An update on model development

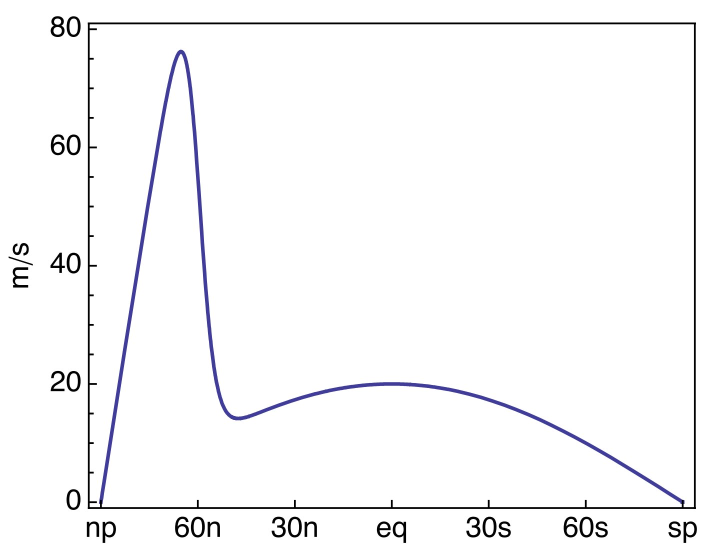
Ross Heikes, C.S. Konor and D. Randall

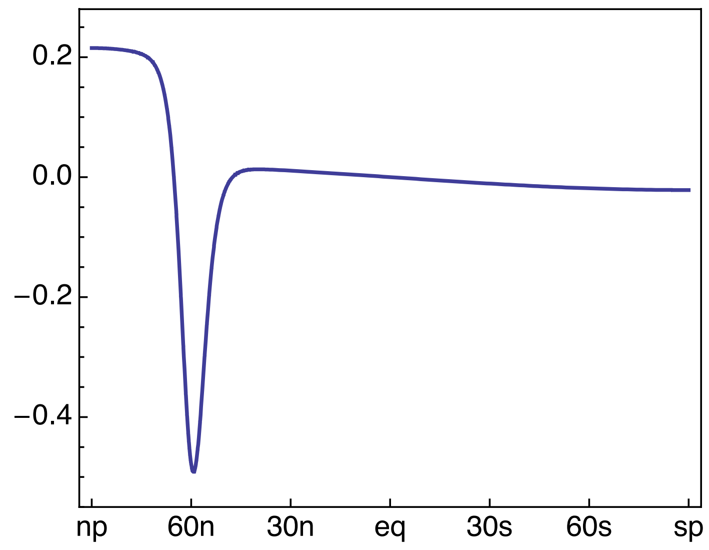
Dept. of Atmospheric Science
Colorado State University

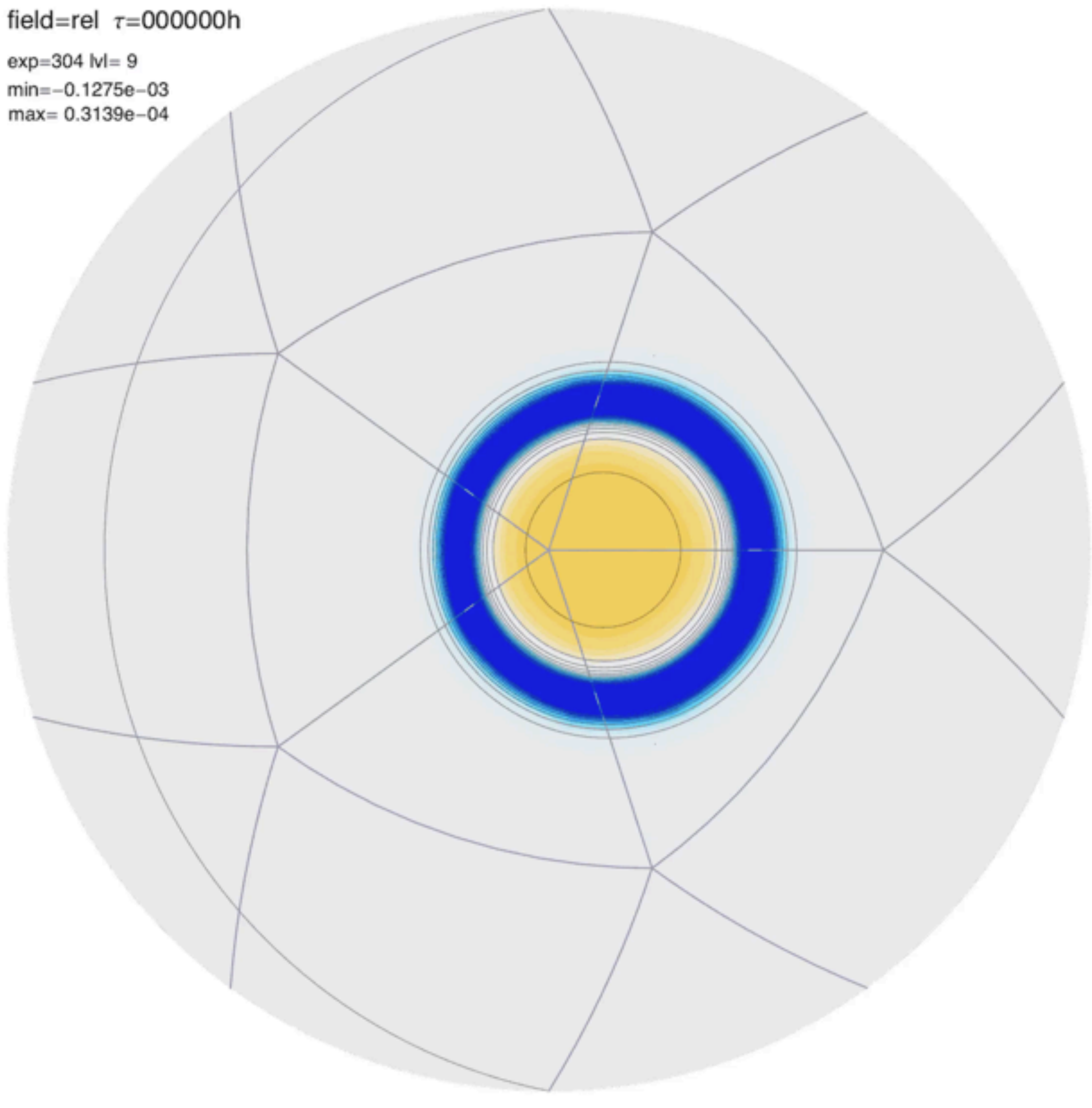
CMMAP winter meeting January 14, 2015

A barotropic vorticity test case

- Two superimposed solid body rotation.
- The faster is slightly offset from vertical.
- Polar stereographic projection







Start

- In the summer meeting we saw instances where grid imprinting seemed to cause a wavenumber 5 (of multiples of 5) errors in the results.
- Since then we have tried to improve the accuracy of the finite-difference operators to remove this error while maintaining properties of the continuous operators.

- Outline:
  1. The origin of wavenumber 5 pattern on the icosahedral grid
  2. Some examples on the grid imprinting
  3. Shallow water equations
  4. Properties of the continuous operators we wish to maintain in the discrete
  5. A proposed new scheme

Icosahedral grid. Projecting to the sphere.

- Our models live on an icosahedral grid.
- Starting with an icosahedron (fig. 1)
- We can project the icosahedron onto a unit sphere (fig. 2) forming 20 spherical triangles.

