# Wave Vortex Dynamics

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## Shallow Water Equations

$$\mathbf{u}_t + \nabla\left(gh + \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) = \zeta\left(v, -u\right)$$

$$h_t = -\nabla\cdot\left(h\mathbf{u}\right)$$

## Wave Vortex Dynamics

$$\mathbf{u}_t + \nabla\left(gh\right) = \frac{h_0}{h}\zeta\left(v, u\right)$$

$$h_t = -\nabla\cdot\left(h_0\mathbf{u}\right)$$

# Shallow Water Dynamics in Hamiltonian Form

$$F = F\left[u, v, h\right]$$

$$\frac{dF}{dt} = \left\{F, H\right\}$$

$$\left\{F, H\right\} \equiv \iint d\mathbf{x} \left( \frac{\zeta}{h} \left( \frac{\delta F}{\delta u}\frac{\delta H}{\delta v} - \frac{\delta H}{\delta u}\frac{\delta F}{\delta v} \right) - \frac{\delta F}{\delta \mathbf{u}} \cdot \nabla \frac{\delta H}{\delta h} + \frac{\delta H}{\delta \mathbf{u}} \cdot \nabla \frac{\delta F}{\delta h} \right)$$

$$H = \iint d\mathbf{x} \left( \tfrac{1}{2} h u^2 + \tfrac{1}{2} h v^2 + \tfrac{1}{2} g h^2 \right)$$

## WV dynamics replaces $H$ with

$$H_{approx} = \iint d\mathbf{x} \left( \tfrac{1}{2} h_0 u^2 + \tfrac{1}{2} h_0 v^2 + \tfrac{1}{2} g h^2 \right)$$

# ADVANTAGE of this: Conservation Laws

Since it involves no approximation to the bracket, WV dynamics conserves:

$$Casimirs = \iint d\mathbf{x} \;\; hG(q), \qquad\qquad q = \frac{\zeta}{h}$$

$$Momentum = \iint d\mathbf{x} \;\; h\mathbf{u}$$

$$Energy = \iint d\mathbf{x} \;\; \left(\tfrac{1}{2}h_0\mathbf{u}\cdot\mathbf{u} + \tfrac{1}{2}gh^2\right)$$

$$\frac{\partial q}{\partial t} + \frac{h_0}{h}\mathbf{u}\cdot\nabla q = 0$$

# WV dynamics with $\alpha \equiv \dfrac{h}{h_0}$

$$\mathbf{u}_t + c^2\nabla\alpha = q(v,-u), \qquad c \equiv \sqrt{gh_0}, \qquad q \equiv \zeta/\alpha$$

$$\alpha_t + \nabla\cdot\mathbf{u} = 0$$

Wave split:

$$\mathbf{u}_t + c^2\nabla\alpha = 0$$
$$\alpha_t + \nabla\cdot\mathbf{u} = 0$$

Vorticity split:

$$\mathbf{u}_t = q(v,-u)$$
$$\alpha_t = 0$$

The waves interact only with vorticity and not with themselves.

1. This suggests an analogy with electrodynamics, which turns out to be surprisingly strong.
2. The splitting is a good basis for a numerical method, especially because $c$ is a constant.

# Analogy with electrodynamics (2 space dimensions)

$$\mathbf{u}_t + c^2\nabla\alpha = q(v,-u), \qquad q \equiv \zeta/\alpha$$
$$\alpha_t + \nabla\cdot\mathbf{u} = 0$$

$$(\partial_t,\partial_x,\partial_y)\cdot(\alpha,u,v) = 0 \quad \Rightarrow \quad (\alpha,u,v) = (\partial_t,\partial_x,\partial_y)\times(\phi,A,B)$$

$$\alpha = B_x - A_y$$
$$u = \phi_y - B_t$$
$$v = A_t - \phi_x$$

The potentials are not unique:

$$(\phi,A,B) \to (\phi,A,B) + (\partial_t,\partial_x,\partial_y)\lambda(x,y,t)$$

This is a gauge symmetry.

