An update on model development

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Start

- Outline:

1. New operators on the icosahedral grid
2. New curl_curl model on the icosahedral grid

Masuda and Ohnishi shallow water equations

- Three operators: **Jacobian**, **divergence** and **Laplacian**

- Masuda, Y., and H. Ohnishi, 1986: An integration scheme of the primitive equation model with an icosahedral-hexagonal grid system and its application to the shallow-water equations. Short- and Medium-Range Numerical Weather Prediction. Japan Meteorological Society, Tokyo, 317-326.

- Prognostic

$$\frac{\partial \eta}{\partial t} - J(\eta, \psi) + \nabla \bullet (\eta \nabla \chi) = 0$$

$$\frac{\partial D}{\partial t} - J(\eta, \chi) - \nabla \bullet (\eta \nabla \psi) + \nabla^2 (K + \phi) = 0$$

$$\frac{\partial \phi}{\partial t} - J(\phi, \psi) + \nabla \bullet (\phi \nabla \chi) = 0$$

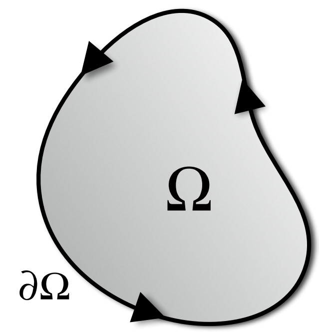
- Diagnostic

$$\nabla^2 \psi = \eta - f \qquad \nabla^2 \chi = D$$

A finite-difference Flux-Divergence operator

- Divergence theorem

$$\int_{\Omega} \nabla \bullet \mathbf{F}\, d\Omega = \oint_{\partial\Omega} \mathbf{F} \bullet \mathbf{n}\, dS$$



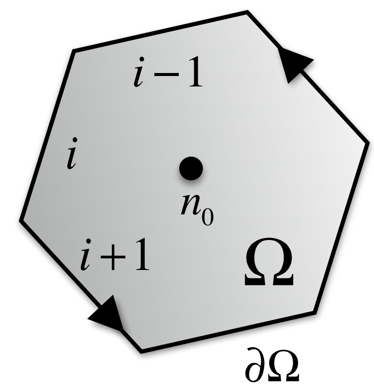
- Define

$$\mathbf{F} \equiv \alpha \nabla \beta$$

- Considering a **hexagonal cell** with cell center n_0.
  With the following observations:

1. $$\int_{\Omega} \nabla \bullet (\alpha \nabla \beta)\, d\Omega = \left[\nabla \bullet (\alpha \nabla \beta) \right]_0 A_0$$

2. $$\oint_{\partial\Omega} \alpha \frac{\partial \beta}{\partial n} dS = \sum_i \left[\oint_{\partial\Omega_i} \alpha \frac{\partial \beta}{\partial n} dS \right]_i$$



- The a template for a **flux-divergence (or Laplacian)** operator is given by

$$\left[\nabla \bullet (\alpha \nabla \beta) \right]_0 = \frac{1}{A_0} \sum_i \left[\oint_{\partial\Omega_i} \alpha \frac{\partial \beta}{\partial n} dS \right]_i$$