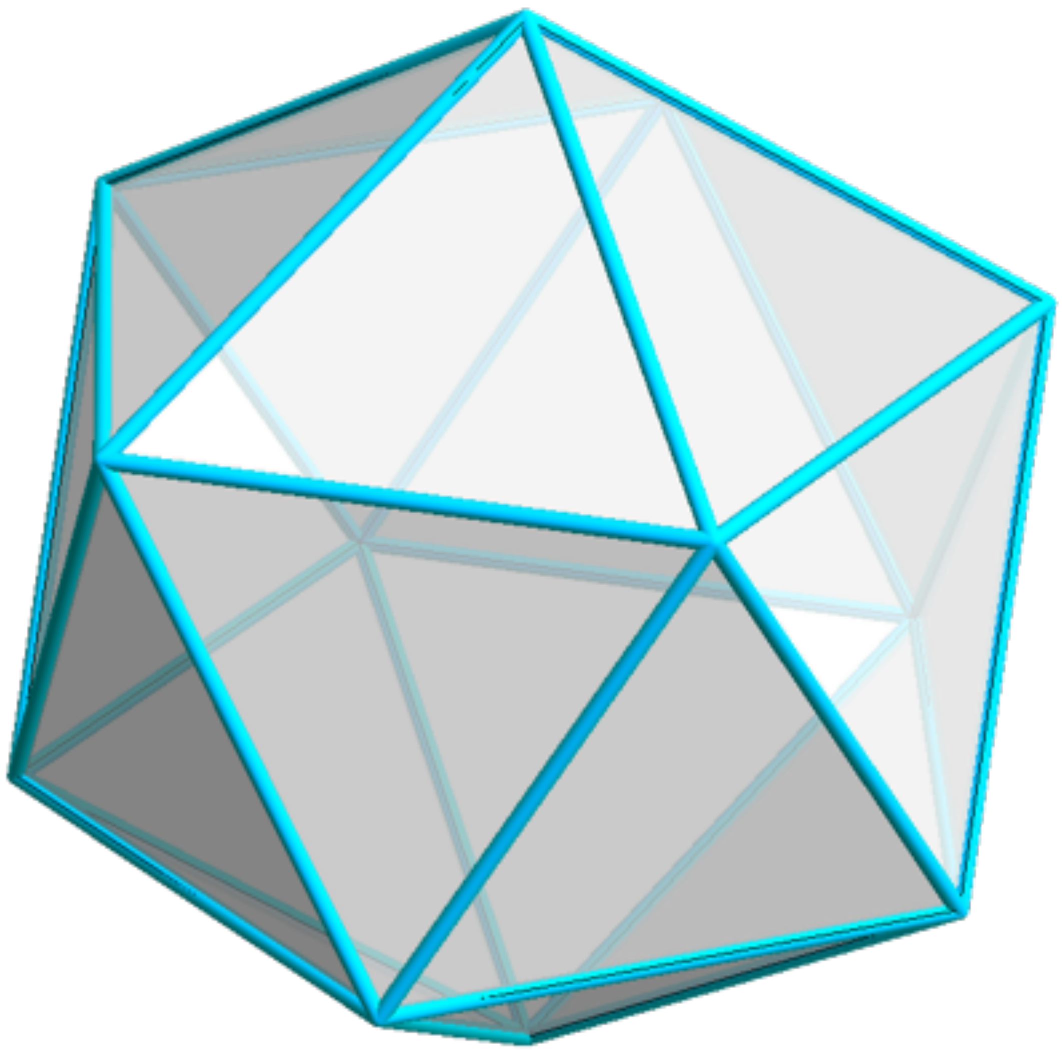


figure 1

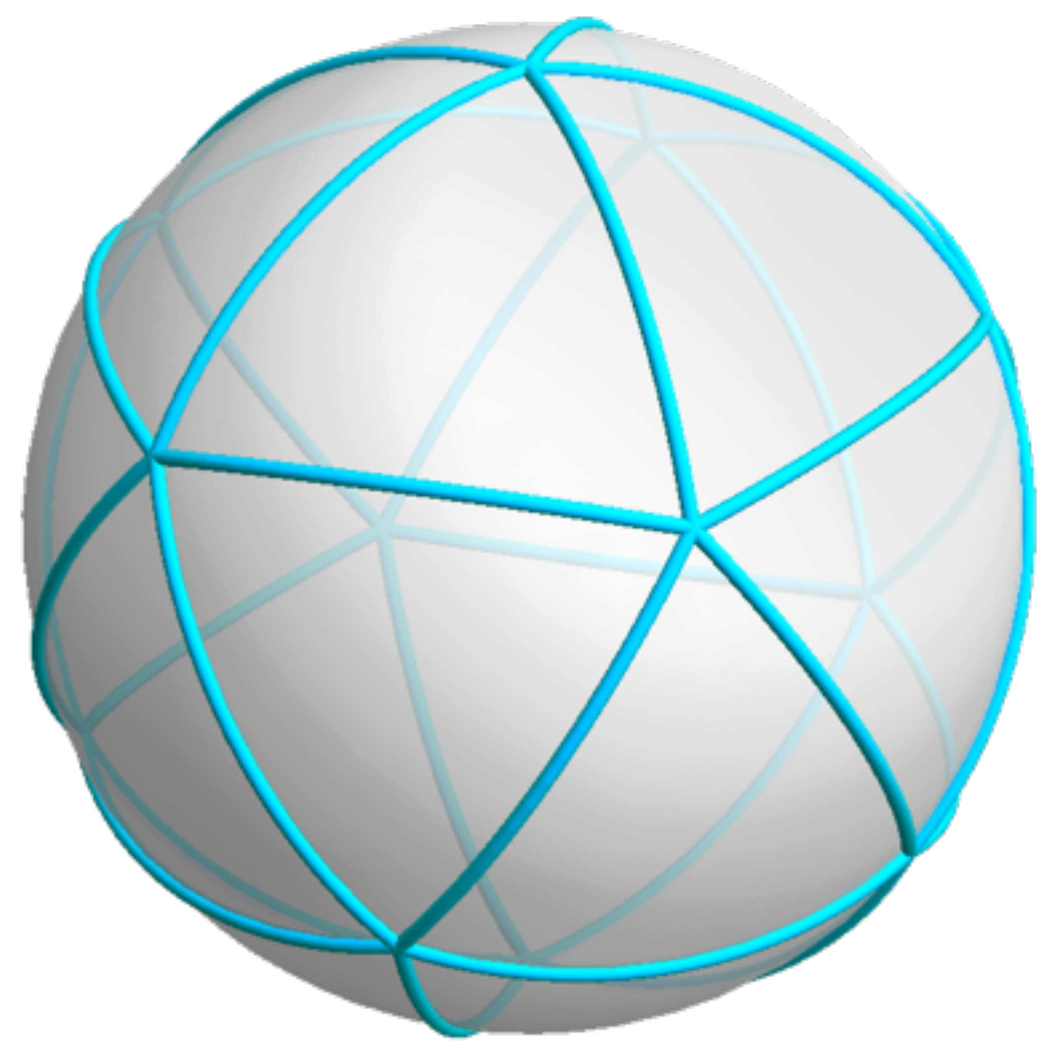
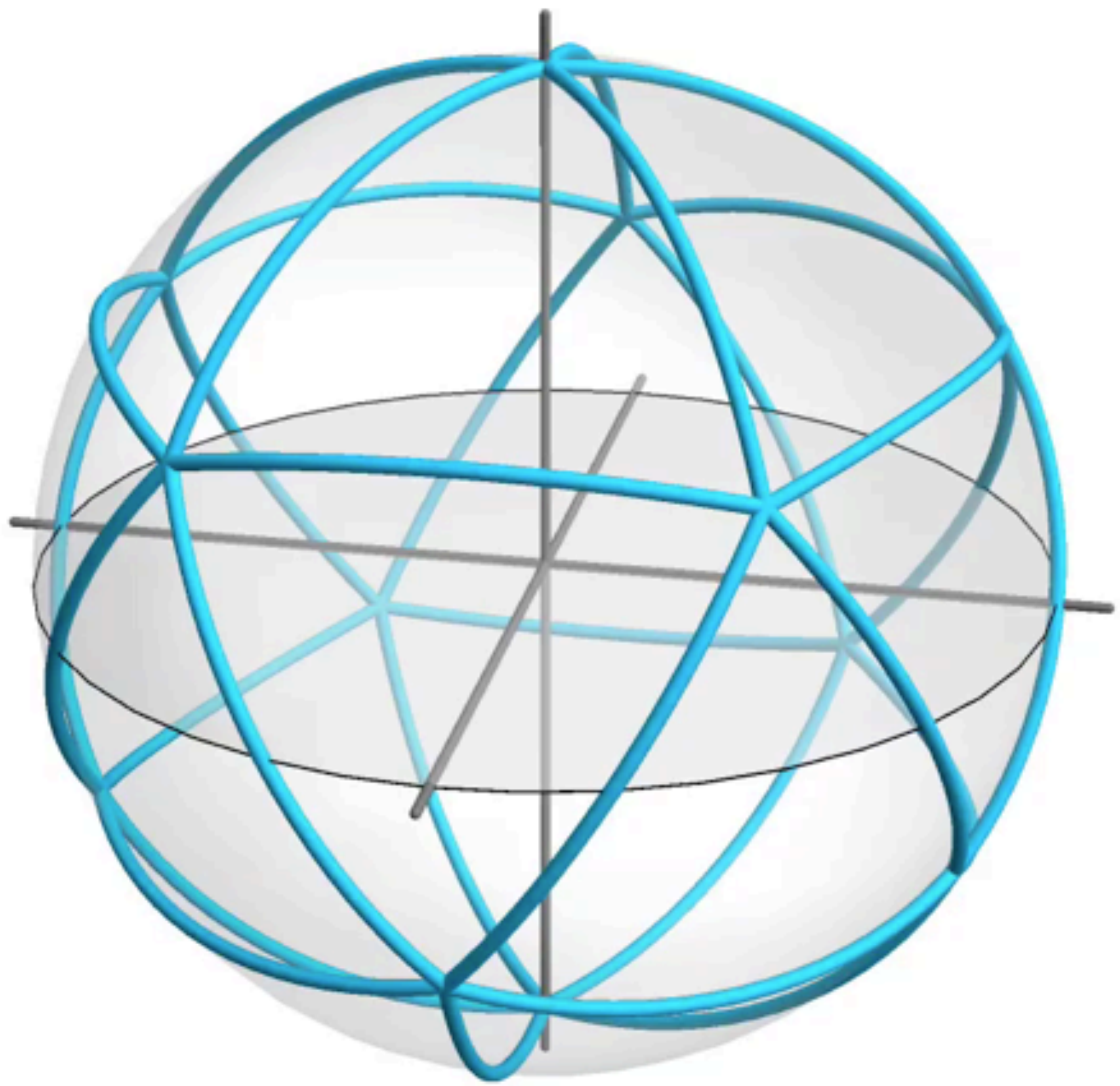


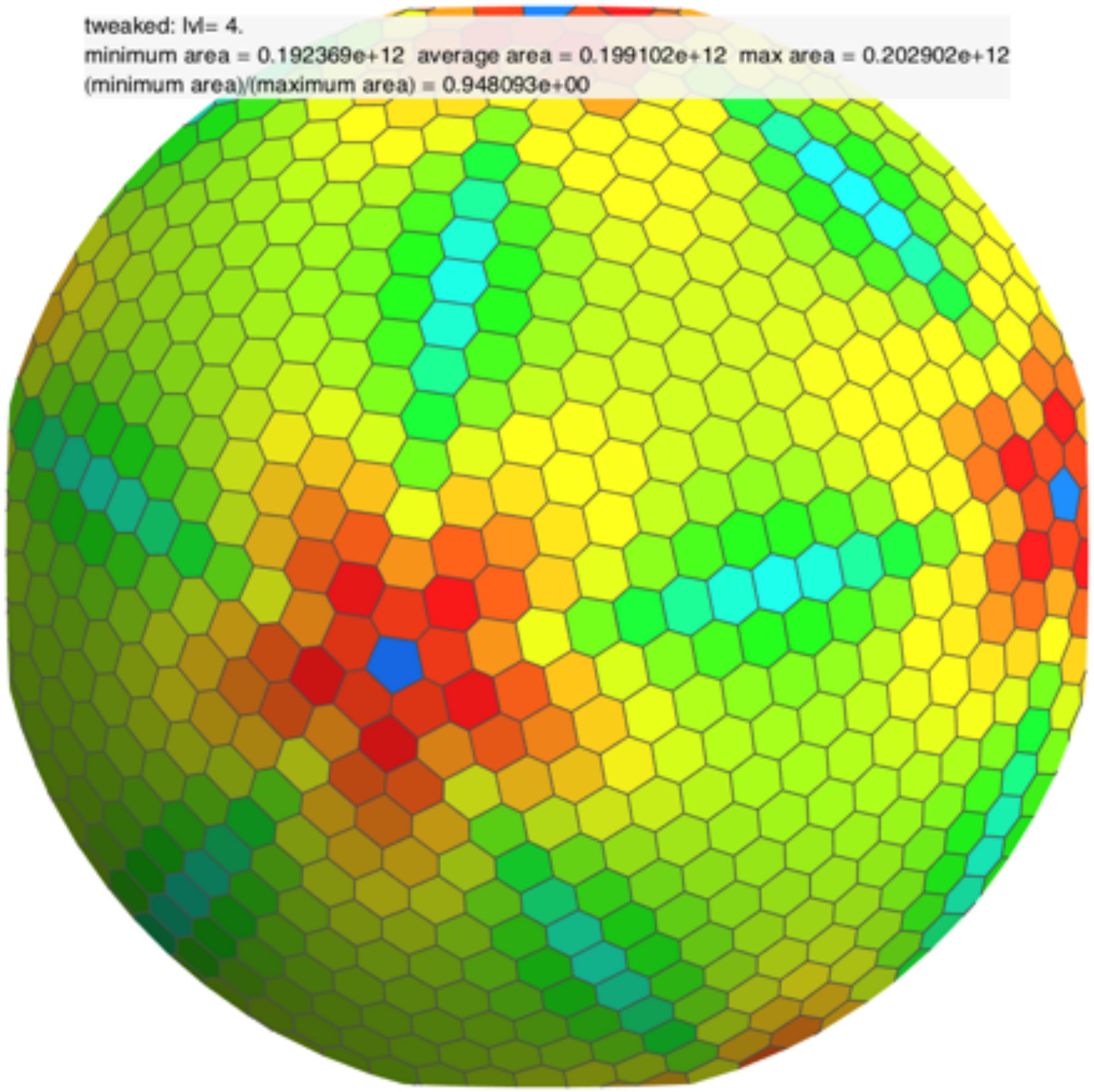
figure 2

Icosahedral grid axis of rotation

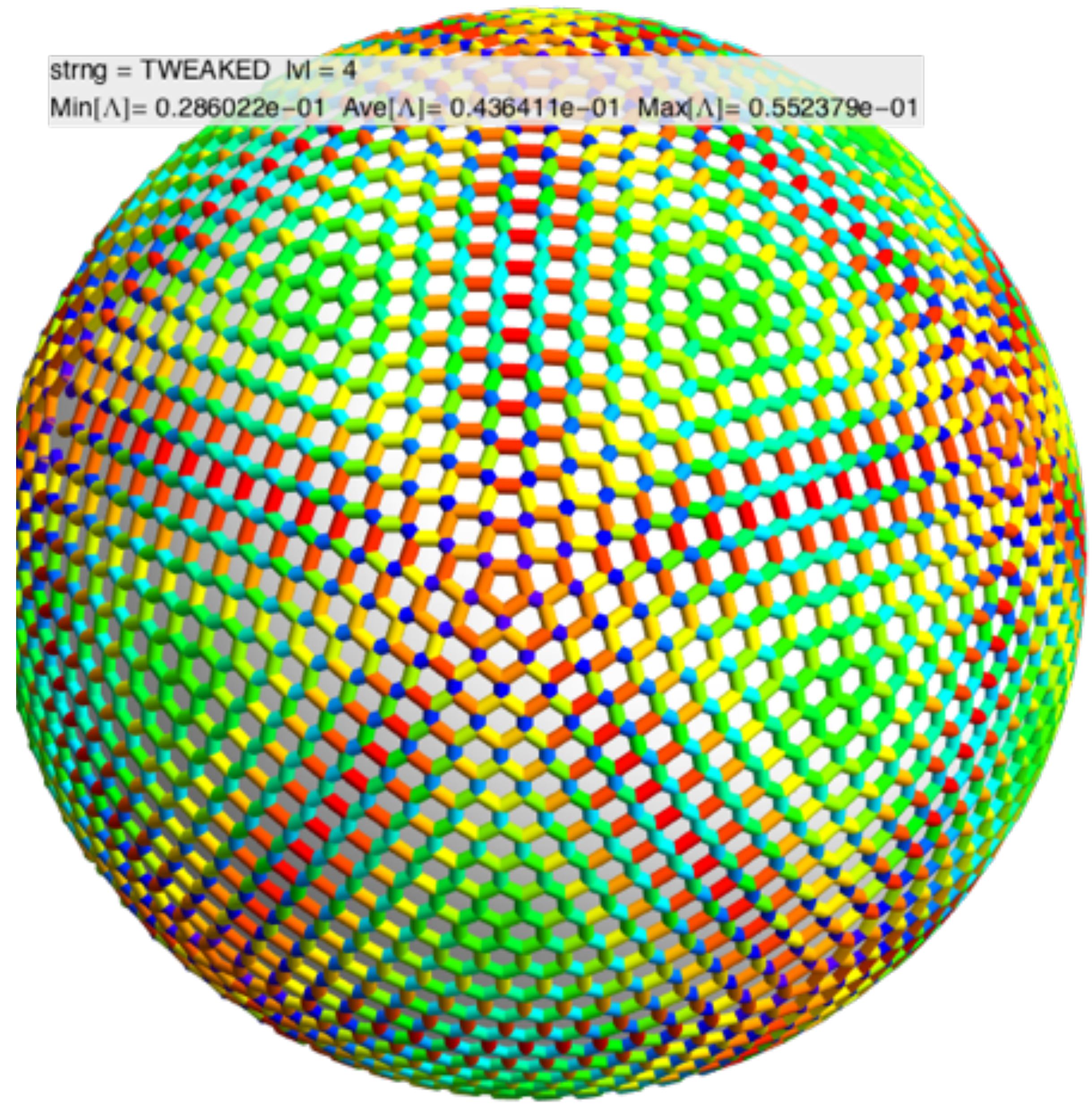
- The axis of rotation passes through two verities of the icosahedron
- Five pentagons on the northern hemisphere and five pentagons on the southern hemisphere



Sources of anisotropy



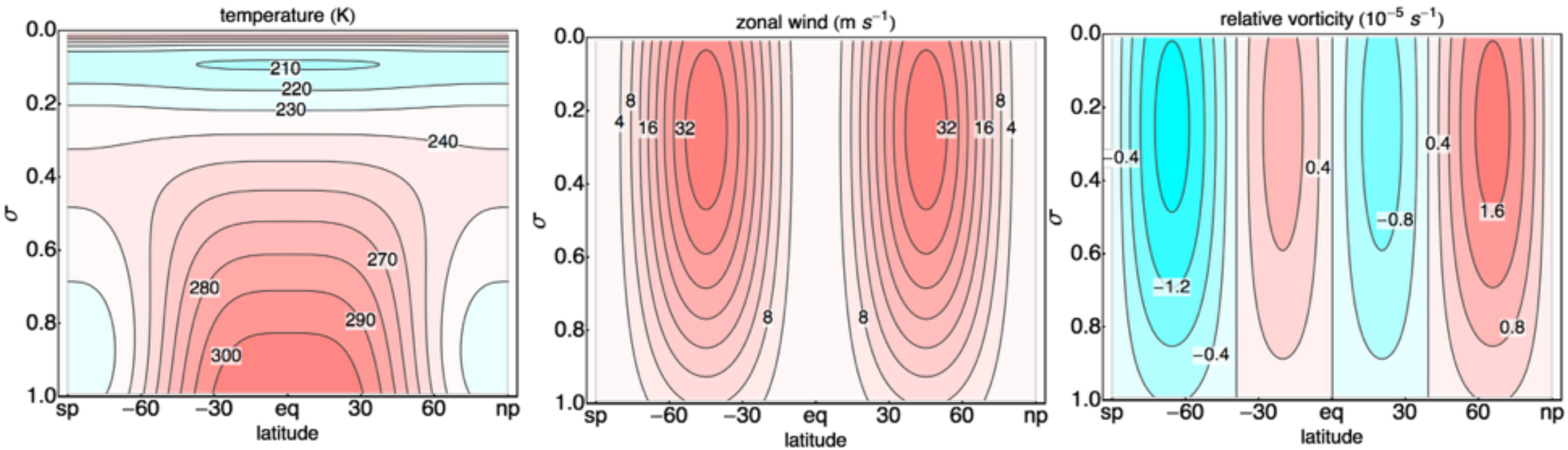
Cell area. The ratio of smallest cell to largest cell is about 0.95