Imposing the gauge condition:

$$\frac{\partial \phi}{\partial t} - c^2 \nabla \cdot \mathbf{A} = 0 \qquad \text{(Lorenz gauge)}$$

WV dynamics takes the form:

$$\left(\partial_{tt} - c^2 \nabla^2\right)\phi = c^2 \zeta$$

$$\left(\partial_{tt} - c^2 \nabla^2\right)\mathbf{A} = -\frac{\zeta}{\alpha}\mathbf{u}$$

To proceed further in the direction of electrodyamics, introduce the analogue of point charges:

$$\zeta(\mathbf{x},t) = \sum_i \Gamma_i \, \delta\left(\mathbf{x} - \mathbf{x}_i(t)\right)$$

This is consistent with the conservation of $\iint d\mathbf{x} \, \zeta$

By the WV potential vorticity equation $\frac{\partial q}{\partial t} + \frac{\mathbf{u}}{\alpha} \cdot \nabla q = 0$

the delta singularities must move with velocity $\frac{\mathbf{u}}{\alpha}$

i.e. $\frac{d\mathbf{x}_i(t)}{dt} = \frac{\mathbf{u}(\mathbf{x}_i(t),t)}{\alpha(\mathbf{x}_i(t),t)}$

It follows that $\frac{\zeta}{\alpha}\mathbf{u} = \sum_i \Gamma_i \, \delta(\mathbf{x} - \mathbf{x}_i(t)) \, \dot{\mathbf{x}}_i(t)$

and the WV dynamics may be written

$$\left(\partial_{tt} - c^2\nabla^2\right)\phi = c^2 \sum_i \Gamma_i \, \delta(\mathbf{x} - \mathbf{x}_i(t))$$

$$\left(\partial_{tt} - c^2\nabla^2\right)\mathbf{A} = -\sum_i \Gamma_i \, \delta(\mathbf{x} - \mathbf{x}_i(t)) \, \dot{\mathbf{x}}_i(t)$$

# Variational Principle

$$L = \left[\phi(\mathbf{x},t), \mathbf{A}(\mathbf{x},t), \mathbf{x}_i(t)\right] = L_1[\phi,\mathbf{A}] + L_2[\phi,\mathbf{A},\mathbf{x}_i]$$

$$L_1 = \frac{1}{2}\iiint dt\, d\mathbf{x}\left[c^2\left(B_x - A_y\right)^2 - \left(A_t - \phi_x\right)^2 - \left(B_t - \phi_y\right)^2\right] \qquad \sim \qquad F_{\mu\nu}F^{\mu\nu}$$

$$L_2 = \iiint dt\, d\mathbf{x}\sum_i \Gamma_i\, \delta\left(\mathbf{x} - \mathbf{x}_i(t)\right)\,\left(\phi(\mathbf{x},t) + \mathbf{A}(\mathbf{x},t)\cdot\dot{\mathbf{x}}_i(t)\right) \qquad \sim \qquad A_\mu j^\mu$$

$\delta\phi,\ \delta\mathbf{A}$ $\Rightarrow$ field equations

$\delta\mathbf{x}_i(t)$ $\Rightarrow$ equations governing the motion of the point vortices

As in electrodynamics, self-interactions of the point vortices are not allowed.

At the leading order in $1/c^2$ this variational principle is equivalent to the familiar one for point vortices.

# Fields generated by a point vortex with a prescribed trajectory $\mathbf{x}_1(t)$

$$\alpha(\mathbf{x},t) = 1 + \frac{\Gamma_1}{4\pi c^2}\int_{-\infty}^{\infty} dz \left( \frac{\mathbf{r}_1 \times \mathbf{a}_1}{cD_1^2} + \frac{\mathbf{r}_1 \times \mathbf{v}_1}{D_1^3}\left(1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_1}{c^2} + \frac{\mathbf{a}_1 \cdot \mathbf{r}_1}{c^2}\right)\right)$$

$$\mathbf{r}_1 = \mathbf{x} - \mathbf{x}_1(t_1), \quad t_1 = t - \sqrt{r_1^2 + z^2}\,/\,c, \quad \mathbf{v}_1 = \dot{\mathbf{x}}_1(t_1), \quad \mathbf{a}_1 = \ddot{\mathbf{x}}_1(t_1), \quad D_1 = \sqrt{r_1^2 + z^2} - \frac{\mathbf{v}_1 \cdot \mathbf{r}_1}{c}$$

(Analogue of the Lienard-Wiechert formula in electrodynamics.)

The corresponding velocity field is divergent.

This is the generalization of point vortex dynamics to the case of compressible flow.

