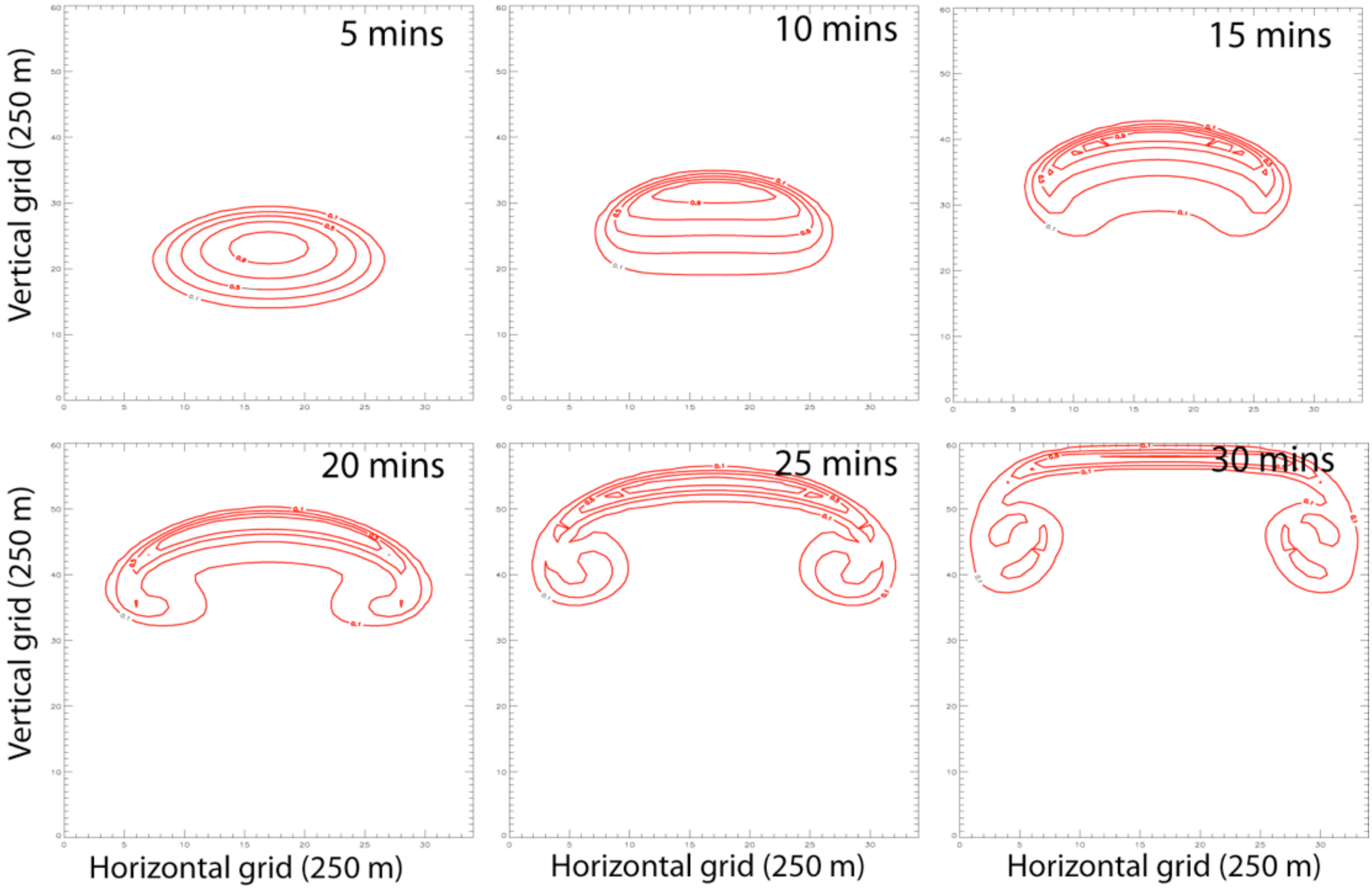

UNIFIED (L-GRID)
THETA–THETA0 (K)
THETA0=300 K
5 mins
10 mins
15 mins
20 mins
25 mins
30 mins
Vertical grid (250 m)
Vertical grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)