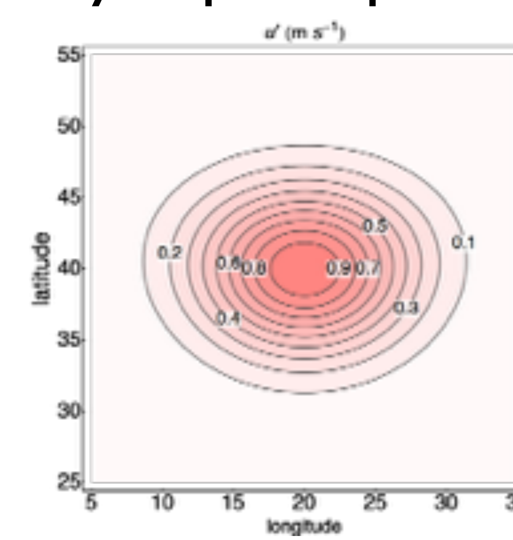
Length of edges. The ratio of shortest edge to longest edge is about 0.55

Extratropical cyclone in the 3D models

- Jablonowski and Williamson (2006) *Quart. J. Roy. Meteor. Soc.*, **132**, 2943-2975
- Prescribed **zonally symmetric** analytic formulas for initial prognostic variables:

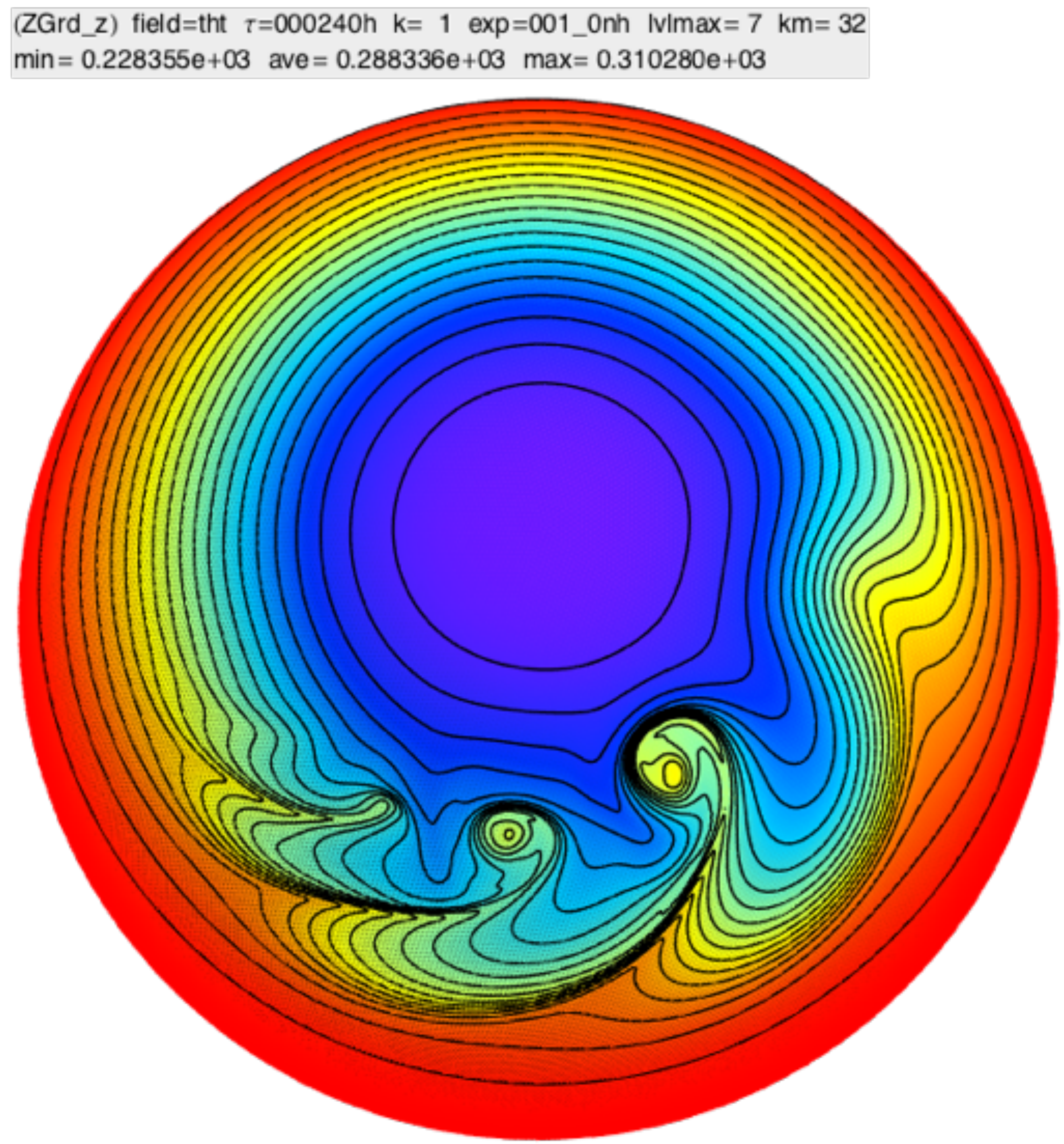


- A baroclinic wave is triggered in the balanced initial conditions by superimposing a perturbation (1 m s^{-1}) in zonal wind at each model level.

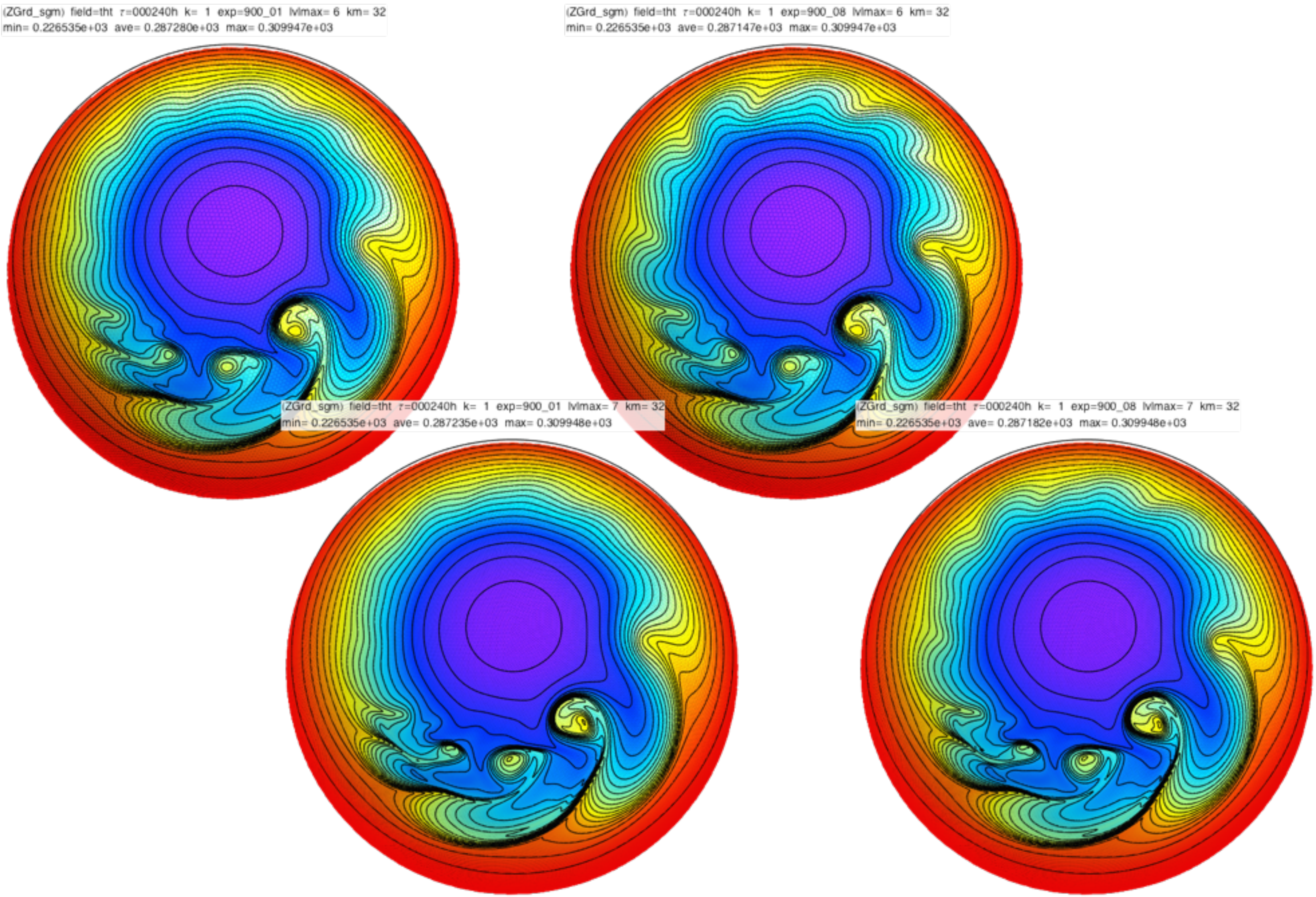


Height-coordinate. Jablonowski-Williamson.

- Surface theta is plotted
- Grid 7 (60 km)
- Day 10
- No visible wave number 5



Hybrid sigma-theta coordinate. Jablonowski-Williamson. Day 10. Grid 6 and 7. α=1 and α=8.



Galewsky test case for shallow water models

- Galewsky (2004) *Tellus*, 56A, 429-440
- Analytically specified, balanced, barotropically unstable, **zonally symmetric**, mid-latitude jet.
- Simple perturbation to initiate the instability

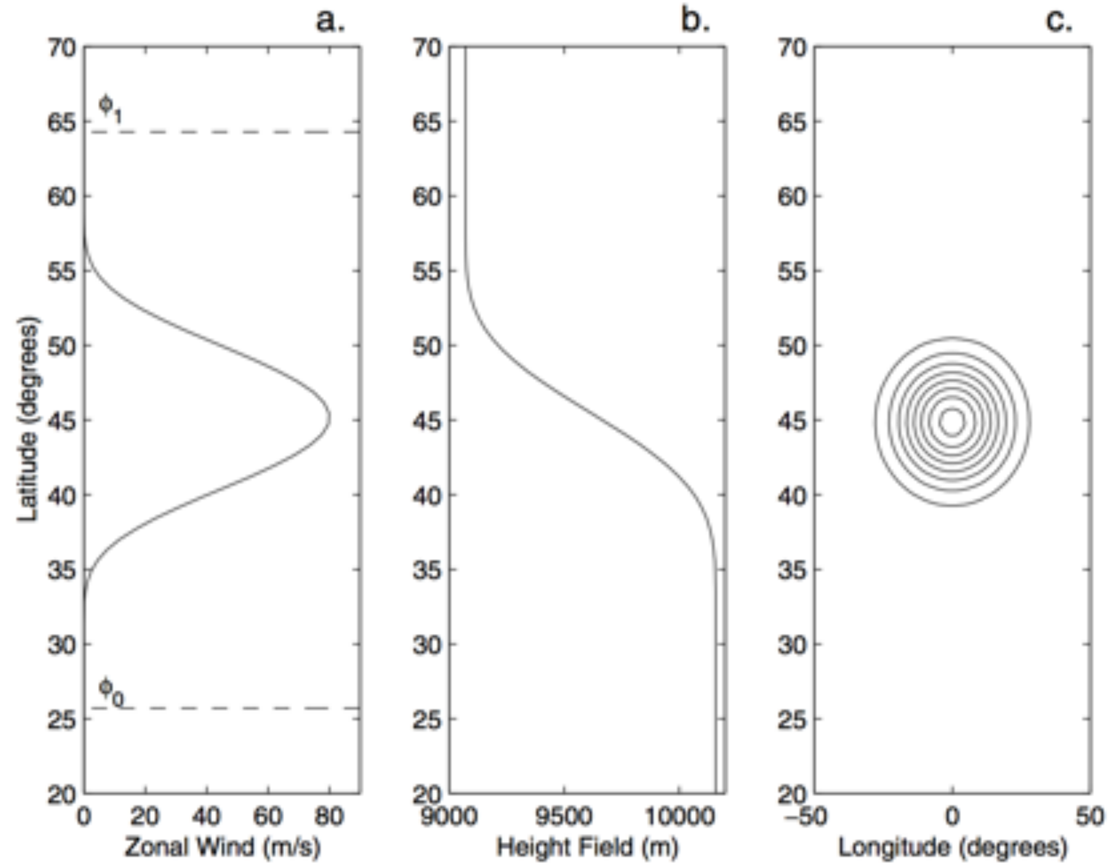


Fig 1. The initial conditions for the new test case. (a) The zonal wind, as defined in eq. (2); (b) the corresponding, balanced height field, calculated using eq. (3); (c) the height field perturbation, as defined in eq. (4), with a contour interval of 10 m; the outermost contour is at 10 m.