# Numerical Application

Wave split:

$$\mathbf{u}_t + c^2 \nabla \alpha = 0$$

$$\alpha_t + \nabla \cdot \mathbf{u} = 0$$

Vorticity split:

$$\mathbf{u}_t = q(v, -u)$$

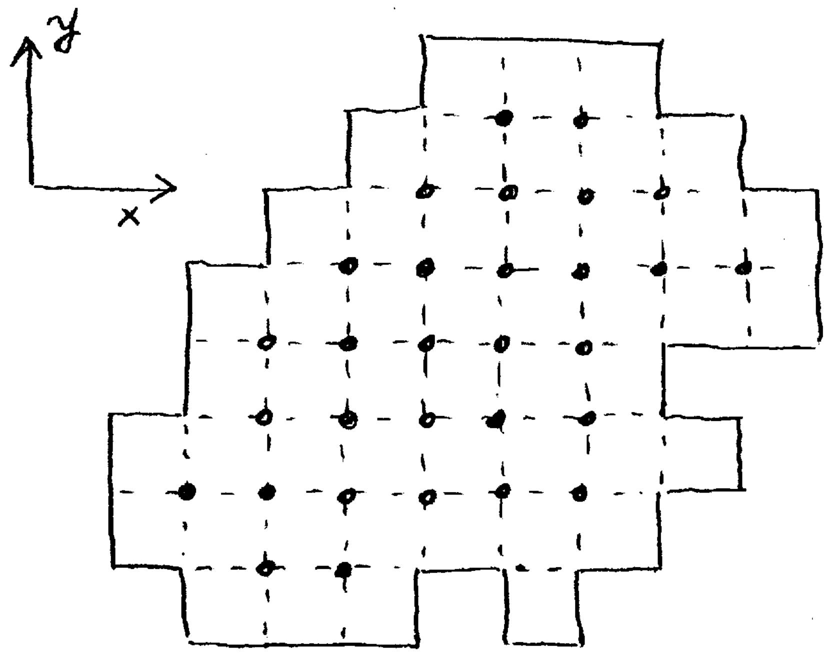
$$\alpha_t = 0$$

Use Strang splitting with the wave split further divided into directional splits:

e.g. $$\frac{\partial u}{\partial t} = -c^2 \frac{\partial \alpha}{\partial x}, \qquad \frac{\partial \alpha}{\partial t} = -\frac{\partial u}{\partial x}$$

$$\Leftrightarrow \qquad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)(u + c\,\alpha) = 0, \qquad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)(u - c\,\alpha) = 0$$

If $c = \Delta x / \Delta t$, these are simple shift operations.




Advantage: No elliptic equations to solve. Irregular domains easily accommodated.

Disadvantage: $c$ must be large compared to $|\mathbf{u}|$ and $fL$ (small time steps).