UNIFIED (L-GRID)
ETA (10^{-4} sec^{-1})
5 mins
10 mins
15 mins
20 mins
25 mins
30 mins
Vertical grid (250 m)
Horizontal grid (250 m)

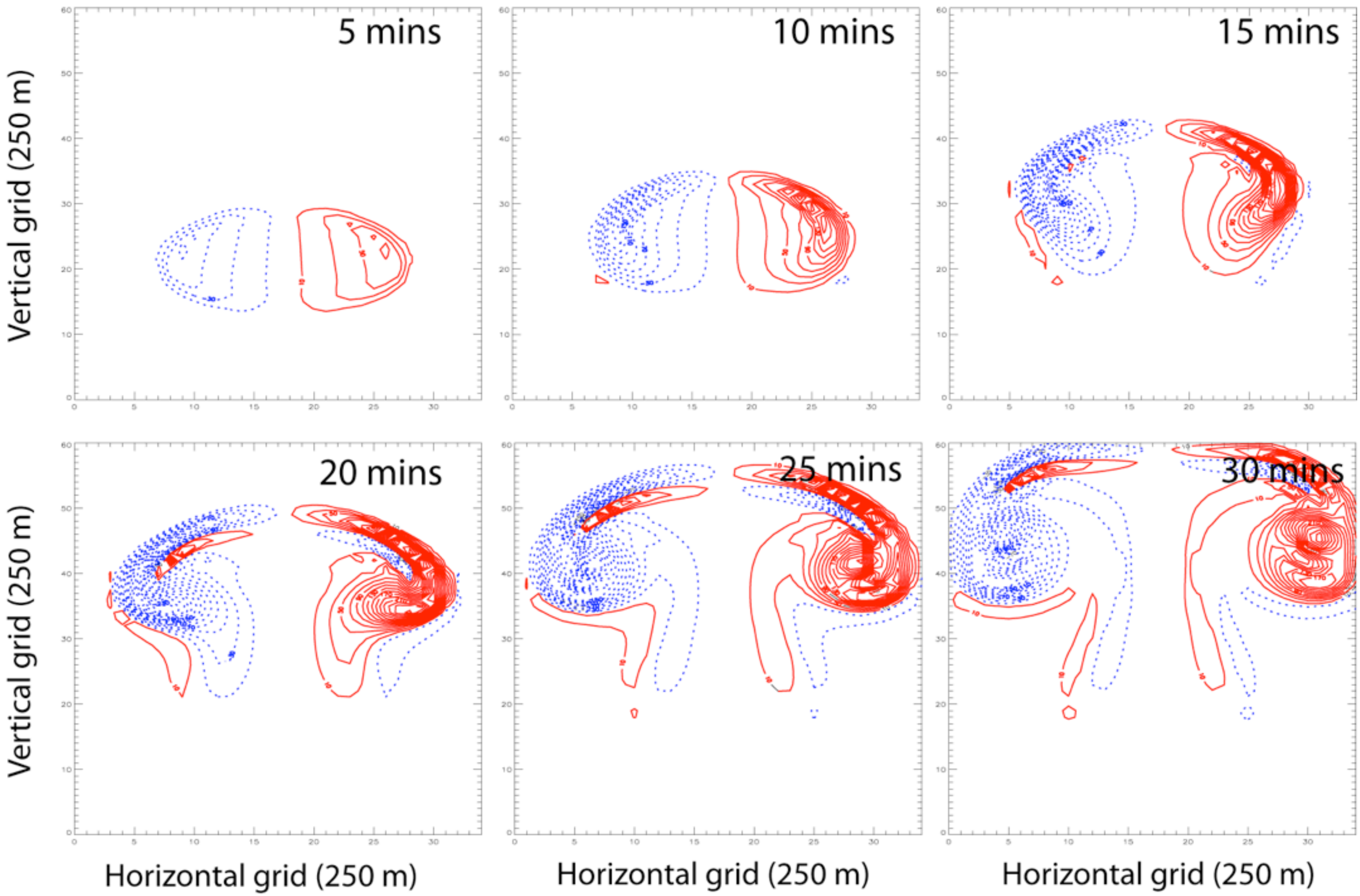

ANELASTIC (L-GRID)
THETA–THETA0 (K)
THETA0=300 K
5 mins
10 mins
15 mins
20 mins
25 mins
30 mins
Vertical grid (250 m)
Vertical grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)