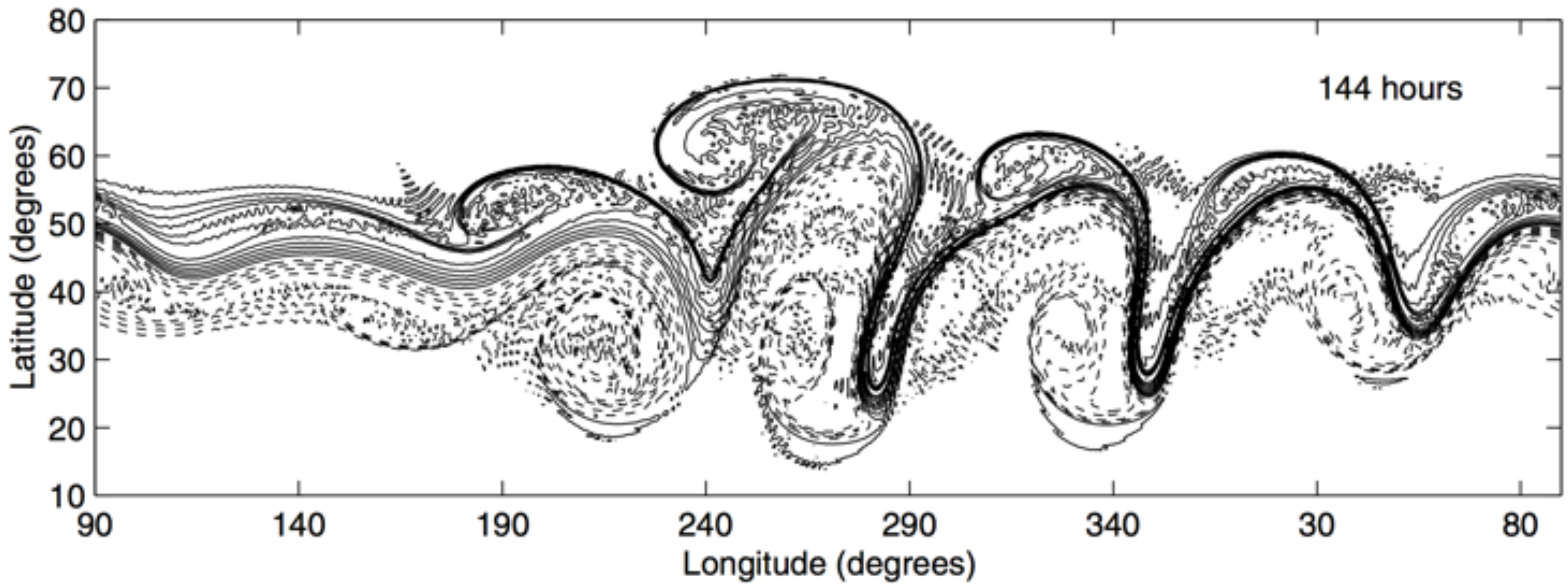
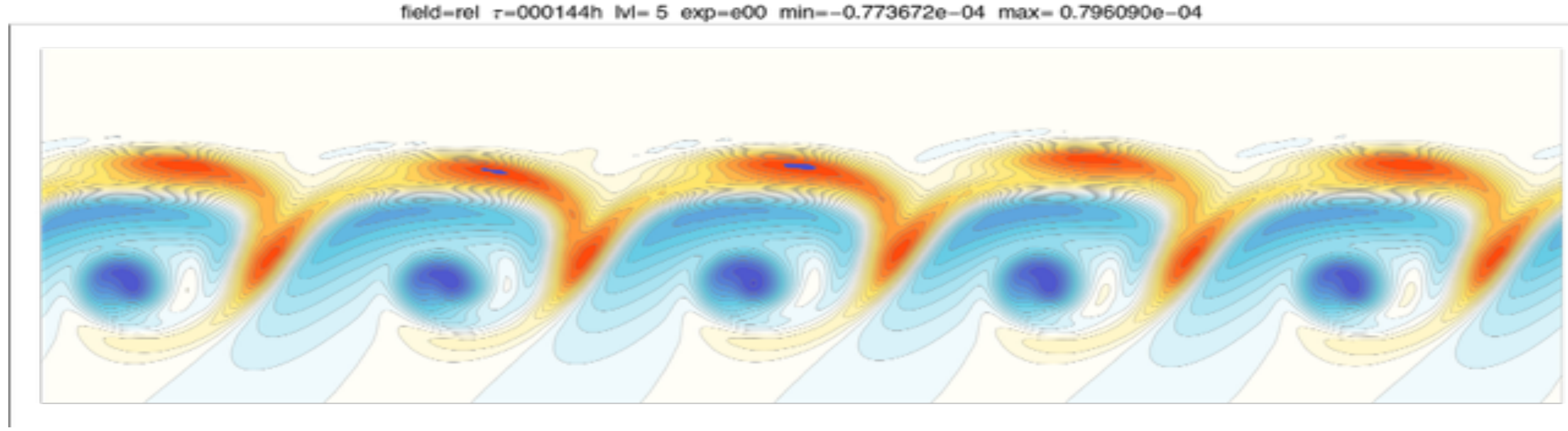


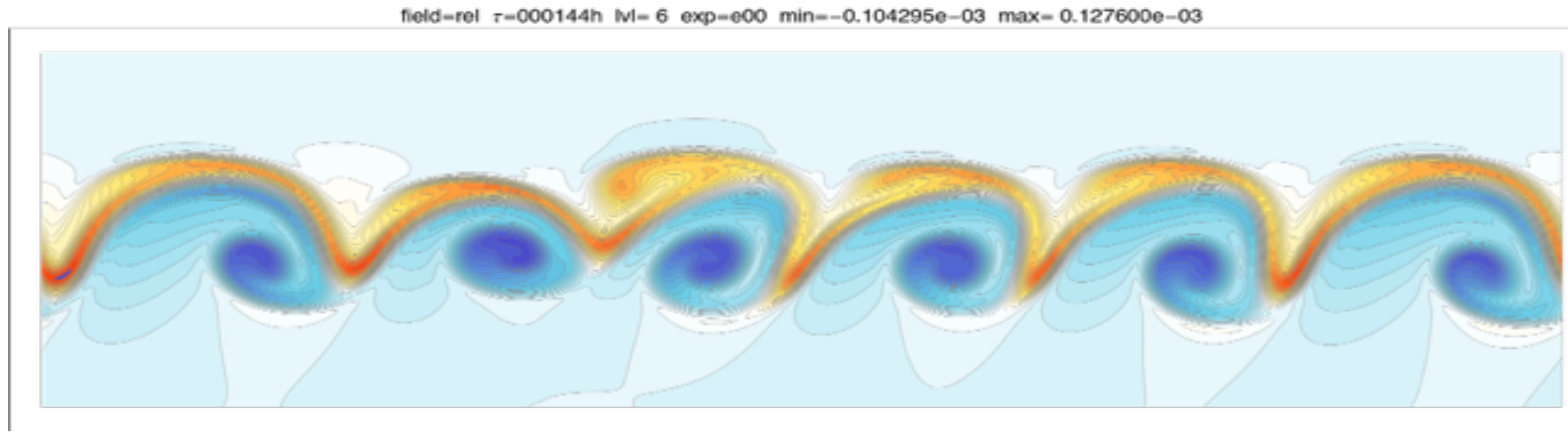
Fig 4. The time evolution of the vorticity field for the inviscid initial-value problem, computed with the FMS-SWM at a resolution of T341 with a 30-s time-step. The contour interval is $2 \times 10^{-5}\ s^{-1}$. Negative contours are dashed. The zero contour is not shown.

- These plots show the relative vorticity on day 6

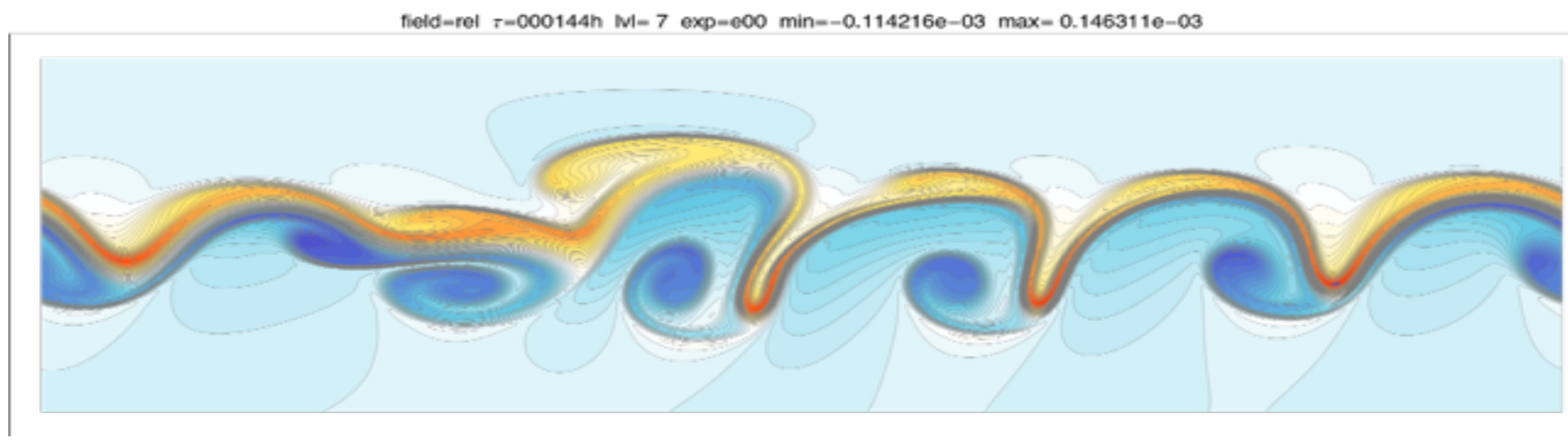
grd05. 240km.



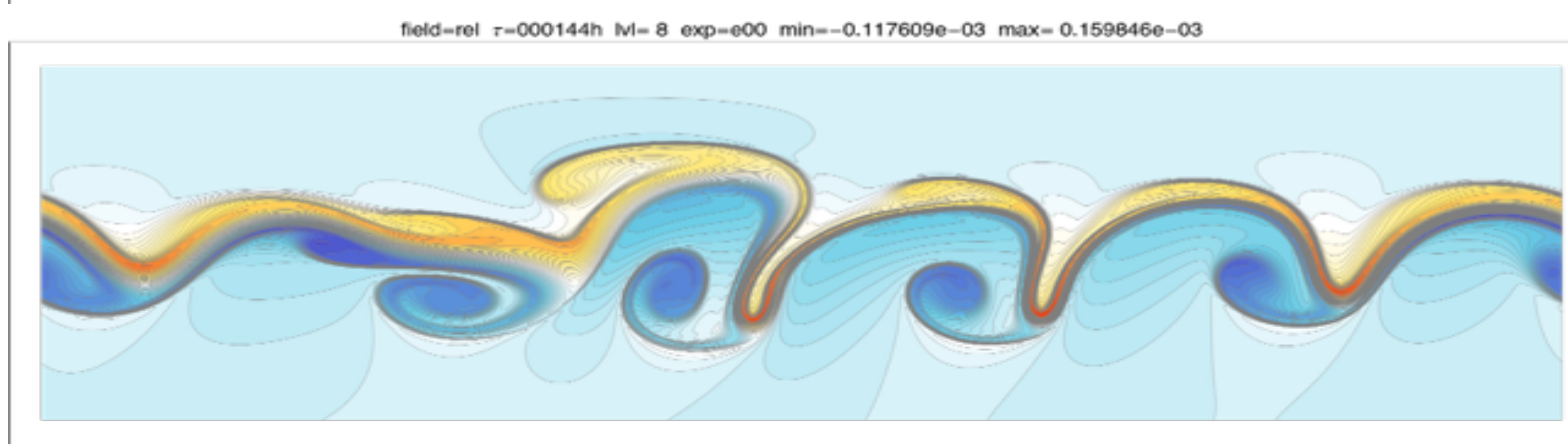
grd06. 120km.



grd07. 60km.



grd08. 30km.



Masuda and Ohnishi shallow water equations

- Masuda, Y., and H. Ohnishi, 1986: An integration scheme of the primitive equation model with an icosahedral-hexagonal grid system and its application to the shallow-water equations. Short- and Medium-Range Numerical Weather Prediction. Japan Meteorological Society, Tokyo, 317-326.

- Prognostic

$$\frac{\partial \eta}{\partial t} - J(\eta,\psi) + \nabla \bullet (\eta \nabla \chi) = 0$$

$$\frac{\partial D}{\partial t} - J(\eta,\chi) - \nabla \bullet (\eta \nabla \psi) + \nabla^2 (K + \phi) = 0$$

$$\frac{\partial \phi}{\partial t} - J(\phi,\psi) + \nabla \bullet (\phi \nabla \chi) = 0$$

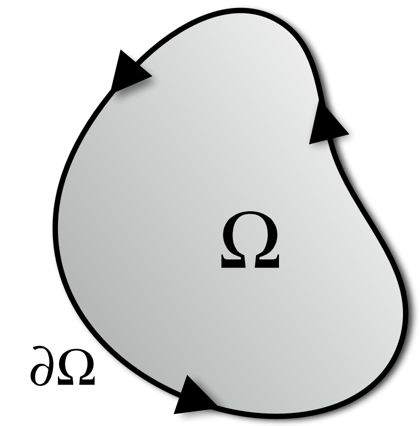
- Diagnostic

$$\nabla^2 \psi = \eta - f \qquad \nabla^2 \chi = D$$

- Three operators: Jacobian, divergence and Laplacian
- We will mostly discuss the Jacobian but the approach could easily apply to the others.