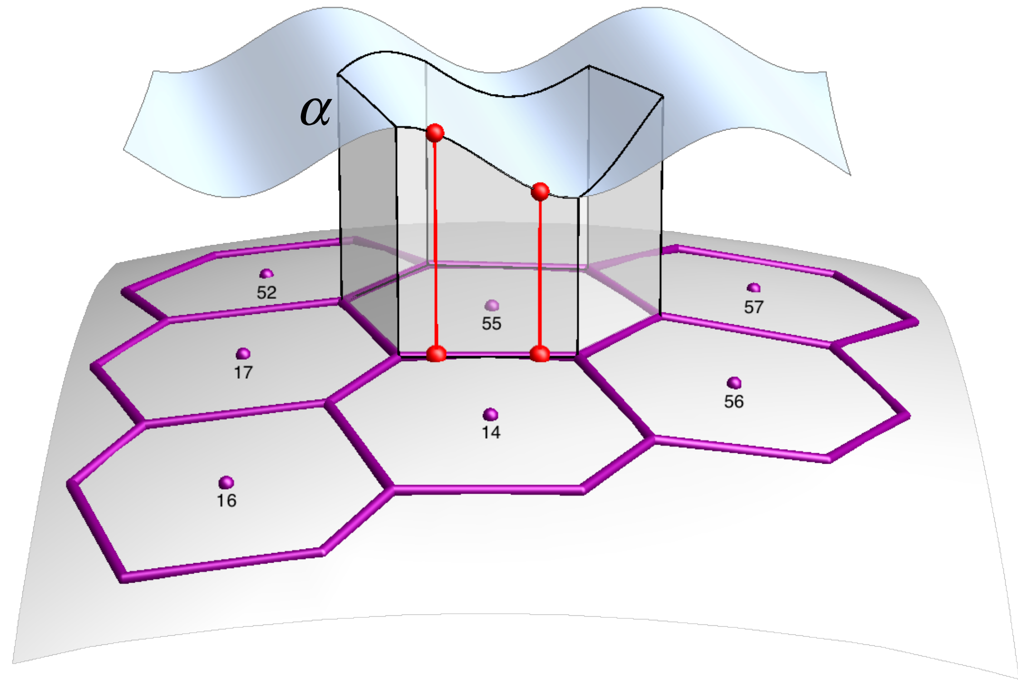
Gaussian quadrature

- **We can use Gaussian quadrature to approximate the line integral along each edge of a cell.**
- Approximate the definite integral of a function as a weighted sum of function values at specified points:

$$\int_{-1}^{1} \alpha(x)dx \approx \sum_{i=1}^{n} w_i \alpha(x_i)$$

- For example:

Number of points (q_m)	Points (x_q)	Weights (w_q)
1	0	2
2	$-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}$	1,1
3	$-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}$	$\frac{5}{9}, \frac{8}{9}, \frac{5}{9}$



Radial Basis Function (RBF)

We will use radial basis functions to approximate functions and derivatives of functions at Gauss points using information from nearby grid points.

Suppose we wish to construct a function s to interpolate data values f_i at scattered distinct grid points $\mathbf{x}_i$ where $i=1,2,\ldots,n$.

Then RBF interpolating function looks like this:

$$s(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \phi\left(\left\|\mathbf{x}-\mathbf{x}_i\right\|\right)$$

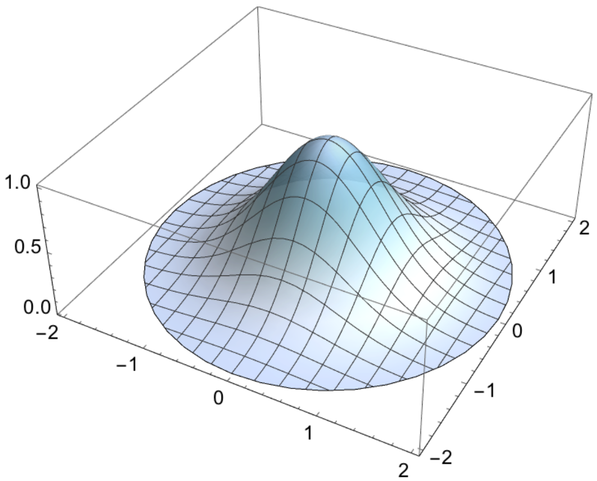
where $\|\bullet\|$ is the Euclidean norm. If we enforce the condition $s(\mathbf{x}_i) = f_i$ for $i=1,2,\ldots,n$ then the expansion coefficients λ_i are determined by the linear system:

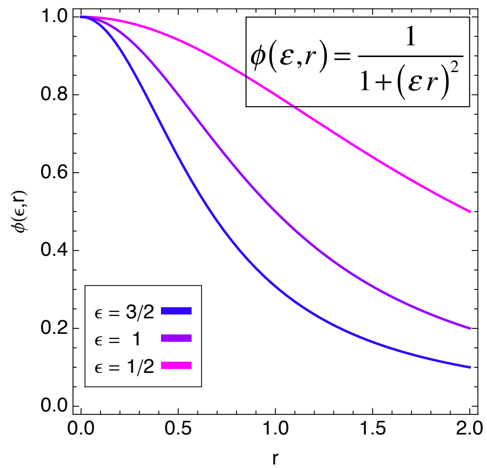
$$\begin{bmatrix} \phi(\varepsilon, r_{1,1}) & \phi(\varepsilon, r_{1,2}) & \cdots & \phi(\varepsilon, r_{1,n}) \\ \phi(\varepsilon, r_{2,1}) & \phi(\varepsilon, r_{2,2}) & \cdots & \phi(\varepsilon, r_{2,n}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\varepsilon, r_{n,1}) & \phi(\varepsilon, r_{n,2}) & \cdots & \phi(\varepsilon, r_{n,n}) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

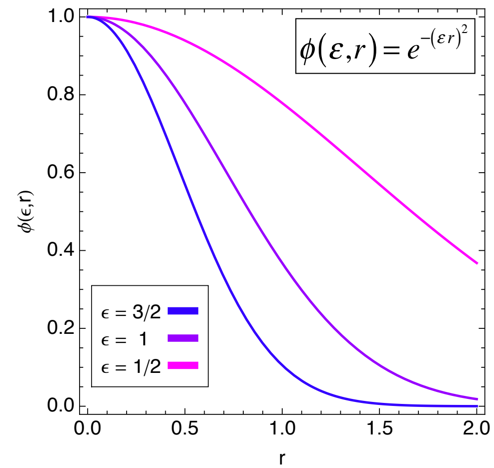
where $r_{i,j} = \|\mathbf{x}_i - \mathbf{x}_j\|$ is the distance between the $\mathbf{x}_i$ and $\mathbf{x}_j$ grid points

Radial Basis Function (RBF)

A radial basis function (RBF) is a real-valued function whose value depends only on the distance from a center point.