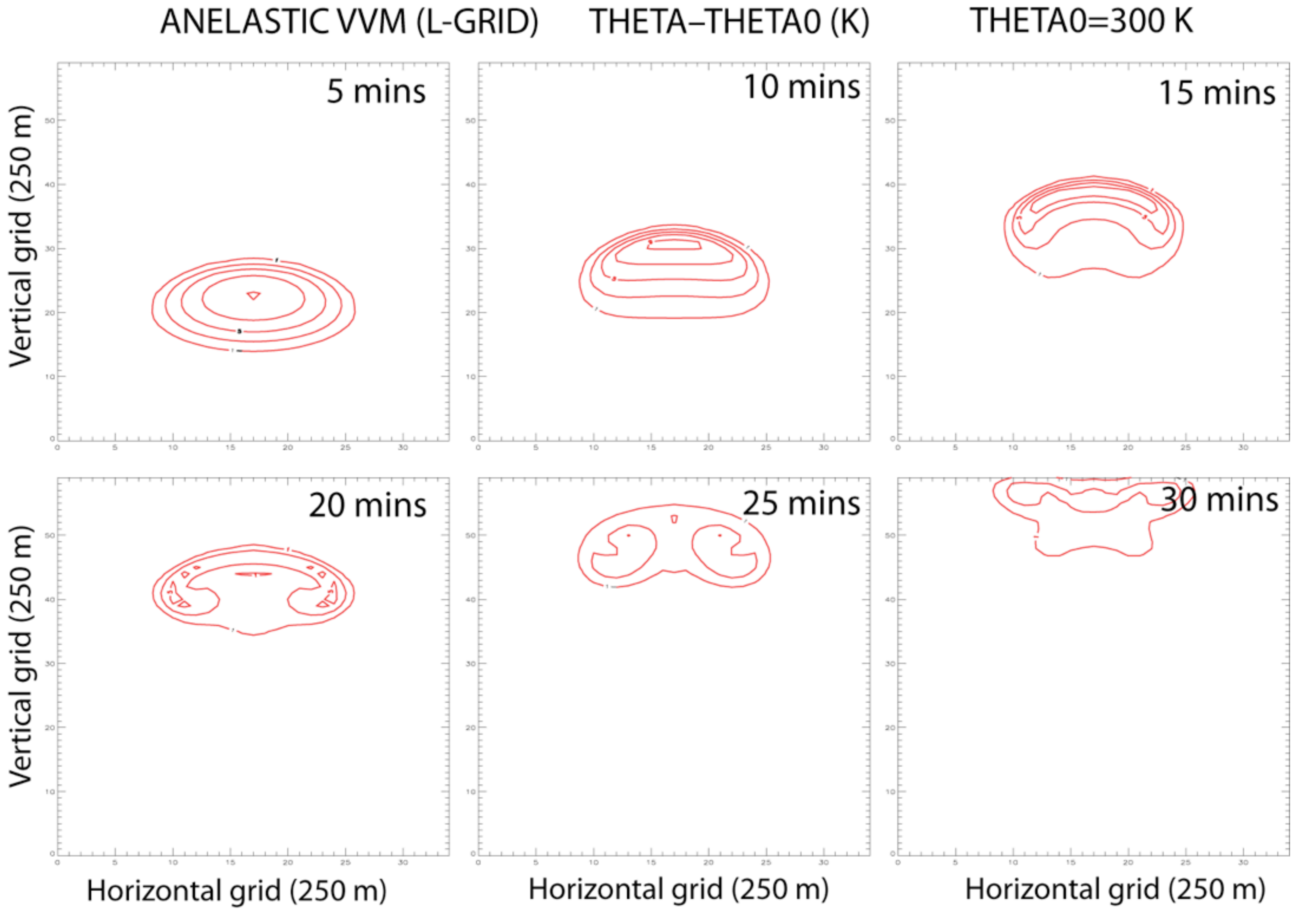

ANELASTIC (L-GRID) ETA (10^{-4} sec^{-1})
5 mins
10 mins
15 mins
20 mins
25 mins
30 mins
Vertical grid (250 m)
Vertical grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)