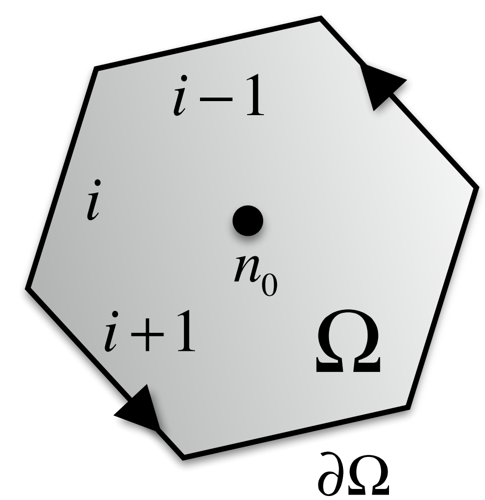
A finite-difference curl operator

- Stokes (Kelvin-Stokes) theorem: The integral of the curl of a vector field over some area equals the line integral of the vector field around the boundary of the area.



$$\int_{\Omega}\left[\mathbf{k}\bullet(\nabla\times\mathbf{F})\right]d\Omega = \oint_{\partial\Omega}(\mathbf{F}\bullet\boldsymbol{\tau})\,dS$$

- Considering a cell and with the following approximations:



1. $\int_{\Omega}\left[\mathbf{k}\bullet(\nabla\times\mathbf{F})\right]d\Omega \approx \left[\mathbf{k}\bullet(\nabla\times\mathbf{F})\right]_0 A_0$
2. $\oint_{\partial\Omega}(\mathbf{F}\bullet\boldsymbol{\tau})\,dS \approx \sum_i (F_\tau)_i l_i$

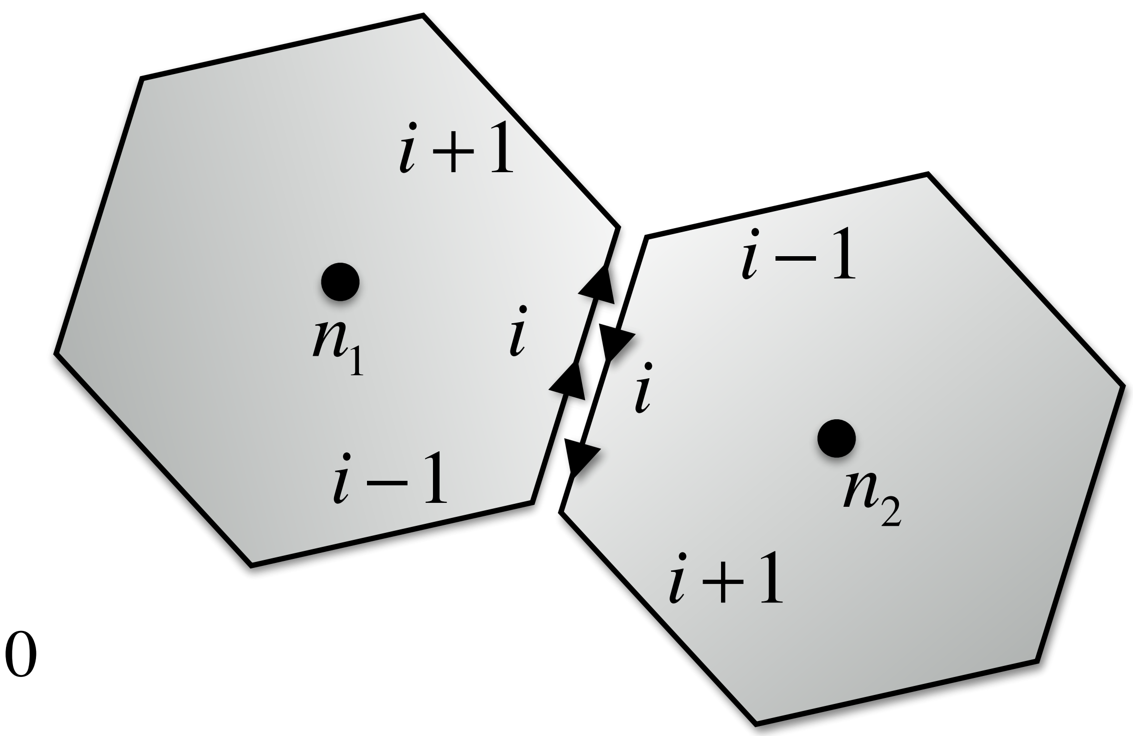
- The finite-difference **curl** operator is given by

$$\left[\mathbf{k}\bullet(\nabla\times\mathbf{F})\right]_0 A_0 \approx \sum_i (F_\tau)_i l_i$$

Properties of the curl operator. **Global integral of the curl is zero.**

- The finite-difference **curl**

$$\left[\mathbf{k}\bullet(\nabla\times\mathbf{F})\right]_0 A_0 = \sum_i \left(F_\tau\right)_i l_i$$



- If we sum over all cells

$$\sum_n^{\text{all cells}} \left\{\left[\mathbf{k}\bullet(\nabla\times\mathbf{F})\right]A\right\}_n = \sum_n^{\text{all cells}} \sum_i \left(F_\tau\right)_{n,i} l_{n,i} = 0$$

since each edge will contribute to the global sum in an equal and opposite fashion.

Properties of the curl operator. **Curl of a gradient is zero.**

- Stokes (Kelvin-Stokes) theorem:

$$\int_{\Omega}\left[\mathbf{k}\bullet(\nabla\times\mathbf{F})\right]d\Omega=\oint_{\partial\Omega}(\mathbf{F}\bullet\boldsymbol{\tau})\,dS$$

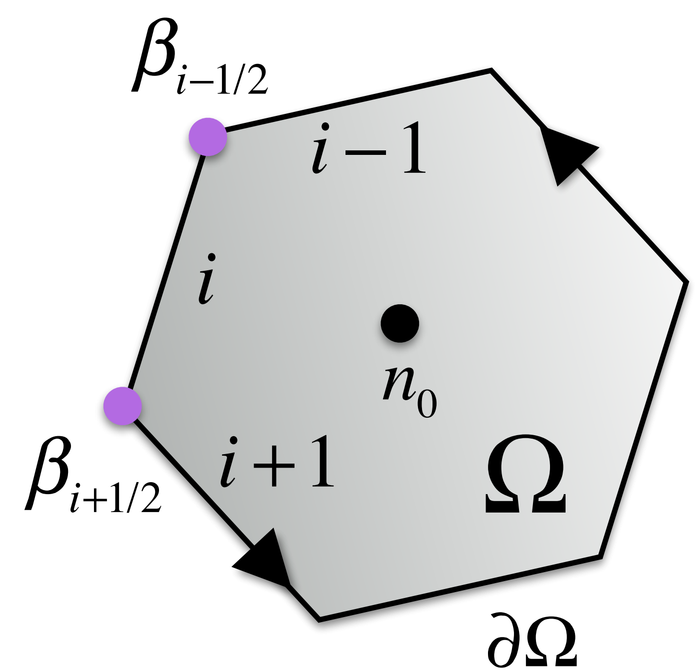
- Suppose $\mathbf{F}$ is defined to be the gradient of some scalar function

$$\mathbf{F}\equiv\nabla\beta$$

- Then

$$\int_{\Omega}\left[\mathbf{k}\bullet(\nabla\times\nabla\beta)\right]d\Omega=\oint_{\partial\Omega}(\nabla\beta\bullet\boldsymbol{\tau})\,dS=\oint_{\partial\Omega}\left(\frac{\partial\beta}{\partial\tau}\right)dS=0$$

$$\sum_i\oint_{\partial\Omega_i}\left(\frac{\partial\beta}{\partial\tau}\right)_i dS_i=\sum_i\left(\beta_{i+1/2}-\beta_{i-1/2}\right)=0$$



The curl of the gradient is zero since the line integral satisfies a sort of antiderivative property along each edge.

Finite-difference Jacobian

- Stokes (Kelvin-Stokes) theorem:

$$\int_{\Omega}\left[\mathbf{k}\bullet(\nabla\times\mathbf{F})\right]d\Omega=\oint_{\partial\Omega}(\mathbf{F}\bullet\boldsymbol{\tau})\,dS$$

- If we pick the vector field in the curl operator to have the form

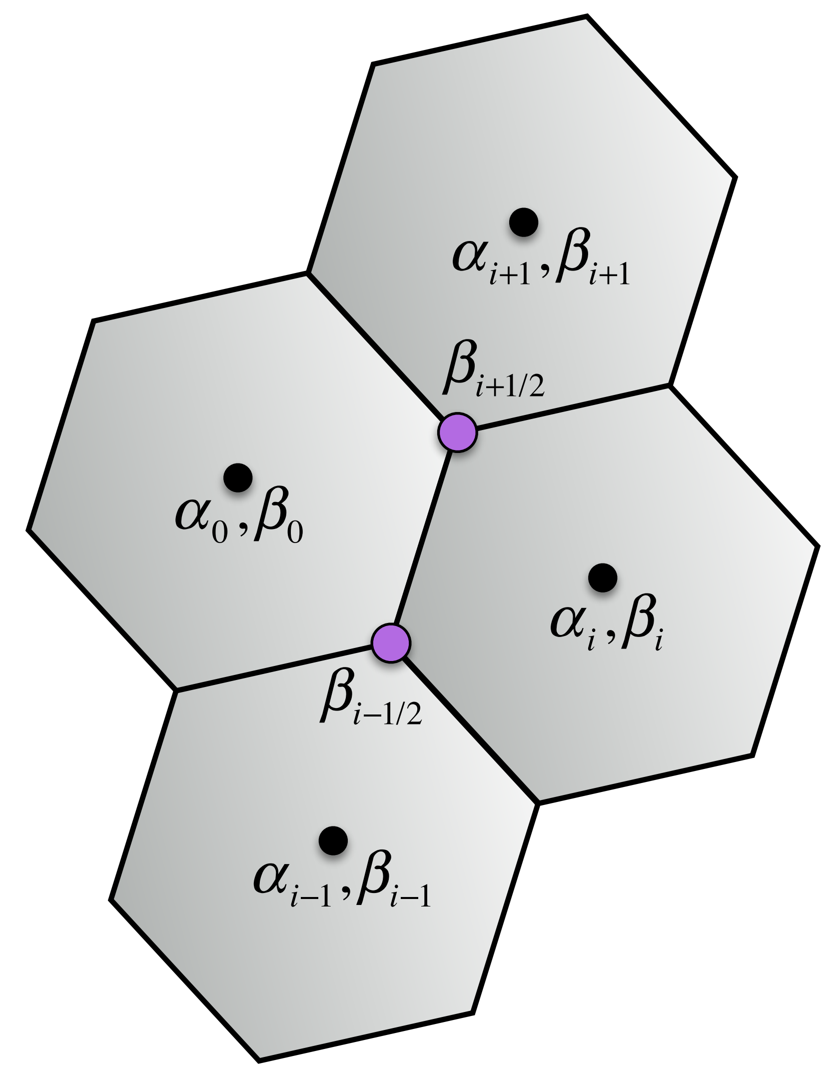
$$\mathbf{F}\equiv\alpha\nabla\beta$$

- Then the finite-difference **Jacobian** is defined by

$$\left[J(\alpha,\beta)\right]_0 A_0\equiv\left[\mathbf{k}\bullet(\nabla\times\alpha\nabla\beta)\right]_0 A_0=\sum_i\bar{\alpha}_i\left(\frac{\partial\beta}{\partial\tau}\right)_i l_i$$

- For example,

$$\left[J(\alpha,\beta)\right]_0=\frac{1}{A_0}\sum_i\frac{\alpha_0+\alpha_i}{2}\,\frac{\beta_{i+1/2}-\beta_{i-1/2}}{l_i}\,l_i$$



- Clearly the global sum of the Jacobian will be zero

- And with $\alpha\equiv$const. the curl of the gradient is zero

Properties of the curl operator. **Antisymmetry property.**

- Since the curl of a gradient is zero, we can write

$$\int_{\Omega}\left[\mathbf{k}\bullet\left(\nabla\times\nabla\alpha\beta\right)\right]d\Omega$$

$$=\int_{\Omega}\left[\mathbf{k}\bullet\left(\nabla\times\alpha\nabla\beta\right)\right]d\Omega+\int_{\Omega}\left[\mathbf{k}\bullet\left(\nabla\times\beta\nabla\alpha\right)\right]d\Omega$$

$$=J\left(\alpha,\beta\right)+J\left(\beta,\alpha\right)=0$$

or, we can write

$$J\left(\alpha,\beta\right)+J\left(\beta,\alpha\right)=\oint_{\partial\Omega}\left(\alpha\frac{\partial\beta}{\partial\tau}\right)dS+\oint_{\partial\Omega}\left(\beta\frac{\partial\alpha}{\partial\tau}\right)dS=\oint_{\partial\Omega}\left(\frac{\partial}{\partial\tau}\alpha\beta\right)dS=0$$

$$\sum_{i}\oint_{\partial\Omega_i}\left(\frac{\partial}{\partial\tau}\alpha\beta\right)_i dS_i=\sum_{i}\left(\alpha_{i+1/2}\beta_{i+1/2}-\alpha_{i-1/2}\beta_{i-1/2}\right)=0$$