Irregular domain shape

$(c = L = 1)$

no-slip boundary conditions

Initially random passive scalar at $t = 0$

Passive scalar at $t = 30$

'Windowed' Vorticity at $t = 30$

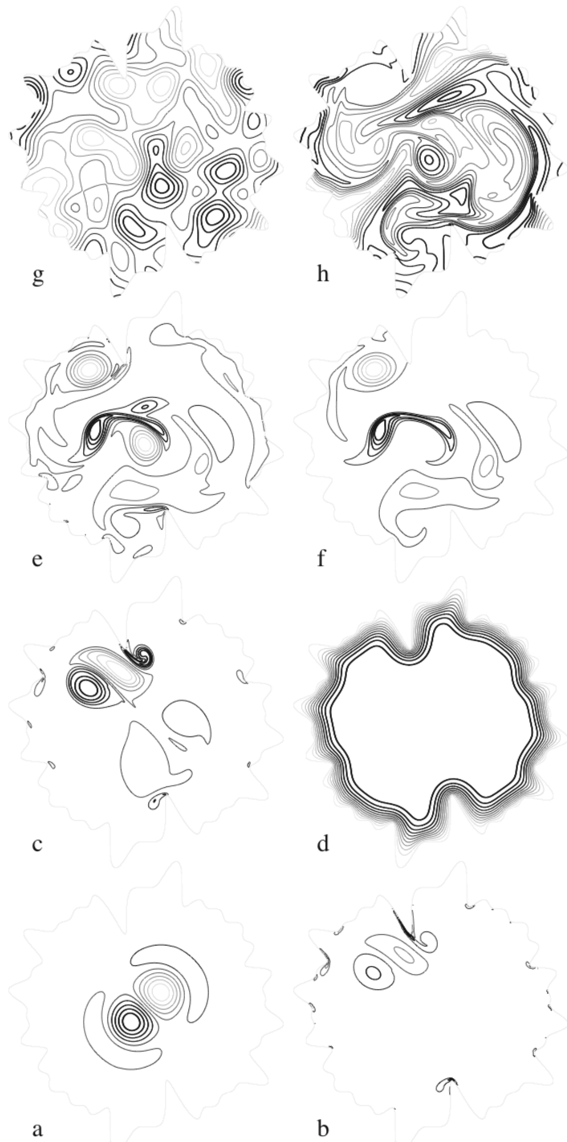
Passive scalar at $t = 30$

'Windowed' Vorticity at $t = 10$

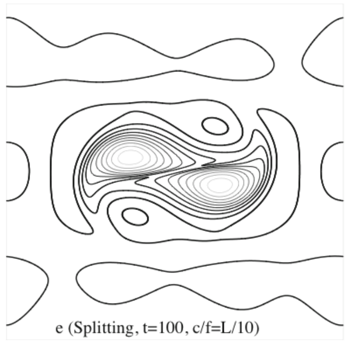
Window

Vorticity at $t = 0$

Vorticity at $t = 10$



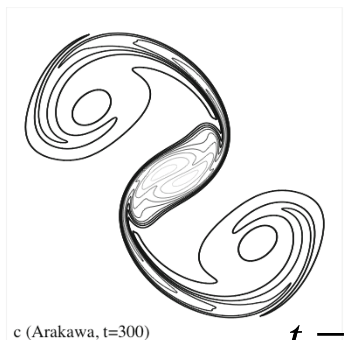
# Comparison

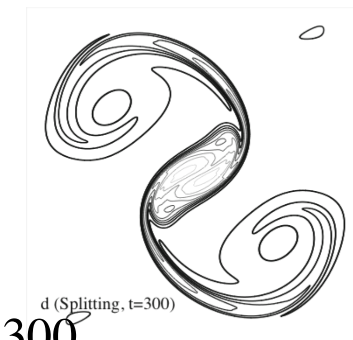



$\Leftarrow \left\{ \begin{array}{c} \text{rotating flow with} \\ c\,/\,f \;\; \text{too small (timestep too large)} \end{array} \right.$

Arakawa

Jacobian

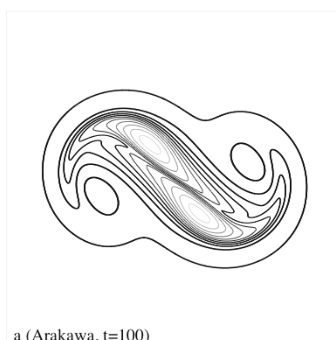


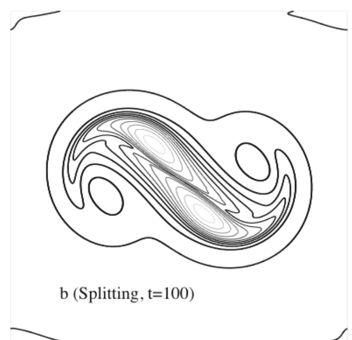




$t = 300$

Splitting

Method




nonrotating vortex merger, $t = 100$

# 3d GENERALIZATION

(becomes a method for solving the nonhydrostatic primitive equations)

Wave split:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla\rho$$

$$\frac{\partial \rho}{\partial t} = -\rho_0 \nabla\cdot\mathbf{v}$$

$$\frac{\partial \theta}{\partial t} = 0$$

Rotation split:

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\rho}{\rho_0}\left(\mathbf{v}\times\omega\right)$$

$$\frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial \theta}{\partial t} = 0$$

Buoyancy split:

$$\frac{\partial \mathbf{v}}{\partial t} = \theta\,\mathbf{k}$$

$$\frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial \theta}{\partial t} = -\frac{\rho_0}{\rho}\mathbf{v}\cdot\nabla\theta$$

# Other uses of this:  Wave/Mean interactions

These tend to be of two kinds:

1. Those not involving vorticity, $\bar{u} \propto A^2$, $\bar{u}$ disappears when the waves disappear
2. Those involving vorticity, $\dfrac{\partial \bar{u}}{\partial t} \propto A^2$, $\bar{u}$ changed permanently via irreversible mixing/stretching of vorticity

WV dynamics admits (2) but not (1)

i.e. it offers a simpler context in which to study the more important category of wave/mean interactions