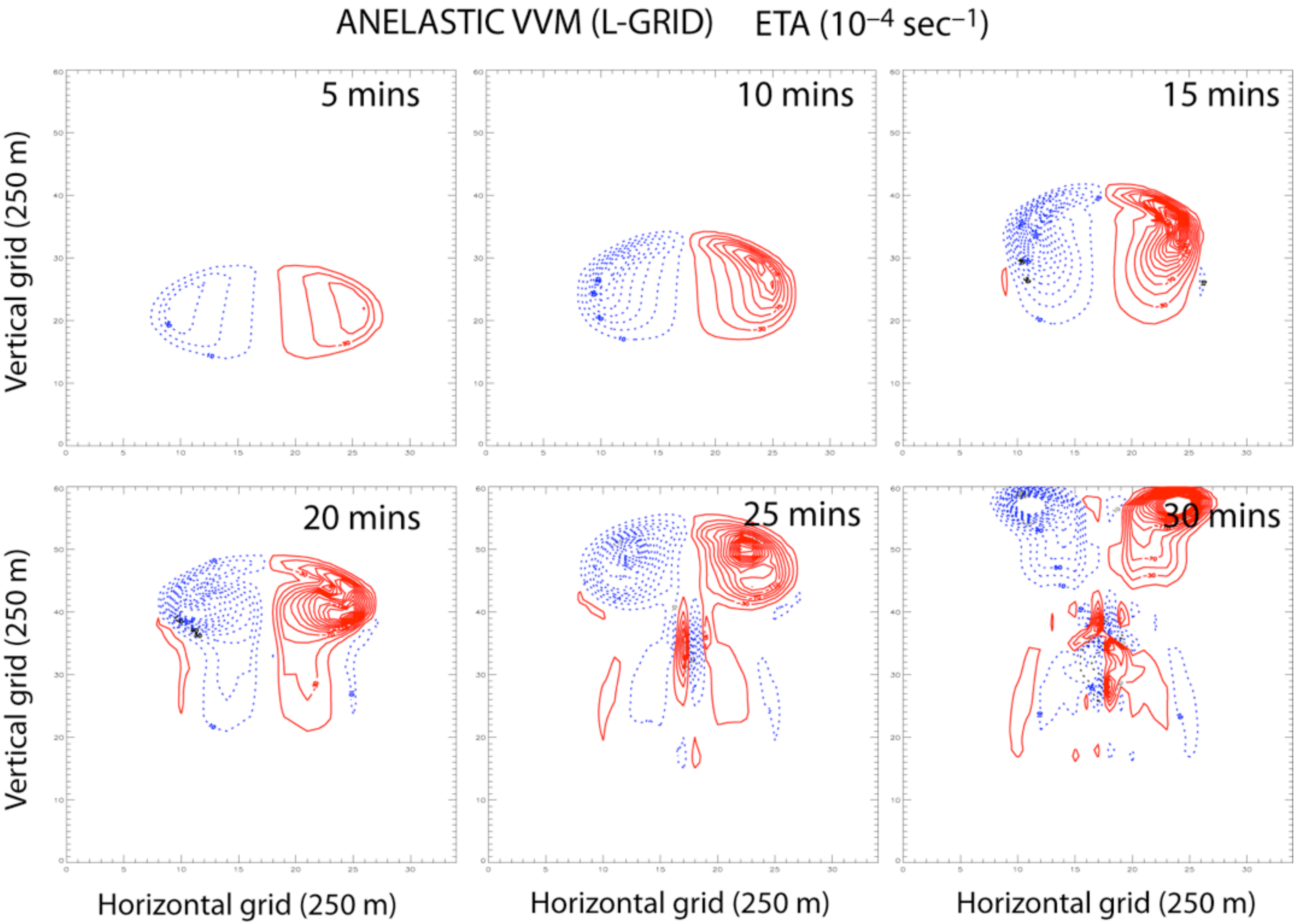
ANELASTIC (CP-GRID) THETA–THETA0 (K) THETA0=300 K