Outline of the new new scheme

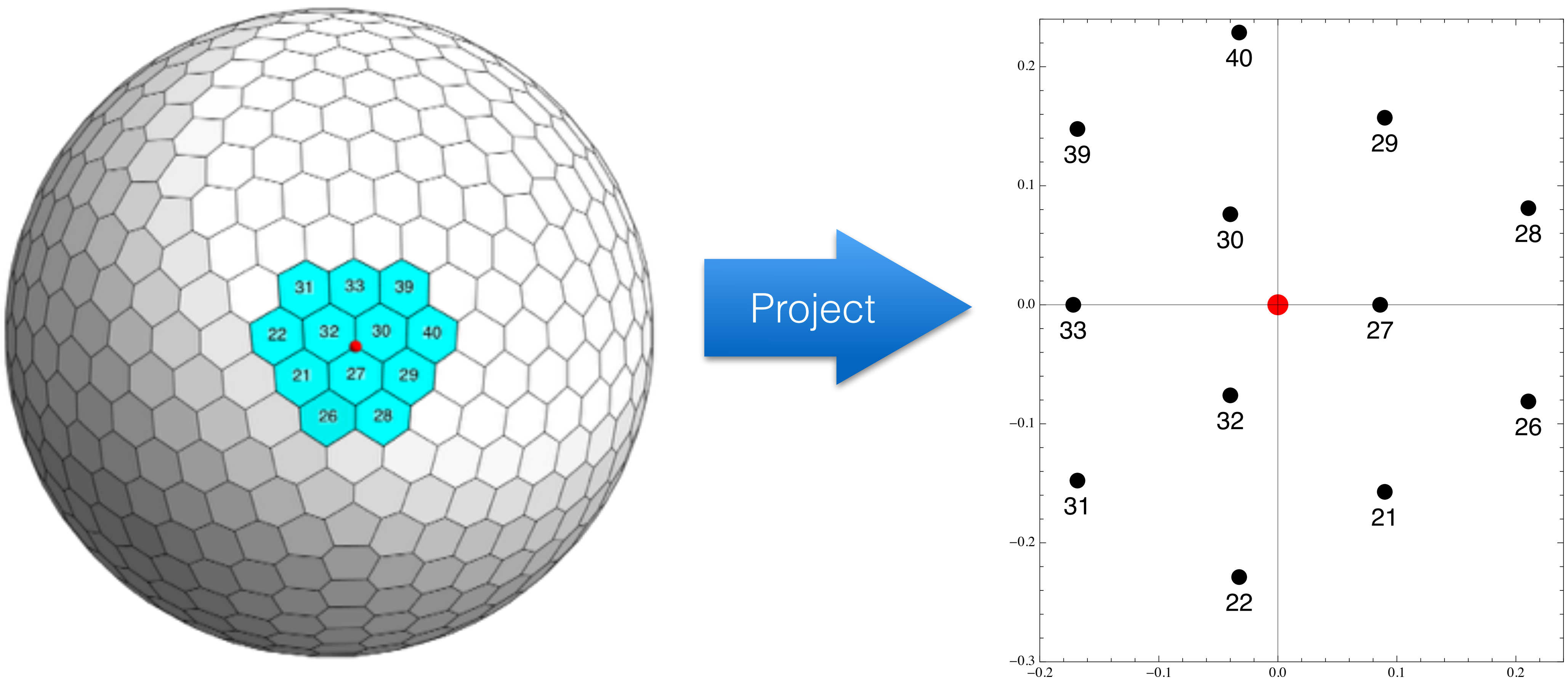
- Design of the new scheme for the **Jacobian**:
  1. Construct a polynomial associated with each cell corner
  2. With the corner polynomials construct a polynomial associated with each edge
  3. Integrate the polynomial along each edge
- In general the new Jacobian will have the form:

$$\left[\mathbf{k}\bullet\left(\nabla\times\alpha\nabla\beta\right)\right]_0 A_0 = \sum_i \oint_{\partial\Omega_i}\left(\alpha\frac{\partial\beta}{\partial\tau}\right)_i dS_i$$

- With the properties that:
  1. Global integral vanishes
  2. Curl of the gradient is zero
  3. Antisymmetry property

On the icosahedral grid

A 12-point stencil is associated with each corner and projected into the plane tangent at that point.



This gives 12 data points of the form:

$$(x_1, y_1, f_1), (x_2, y_2, f_2), \ldots, (x_{12}, y_{12}, f_{12})$$

Least squares for the corner polynomials

The classic textbook (or wikipedia) approach to the **least squares** problem.

Given N data points:

$$\left(x_1, y_1, f_1\right), \left(x_2, y_2, f_2\right), \ldots, \left(x_N, y_N, f_N\right)$$

Given a set of M basis functions (in this case polynomials):

$$X_1\left(x, y\right), X_2\left(x, y\right), \ldots, X_M\left(x, y\right)$$

We can write a linear combination of the basis functions:

$$f\left(x, y\right) = \sum_{k=1}^{M} a_k X_k\left(x, y\right)$$

We wish to determine the coefficients $a_1, a_2, \ldots, a_M$ which will minimize the merit function.

Where we define the define a merit function

$$\chi^2\left(a_1, a_2, \ldots, a_M\right) = \sum_{i=1}^{N}\left[f_i\left(x, y\right) - \sum_{k=1}^{M} a_k X_k\left(x, y\right)\right]^2$$

Least squares for the corner polynomials

Design matrix

$$\mathbf{A} = \begin{bmatrix} X_1(x_1,y_1) & X_2(x_1,y_1) & \cdots & X_M(x_1,y_1) \\ X_1(x_2,y_2) & X_2(x_2,y_2) & & \vdots \\ \vdots & & \ddots & \\ X_1(x_N,y_N) & \cdots & & X_M(x_N,y_N) \end{bmatrix}$$

Define

$$\mathbf{a} = (a_1, a_2, \ldots, a_M) \qquad \mathbf{b} = (f_1, f_2, \ldots, f_N)$$

Define the so called normal equations

$$\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{a} = \mathbf{A}^T\mathbf{b}$$

For a given set of input values $f_1, f_2, \ldots, f_N$, the normal equations provide the weights $a_1, a_2, \ldots, a_M$ in the linear combination of basis functions that will minimize the merit function.

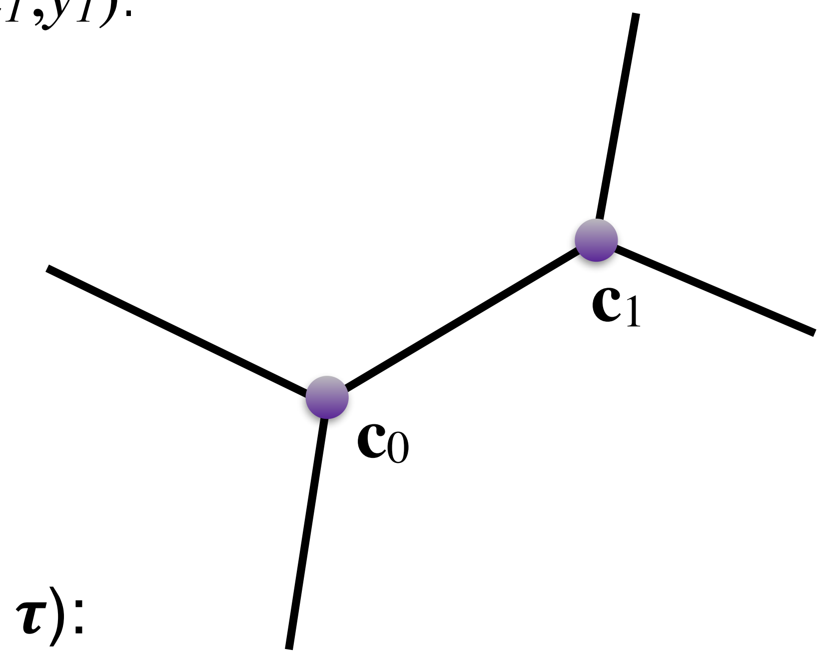
Polynomial approximation along each edge

Consider a cell edge with endpoints $\mathbf{c}_0 = (x_0, y_0)$ and $\mathbf{c}_1 = (x_1, y_1)$.

We associate **polynomials** with each endpoint:

$$\mathbf{c}_0 \Rightarrow \alpha_0(x,y), \beta_0(x,y)$$

$$\mathbf{c}_1 \Rightarrow \alpha_1(x,y), \beta_1(x,y)$$



Define the derivative in the direction of the edge (denoted $\boldsymbol{\tau}$):

$$\frac{\partial \beta_0}{\partial \tau}(x,y) = \nabla \beta_0 \bullet \boldsymbol{\tau} \qquad \frac{\partial \beta_1}{\partial \tau}(x,y) = \nabla \beta_1 \bullet \boldsymbol{\tau} \qquad \text{where} \qquad \boldsymbol{\tau} \equiv \frac{\mathbf{c}_1 - \mathbf{c}_0}{\sqrt{(\mathbf{c}_1 - \mathbf{c}_0) \bullet (\mathbf{c}_1 - \mathbf{c}_0)}}$$

Parameterize the path along a cell edge from $\mathbf{c}_0$ to $\mathbf{c}_1$.

$$\boldsymbol{\sigma}(\varepsilon) = \frac{1}{2}\left[(1-\varepsilon)\mathbf{c}_0 + (1+\varepsilon)\mathbf{c}_1\right] \qquad \text{where} \qquad \varepsilon \in [-1,1]$$

So

$$\boldsymbol{\sigma}(-1) = \mathbf{c}_0 \qquad \text{and} \qquad \boldsymbol{\sigma}(1) = \mathbf{c}_1$$

Polynomial approximation along each edge

We can form a **linear combination of corner functions** to create a function associated with each edge:

$$\beta(\varepsilon) = \frac{1}{2}\Big[(1-\varepsilon)\beta_0\big(\boldsymbol{\sigma}(\varepsilon)\big) + (1+\varepsilon)\beta_1\big(\boldsymbol{\sigma}(\varepsilon)\big)\Big] \quad \text{where} \quad \varepsilon \in [-1,1]$$

So

$$\beta(-1) = \beta_0(\mathbf{c}_0) \quad \text{and} \quad \beta(1) = \beta_1(\mathbf{c}_1)$$

This function has the antiderivative property along the edge:

$$\int_{-1}^{1} \frac{\partial\beta}{\partial\tau}(\varepsilon)\,d\varepsilon = \beta(-1) - \beta(1) = \beta_1(\mathbf{c}_1) - \beta_0(\mathbf{c}_0)$$

So the integral around a closed path (along the walls of a cell) is zero.

The curl of a gradient is zero.

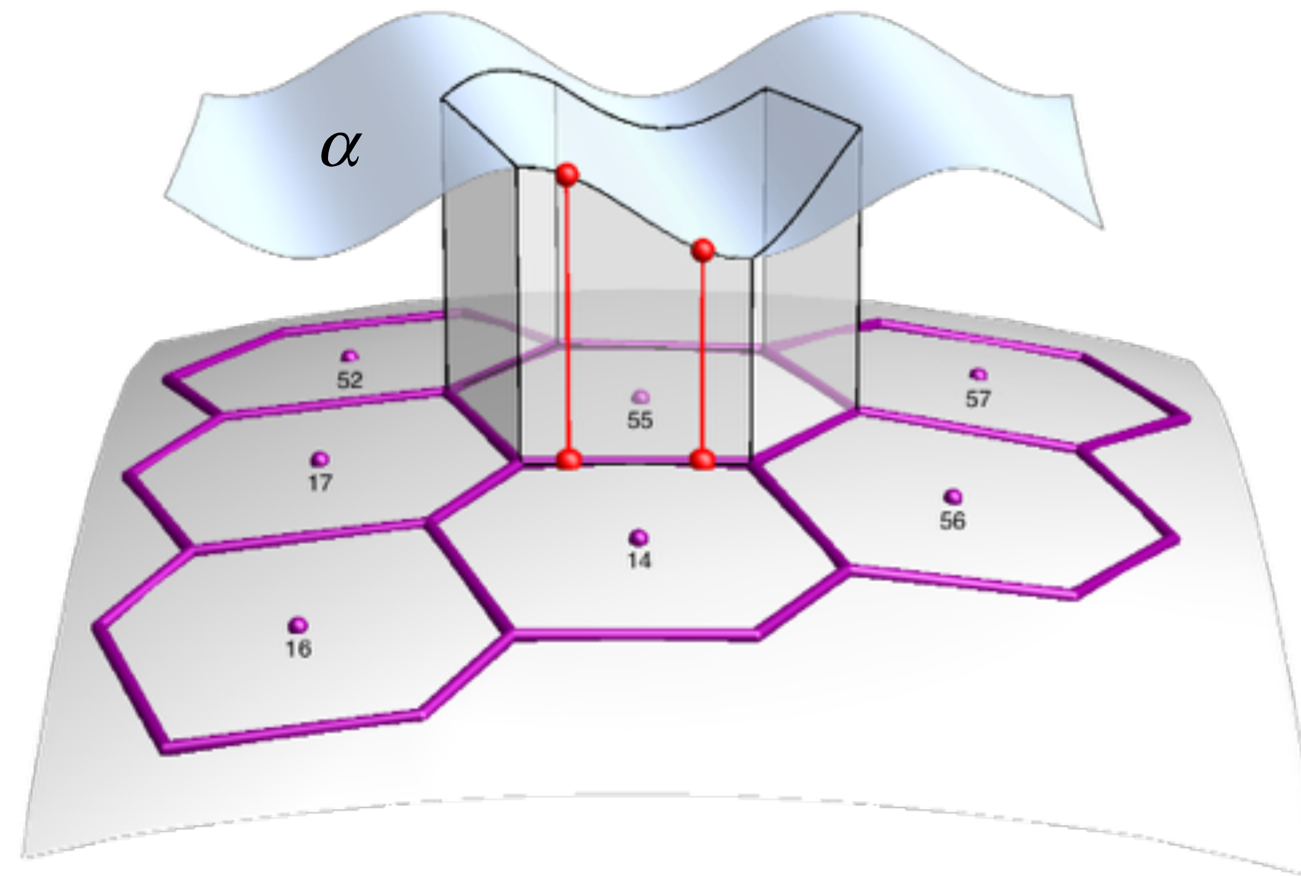
Gaussian quadrature

- Approximate the definite integral of a function as a weighted sum of function values at specified points:

$$\int_{-1}^{1} \alpha(x)dx \approx \sum_{i=1}^{n} w_i \alpha(x_i)$$

- We use this approach to approximate the line integral along each edge

- The sum will yield an exact result for polynomials of degree $2n-1$.
  So, for $n = 2$, the quadrature will be exact for 3rd degree polynomials.



New Scheme

The new scheme has the form:

$$\left[J(\alpha,\beta)\right]_0 A_0 = \sum_i \sum_q \omega_q \alpha_{i,q} \left(\frac{\partial\beta}{\partial\tau}\right)_{i,q}$$

with the properties that

1. Global sum vanishes
2. Curl of the gradient is zero
3. Antisymmetry (with higher-order integration)

Some preliminary numerical results with the Williamson shallow water test case 2.