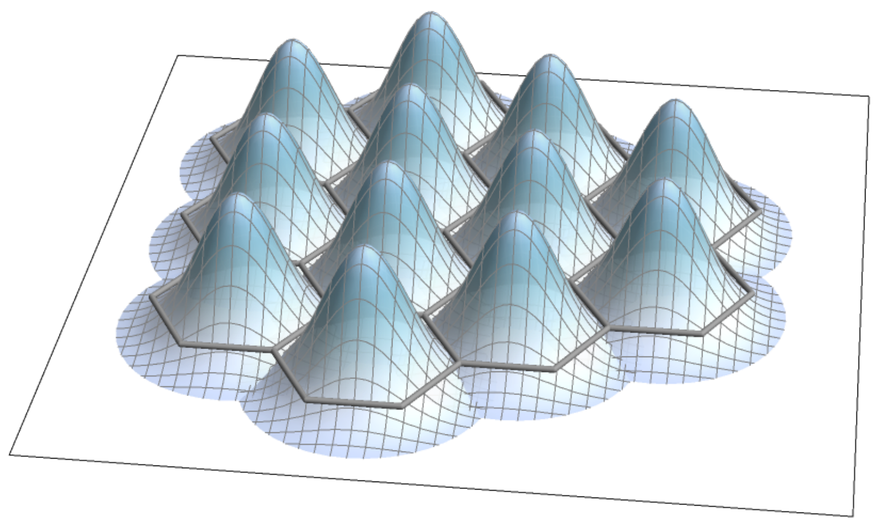




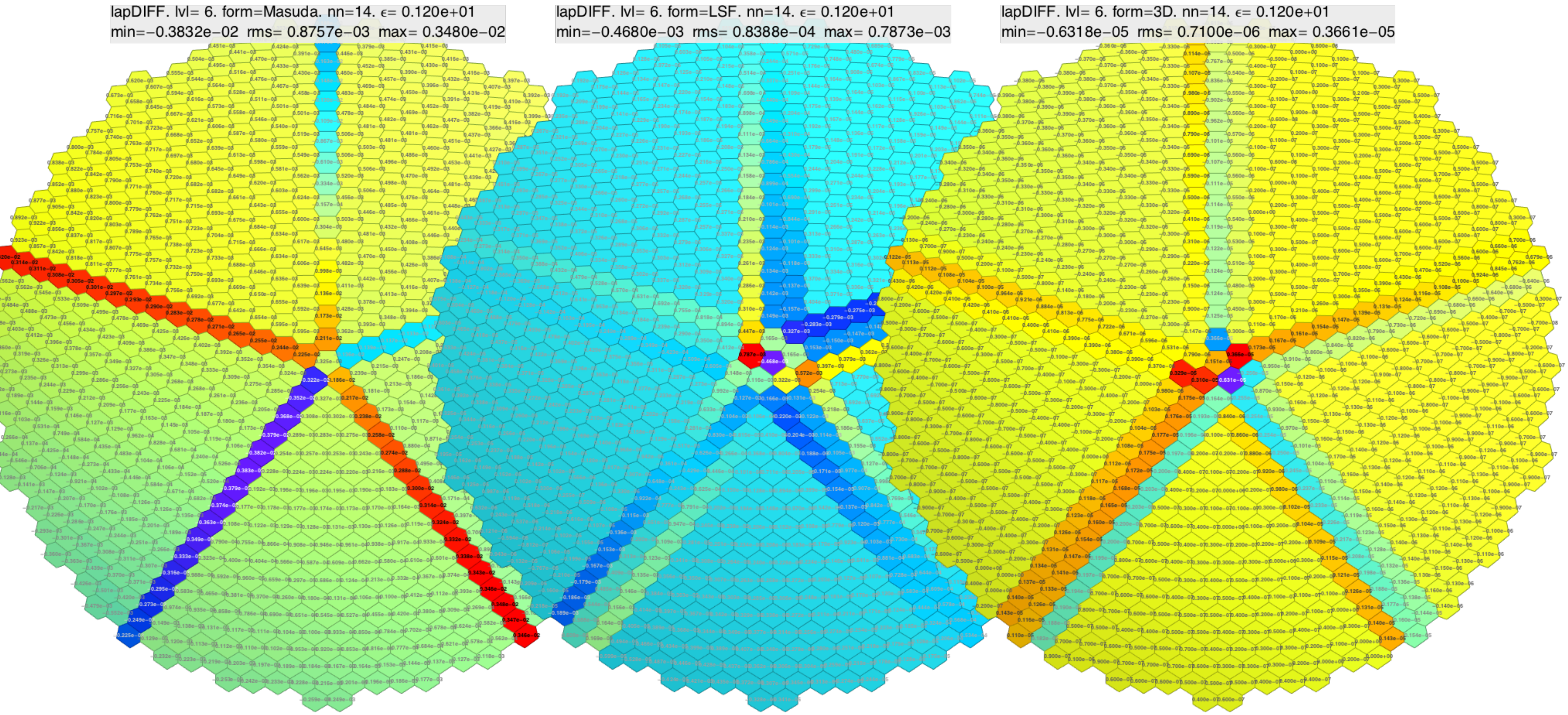


The function s is a super-position of basis functions each defined at surrounding grid points

$$s(\mathbf{x})=\sum_{i=1}^{n}\lambda_i\phi\left(\left\|\mathbf{x}-\mathbf{x}_i\right\|\right)$$