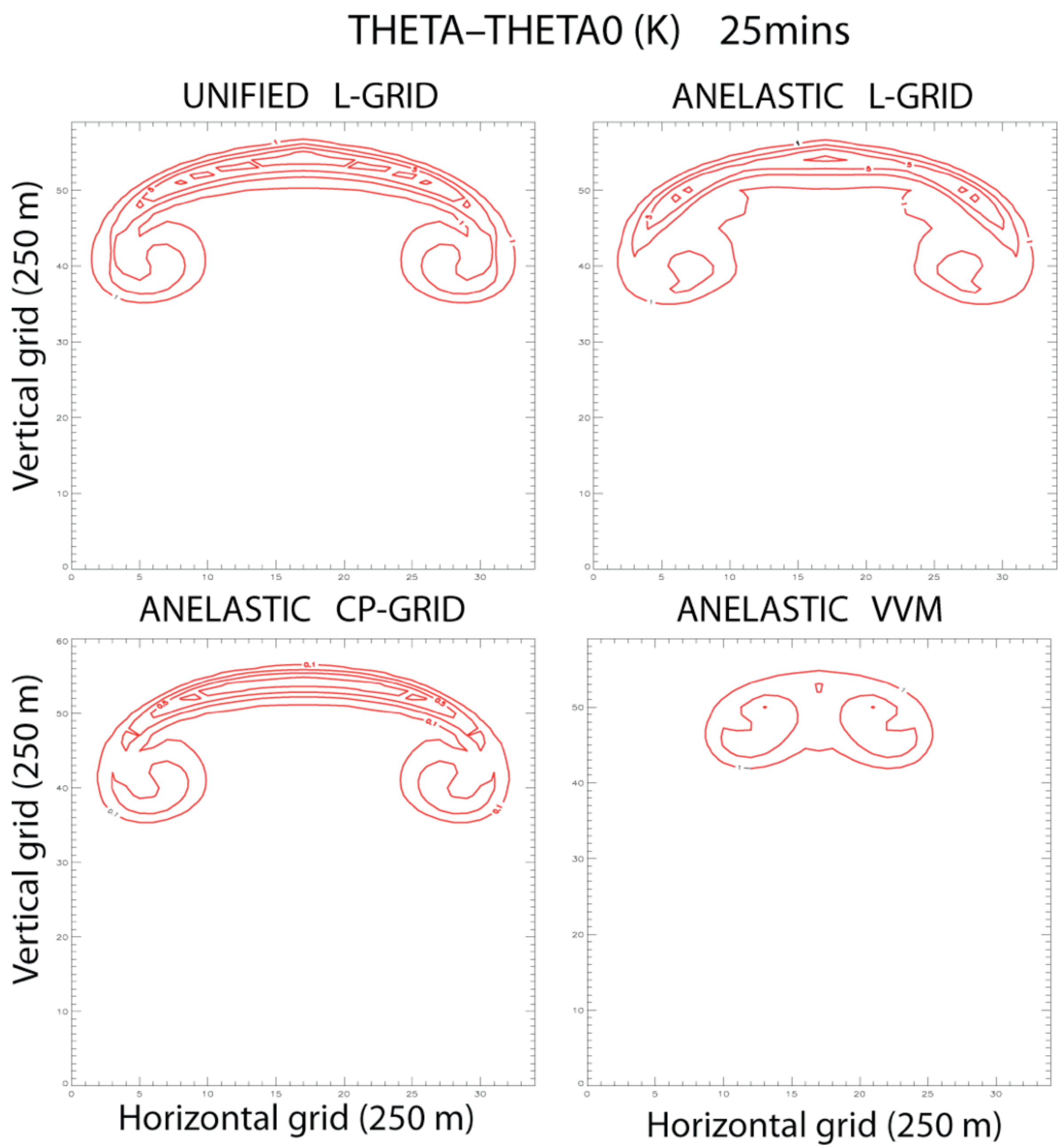
ANELASTIC (CP-GRID) ETA (10^{-4} sec^{-1})