- Williamson *et al.* (1992)

- Solid body rotation (or purely zonal flow) with the corresponding geostrophic height field.

$$\phi(\lambda,\varphi) = gh_0 - \left(a\Omega u_0 + \frac{u^2}{2}\right)\sin^2\varphi$$

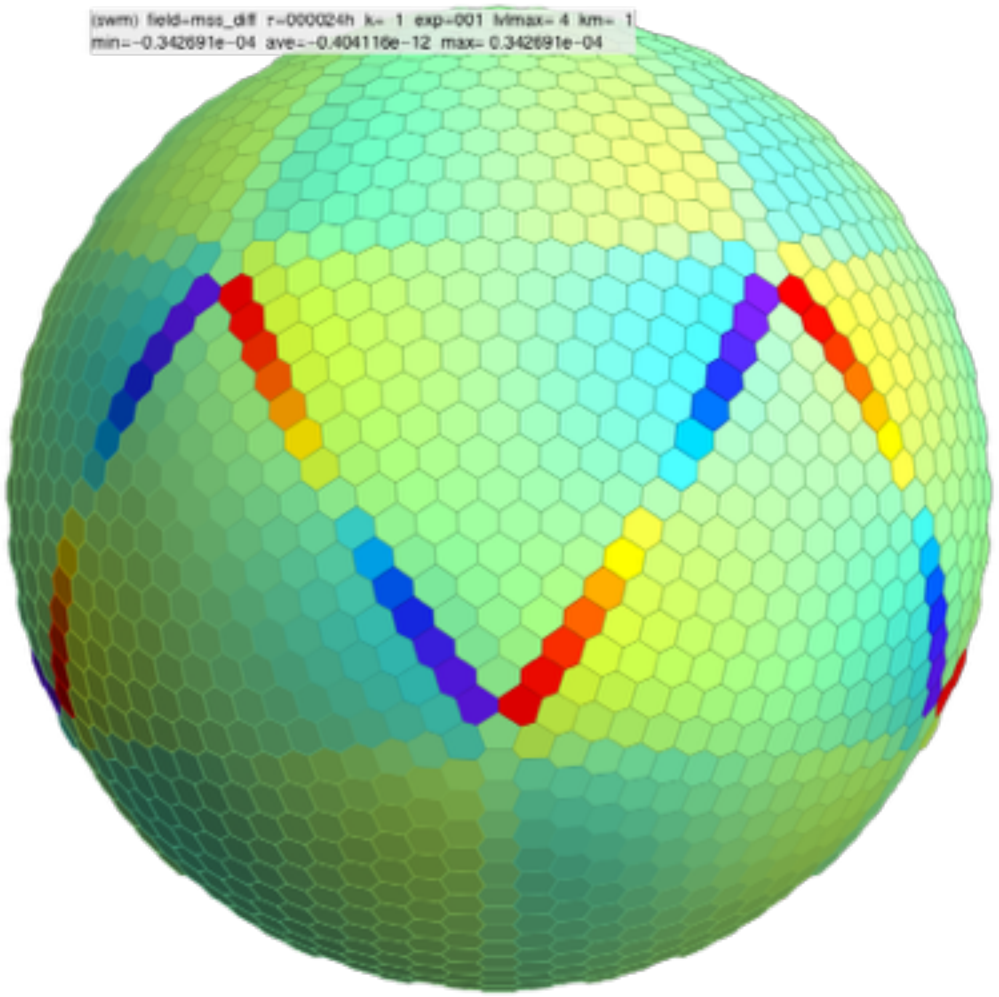
$$\psi(\lambda,\varphi) = -au_0 \sin\varphi$$

- The system is steady state and nondivergent:

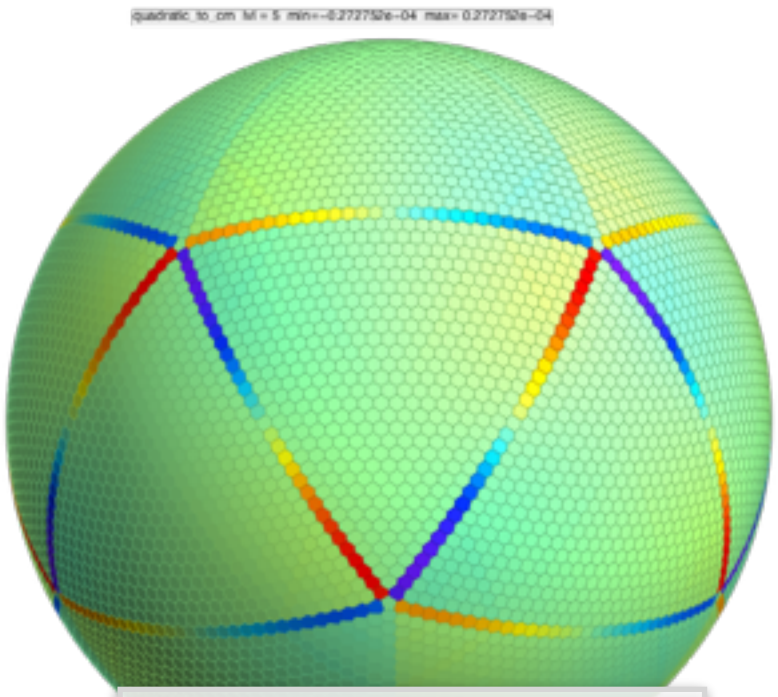
$$\frac{\partial\eta}{\partial t} - J(\eta,\psi) = 0$$

- We want to show how well this is true:

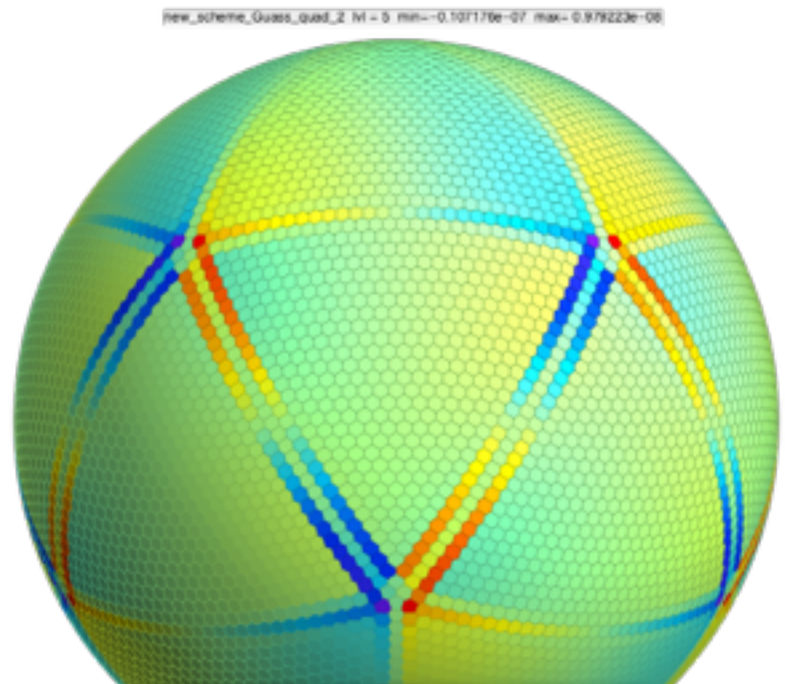
$$J(\phi,\psi) = 0$$



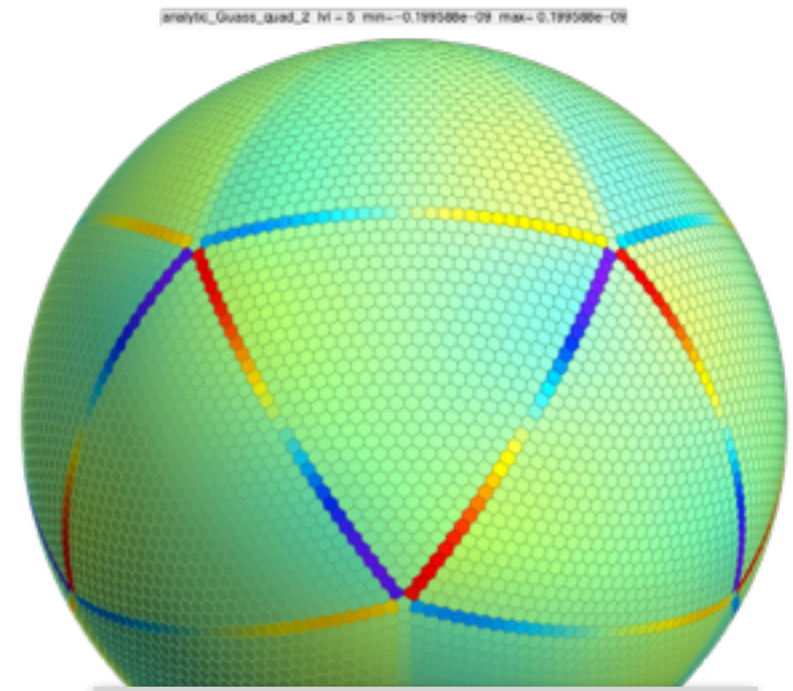
Williamson Test Case 2. Tweaked grid.



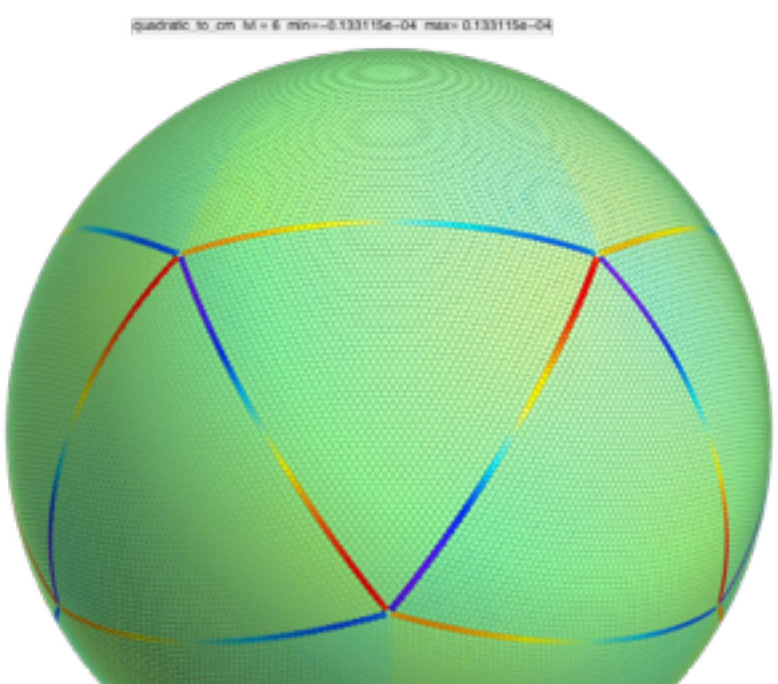
quadratic_to_crn. lvl=5.
max error 0.272E-04



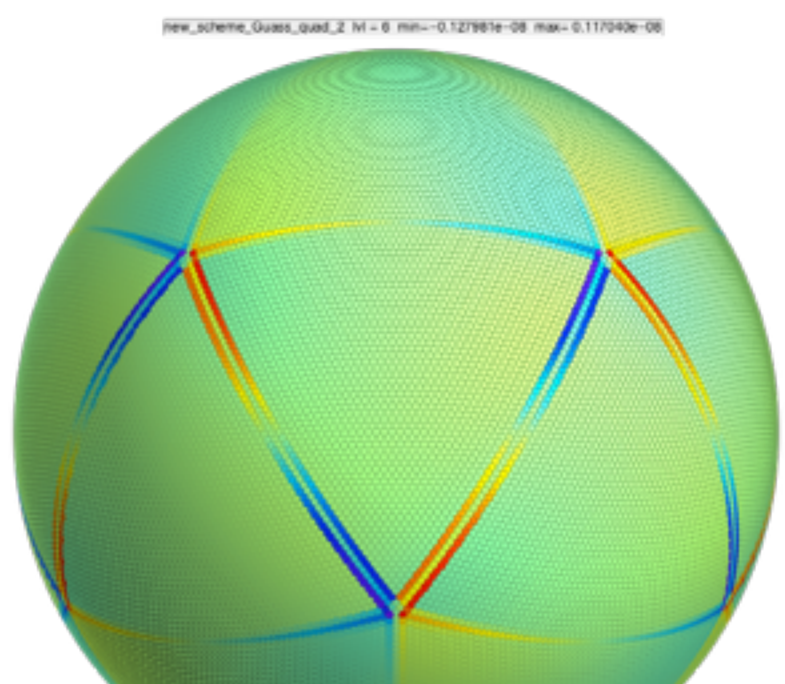
new scheme. lvl=5.
max error 0.107E-07



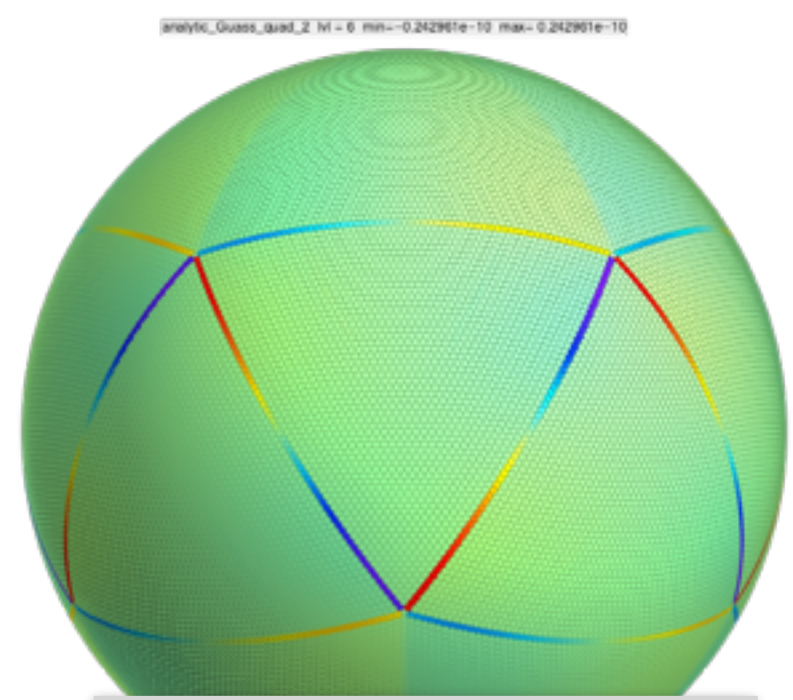
analytic_Gauss_quad. lvl=5.
max error 0.200E-09