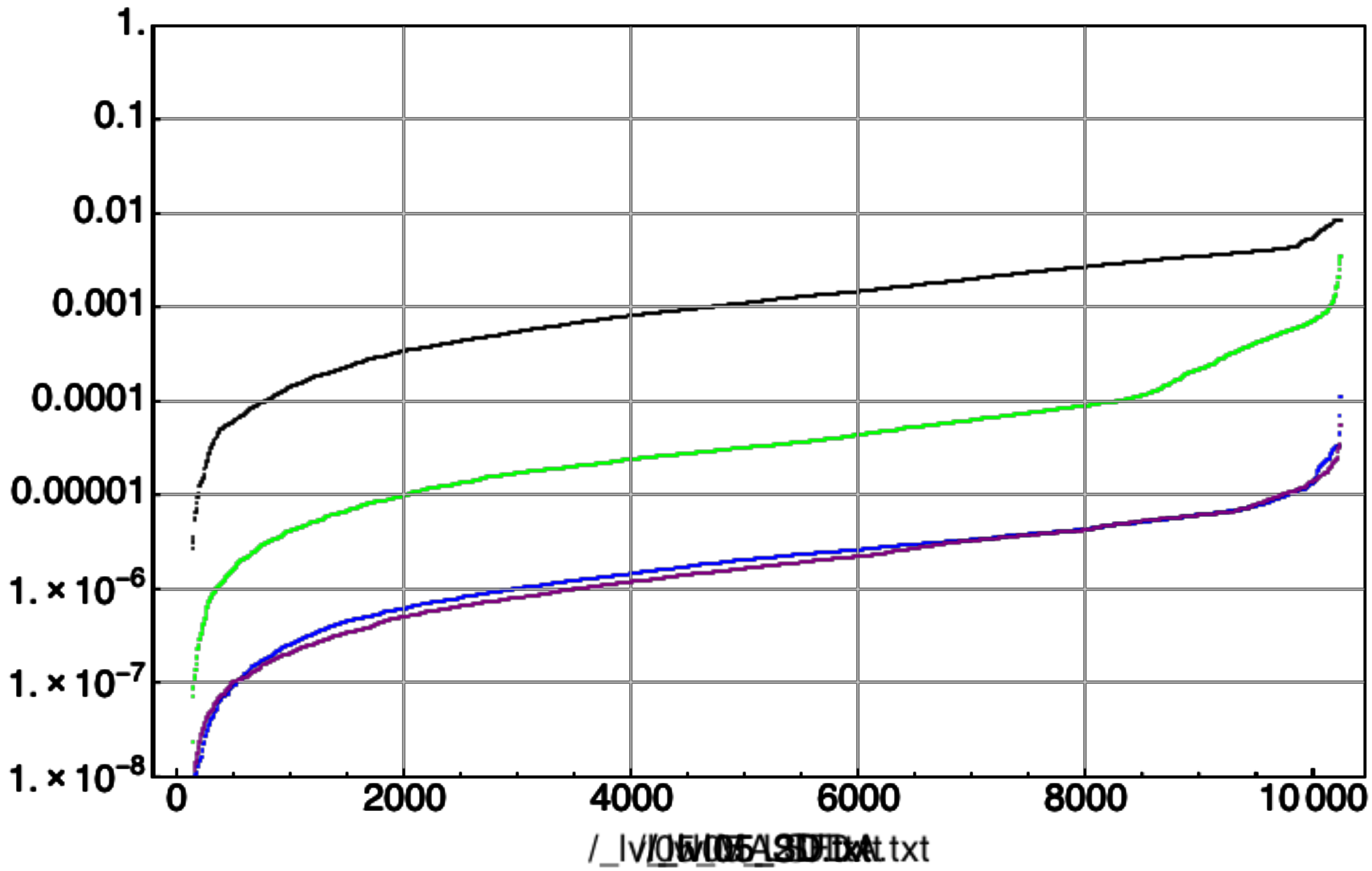


New Operators. Laplacian. Difference plots.



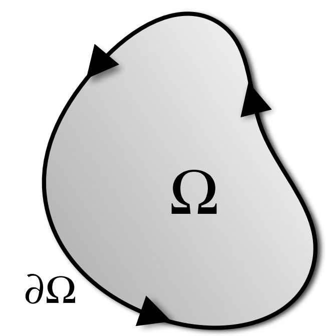
Masuda	Least squares fit	RBF 3D
max error = 0.382E-02	**max error = 0.787E-03**	**max error = 0.631E-05**
rms error = 0.876E-03	**rms error = 0.839E-04**	**rms error = 0.710E-06**

New Operators. Laplacian



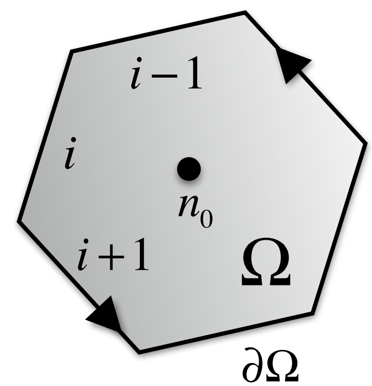
A finite-difference curl operator

- Stokes (Kelvin-Stokes) theorem: The integral of the curl of a vector field over some area equals the line integral of the vector field around the boundary of the area.