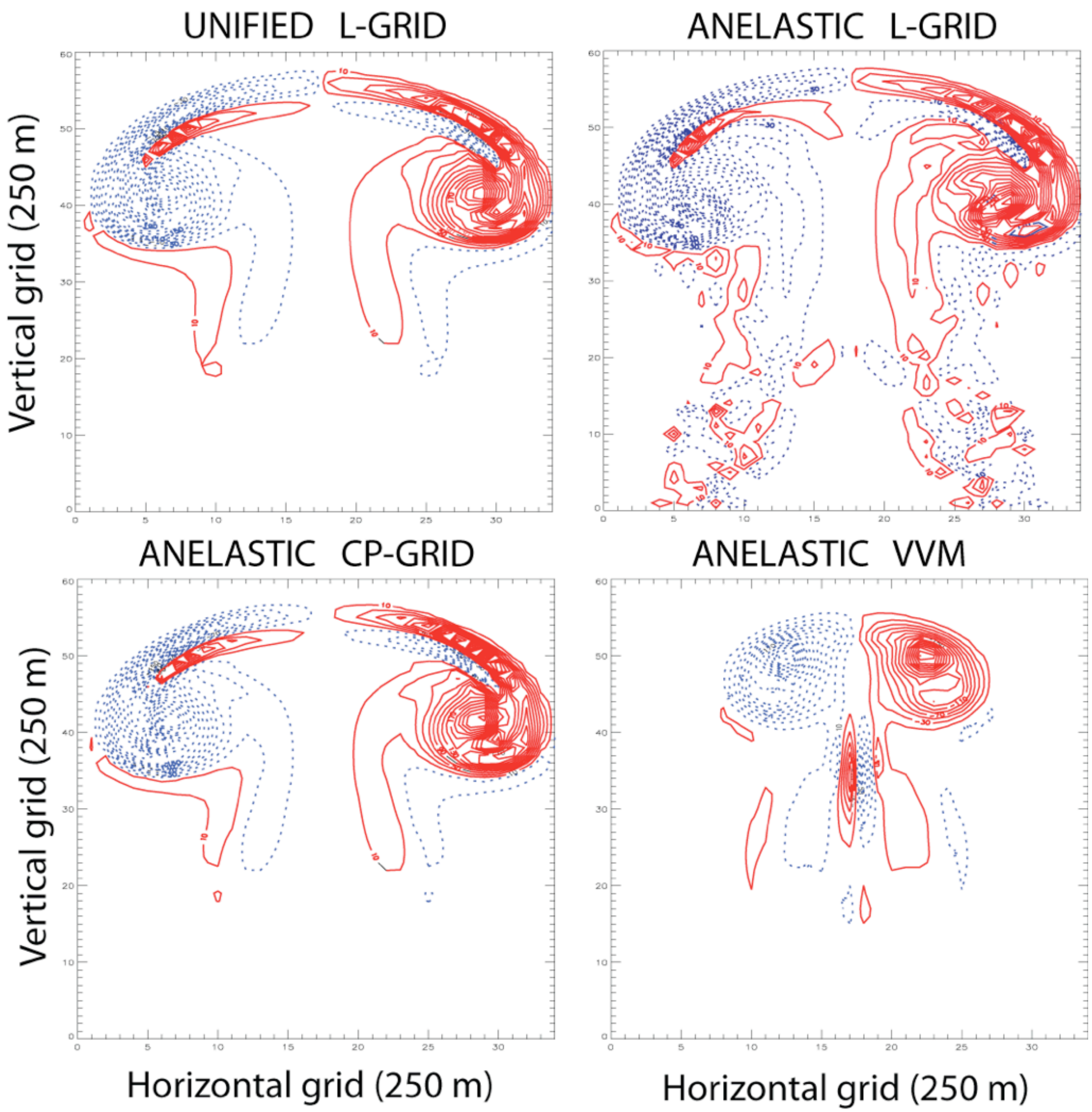

ANELASTIC VVM (L-GRID)
THETA–THETA0 (K)
THETA0=300 K
5 mins
10 mins
15 mins
20 mins
25 mins
30 mins
Vertical grid (250 m)
Vertical grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)