quadratic_to_crn. lvl=6.
max error 0.133-04



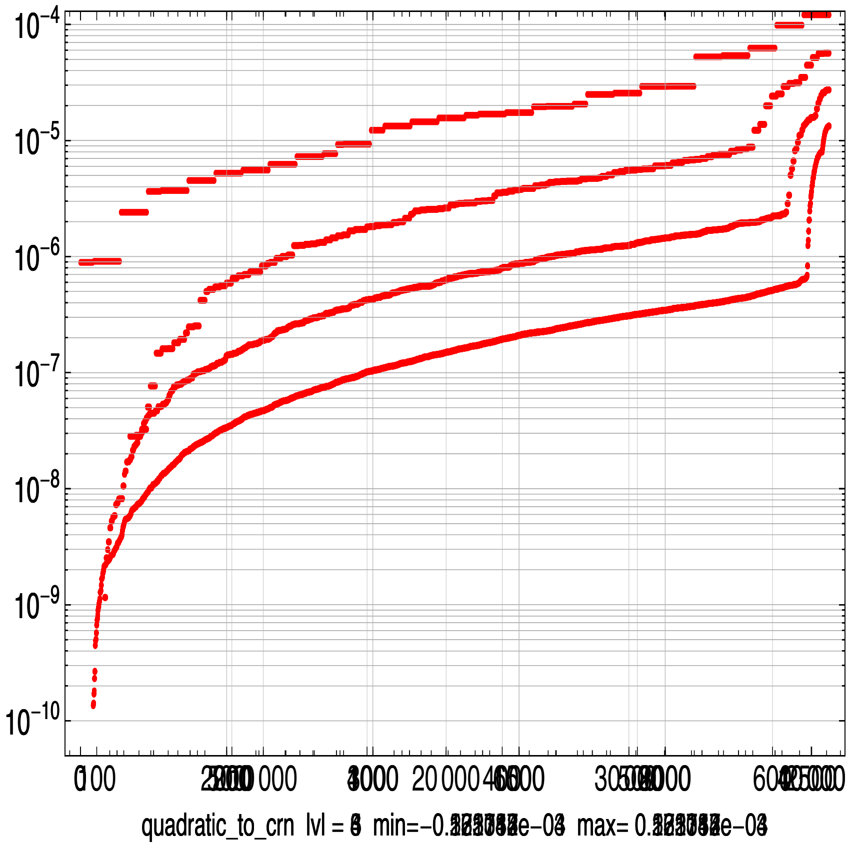
new scheme. lvl=6.
max error 0.128E-08



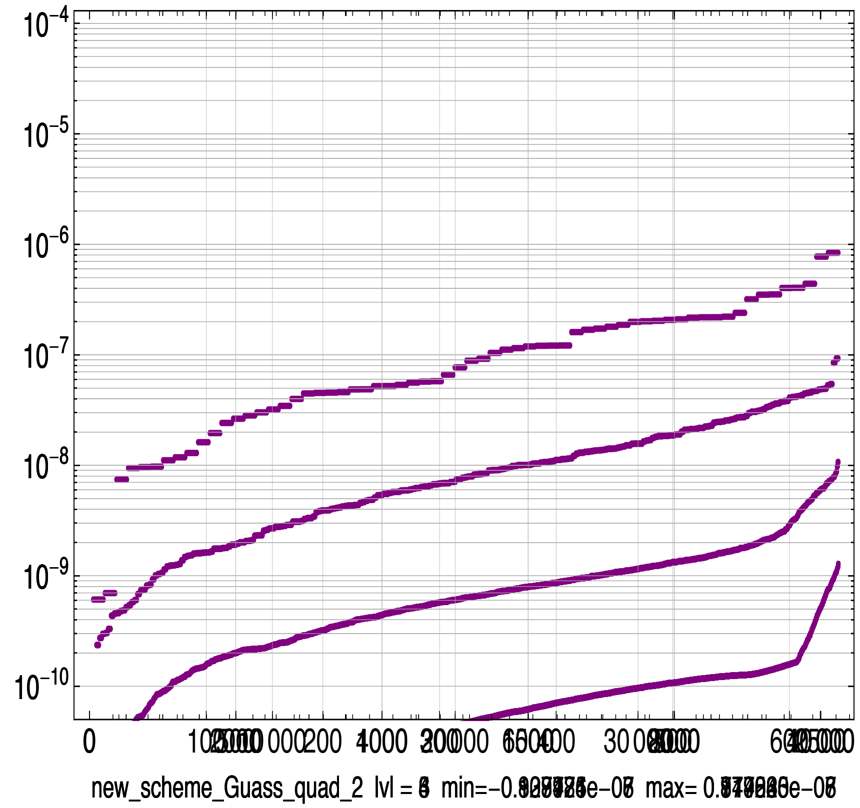
analytic_Gauss_quad. lvl=6.
max error 0.243E-10

Williamson Test Case 2. Tweaked grid.

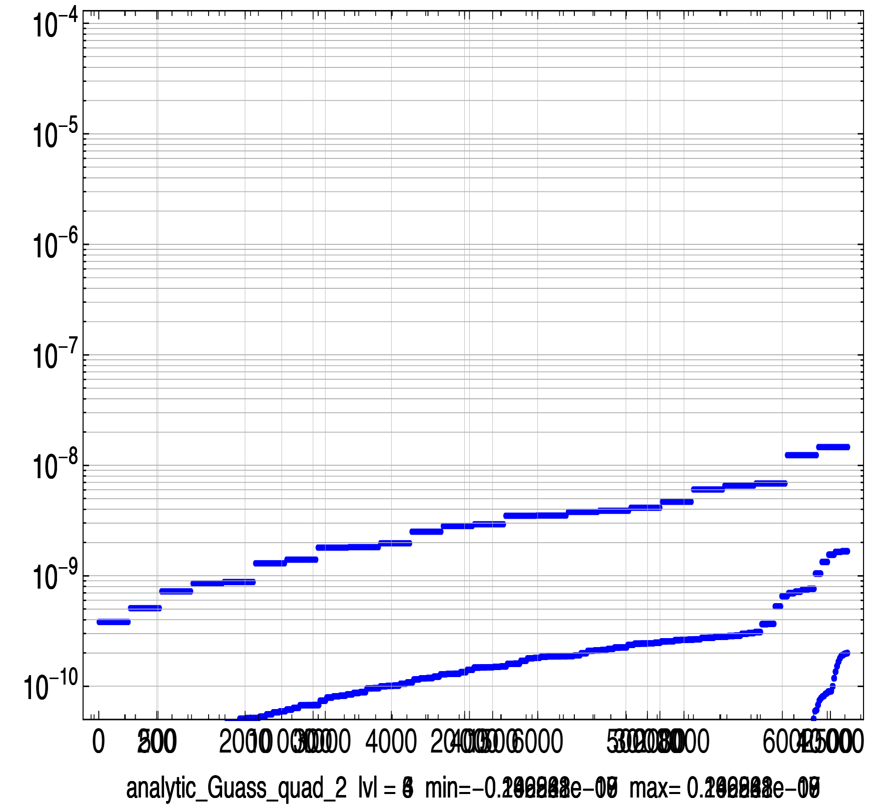
- The error for grids 3,4,5 and 6 are sorted and plotted.



old scheme



new scheme



analytic Gaussian quadrature

Conclusions

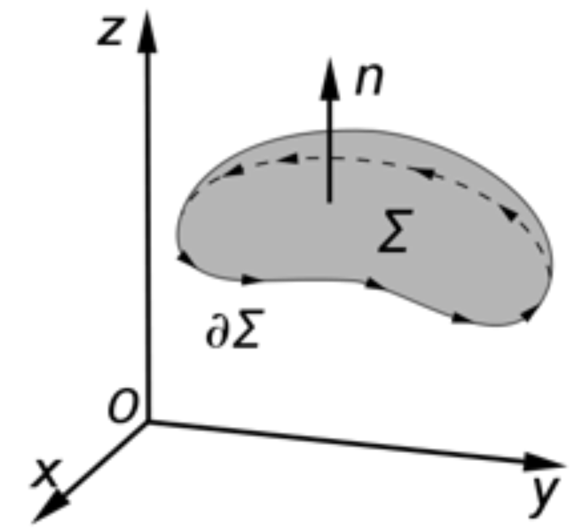
We have developed a mimetic discrete Jacobian operator that maintains several properties of the continuous operator.

The new operator has a higher order of accuracy than the old scheme.

The same approach could be applied the the divergence and Laplacian

Properties of the continuous curl operator

- Stokes' (Kelvin-Stokes) theorem: The integral of the curl of the vector field over some surface equals the line integral of the vector field around the boundary of the surface.



$$\int_{\Sigma}(\nabla\times\mathbf{F})\bullet d\Sigma=\oint_{\partial\Sigma}\mathbf{F}\bullet d\mathbf{r}$$

- Define

$$\mathbf{F}\equiv\left(F_x,F_y,F_z\right)$$

- Then the above integral equation (1) can be written

$$\iint_{\Sigma}\left\{\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)dydz-\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)dzdx+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)dxdy\right\}=\oint_{\partial\Sigma}\left\{F_x dx+F_y dy+F_z dz\right\}$$

- Suppose $\mathbf{F}$ is only a function x and y and the z-component is zero

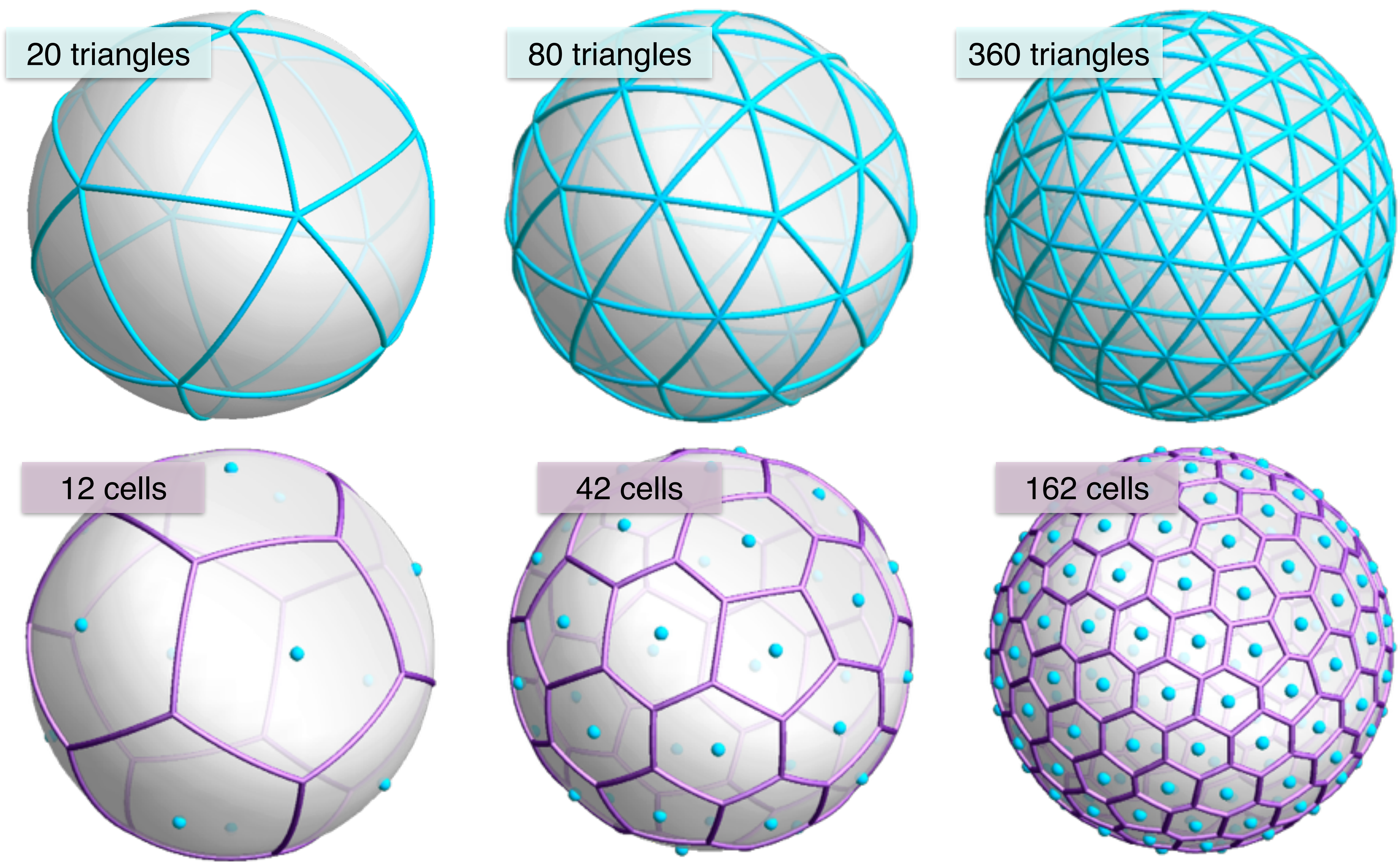
$$\mathbf{F}\equiv\left(F_x(x,y),F_y(x,y),0\right)$$

- Then the above integral equation (2) reduces to this

$$\iint_{\Sigma}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)dxdy=\iint_{\Sigma}\left[\mathbf{k}\bullet(\nabla\times\mathbf{F})\right]dxdy=\oint_{\partial\Sigma}\left\{F_x dx+F_y dy+F_z dz\right\}$$

Icosahedral grid

- Each spherical triangle is further partitioned into four spherical triangles.
- The vetches of these polyhedrons are used to generate the icosahedral grid.
- An area (Voronoi cell) is associated with each generating point.



Counting the cells.

- Let r denote the number of applications of the subdivision algorithm, that is partitioning one triangle into four triangles.
- Our target resolutions are:

resolution (r)	number of cells	global grid point spacing (km)
5	10,242	239.8
6	40,962	119.9
7	163,842	59.95
8	655,362	29.97
9	2,621,442	14.99
10	10,485,762	7.495
11	41,943,042	3.747
12	167,772,162	1.874