$$\int_{\Omega}\left[\mathbf{k}\bullet\left(\nabla\times\mathbf{F}\right)\right]d\Omega=\oint_{\partial\Omega}\left(\mathbf{F}\bullet\boldsymbol{\tau}\right)dS$$

- Considering a cell and with the following approximations:



1. $\int_{\Omega}\left[\mathbf{k}\bullet\left(\nabla\times\mathbf{F}\right)\right]d\Omega=\left[\mathbf{k}\bullet\left(\nabla\times\mathbf{F}\right)\right]_0 A_0$
2. $\oint_{\partial\Omega}\left(\mathbf{F}\bullet\boldsymbol{\tau}\right)dS=\sum_i\left(F_\tau\right)_i l_i$

- The finite-difference **curl** operator is given by

$$\left[\mathbf{k}\bullet\left(\nabla\times\mathbf{F}\right)\right]_0 A_0\approx\sum_i\left(F_\tau\right)_i l_i$$

Curl Curl on the icosahedral grid. Equations 1.

The three-dimensional momentum equations are given by:

$$\frac{\partial \mathbf{V}}{\partial t} + \boldsymbol{\omega}_a \times \mathbf{V} + \frac{1}{2}\nabla(\mathbf{V} \bullet \mathbf{V}) = -\alpha\nabla p + \mathbf{F}$$

where

$$\boldsymbol{\omega}_a = \boldsymbol{\eta}_a + \zeta_a \mathbf{k} = (\boldsymbol{\eta} + 2\boldsymbol{\Omega}_H) + (\zeta + 2\Omega_z)\mathbf{k}$$

Take the curl:

$$\frac{\partial \boldsymbol{\omega}_a}{\partial t} + \nabla \times (\boldsymbol{\omega}_a \times \mathbf{V}) = \nabla p \times \nabla \alpha - \nabla \times \mathbf{F}$$

$$\frac{\partial \boldsymbol{\eta}_a}{\partial t} + \nabla_H \times (\boldsymbol{\eta}_a \times \mathbf{v}) - \mathbf{k}\frac{\partial}{\partial z}[\zeta_a \mathbf{v} - w\boldsymbol{\eta}_a] = \left(\nabla_H p \times \frac{\partial \alpha}{\partial z}\mathbf{k} + \frac{\partial p}{\partial z}\mathbf{k} \times \nabla_H \alpha\right) - (\nabla \times \mathbf{F})_H$$

$$\frac{\partial \zeta_a}{\partial t} + \nabla_H \bullet [\zeta_a \mathbf{v} - w\boldsymbol{\eta}_a] = \mathbf{k} \bullet \left[(\nabla_H p \times \nabla_H \alpha) - (\nabla \times \mathbf{F})\right]$$

Curl Curl on the icosahedral grid. Equations 2.

Horizontal vorticity equation:

$$\frac{\partial \boldsymbol{\eta}_a}{\partial t} + \nabla_H \times (\boldsymbol{\eta}_a \times \mathbf{v}) - \mathbf{k}\frac{\partial}{\partial z} \times [\zeta_a \mathbf{v} - w\boldsymbol{\eta}_a] = \left(\nabla_H p \times \frac{\partial \alpha}{\partial z}\mathbf{k} + \frac{\partial p}{\partial z}\mathbf{k} \times \nabla_H \alpha \right) - (\nabla \times \mathbf{F})_H$$

Define:

$$\Gamma_a \equiv \mathbf{k} \bullet \nabla_H \times \boldsymbol{\eta}_a$$

Define:

$$d\mathbf{k} \equiv \boldsymbol{\eta}_a \times \mathbf{v}$$

$$\mathbf{k} \bullet \nabla_H \times \left[\nabla_H \times (\boldsymbol{\eta}_a \times \mathbf{v}) \right] = -\nabla_H^2 d$$