ANELASTIC VVM (L-GRID) ETA (10⁻⁴ sec⁻¹)
5 mins
10 mins
15 mins
20 mins
25 mins
30 mins
Vertical grid (250 m)
Vertical grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)


THETA–THETA0 (K) 25mins
UNIFIED L-GRID
ANELASTIC L-GRID
ANELASTIC CP-GRID
ANELASTIC VVM
Vertical grid (250 m)
Vertical grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)

ETA (25 mins)


UNIFIED L-GRID
ANELASTIC L-GRID
ANELASTIC CP-GRID
ANELASTIC VVM
Vertical grid (250 m)
Vertical grid (250 m)
Horizontal grid (250 m)
Horizontal grid (250 m)

INTEGRATION PROCEDURES OF THE UNIFIED SYSTEM OF EQUATIONS (2)

Momentum equation

$$\frac{\left(\rho_{qs}\mathbf{V}_H\right)^{n+1}-\left(\rho_{qs}\mathbf{V}_H\right)^{n}}{\Delta t}=\left(\mathbf{G}_3\right)^n-\left(\rho_{qs}c_p\theta\right)^n\nabla_H\left(\delta\pi\right)$$

$$\frac{\left(\rho_{qs}w\right)^{n+1}-\left(\rho_{qs}w\right)^{n}}{\Delta t}=\left(G_4\right)^n-\left(\rho_{qs}c_p\theta\right)^n\frac{\partial\left(\delta\pi\right)}{\partial z}$$

Using existing $(\delta\pi)$

Continuity equation

$$\frac{\left(\rho_{qs}\right)^{n+1}-\left(\rho_{qs}\right)^{n}}{\Delta t}=-\nabla_H\cdot\left(\rho_{qs}\mathbf{V}_H\right)^{n+1}-\frac{\partial}{\partial z}\left(\rho_{qs}w\right)^{n+1}$$

Thermodynamic equation

$$\frac{\theta^{n+1}-\theta^{n}}{\Delta t}=\left(G_5\right)^n+\frac{Q^n}{\left(\pi_{qs}\right)^n}$$

Quasi-hydrostatic variables

$$\left(\pi_{qs}^{1/\kappa}\right)_T^{n+1}=\left[\left(\pi_{qs}\right)_T^{n+1}-\frac{g}{c_p}\int_{z_T}^{z_S}\frac{1}{\theta^{n+1}}dz\right]^{1/\kappa}+\frac{g}{p_{00}}\int_{z_T}^{z_S}\left(\rho_{qs}\right)^{n+1}dz$$

$$\left(\pi_{qs}\right)^{n+1}=\left(\pi_{qs}\right)_T^{n+1}-\frac{g}{c_p}\int_{z_T}^{z}\frac{1}{\theta^{n+1}}dz$$

Nonhydrostatic variables

$$\nabla_H\cdot\left[\left(\rho_{qs}c_p\theta\right)^n\nabla_H\left(\delta\pi\right)\right]+\frac{\partial}{\partial z}\left(\rho_{qs}c_p\theta\right)^n\frac{\partial\left(\delta\pi\right)}{\partial z}=$$

$$\nabla_H\cdot\left(\mathbf{G}_3\right)^n+\frac{\partial}{\partial z}\left(G_4\right)^n+\frac{\left(\rho_{qs}\right)^{n+1}-2\left(\rho_{qs}\right)^{n}+\left(\rho_{qs}\right)^{n-1}}{\Delta t^2}$$

BUOYANT BUBBLE SIMULATIONS WITH THE UNIFIED, ANELASTIC AND VVM MODELS

- Improve the potential temperature advection in the VVM.
- Construct an US-VVM.
- Construct the versions of the US and VVM with the CP-grid.
- Try to explain the differences between the simulations obtained by directly predicting momentum and vorticity.
- Construct an experimental fully-compressible model for comparison purposes.

Simulations with the VVCM using different radiation schemes (Tropical Warm Pool Experiment (TWP-ICE) Forcing)

Thomas Cram

Experiments: EXP5S, CAM5S, RRT5S (v 3.4 sw and v 4.4 lw) and RRTMG1 (v 3.8 sw and v 4.8 lw)