Gamma equation:

$$\frac{\partial \Gamma_a}{\partial t} - \nabla_H^2 d - \frac{\partial}{\partial z}\left[\mathbf{k} \bullet \nabla_H \times (\zeta_a \mathbf{v} - w\boldsymbol{\eta}_a) \right] = \nabla_H \bullet \left(-\frac{\partial \alpha}{\partial z}\nabla_H p + \frac{\partial p}{\partial z}\nabla_H \alpha \right) - \mathbf{k} \bullet \nabla_H \times (\nabla \times \mathbf{F})_H$$

Curl Curl on the icosahedral grid. Time step flow chart.

Predict the potential temperature.

↓

Predict the vertical component of vorticity.

↓

Predict the curl of the horizontal component of vorticity.

↓

Elliptic solve for vertical velocity:

$$\nabla_H^2 w + \frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w)\right] = -\Gamma$$

↓

Diagnose the divergence of horizontal vorticity from the vertical divergence of vertical vorticity:

$$\nabla_H \bullet \boldsymbol{\eta} + \frac{\partial \zeta}{\partial z} = 0 \qquad \delta = -\frac{\partial \zeta}{\partial z}$$

Construct horizontal vorticity from the curl and divergence of horizontal vorticity:

$$\nabla_H^2 \psi_\eta = \Gamma \text{ and } \nabla_H^2 \chi_\eta = \delta$$

and

$$\boldsymbol{\eta} = k \times \nabla_H \psi_\eta + \nabla_H \chi_\eta$$

→

Diagnose the divergence of horizontal velocity in the top layer:

$$\delta_{\mathbf{v}} \equiv (\nabla_H \bullet \mathbf{v})_T$$

$$= -\left[\frac{1}{\rho_{qs}}\frac{\partial}{\partial z}(\rho_{qs} w)\right]_T - \left[\frac{1}{\rho_{qs}}\left(\frac{\partial}{\partial t} + \nabla_H\right)\rho_{qs}\right]_T$$

Construct horizontal velocity in the top layer:

$$\nabla_H^2 \psi_{\mathbf{v}} = \zeta_T \text{ and } \nabla_H^2 \chi_{\mathbf{v}} = \delta_{\mathbf{v}}$$

and

$$\mathbf{v}_T = k \times \nabla_H \psi_{\mathbf{v}} + \nabla_H \chi_{\mathbf{v}}$$

↓

Diagnose horizontal velocity:

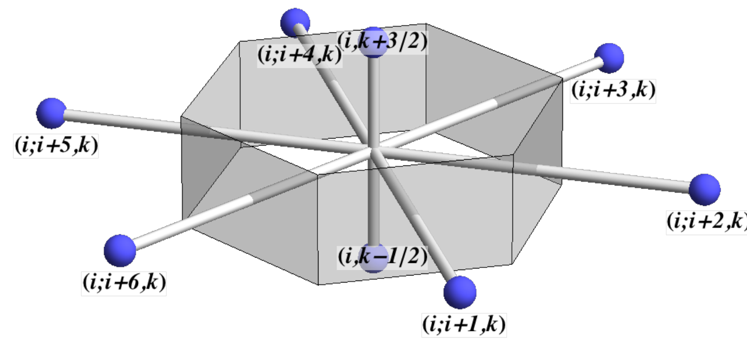
$$\mathbf{v} - \mathbf{v}_T = \int_{z_T}^{z} \left(\nabla_H w - \mathbf{k} \times \boldsymbol{\eta}\right) dz$$

3D elliptic solver

The continuous equation:

$$\nabla^2 w + \frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial z}(\rho w)\right] = -\Gamma$$

The discrete equations:



$$\frac{1}{A_i}\sum_{i'}\frac{w^{(\kappa)}_{i+i',3/2} - w^{(\kappa+1)}_{i,3/2}}{L_{i;i+i'}} l_{i;i+i'} + \frac{1}{\delta z_{3/2}}\left[\frac{1}{\rho_2 \delta z_2}\left(\rho_{5/2} w^{(\kappa+1)}_{i,5/2} - \rho_{3/2} w^{(\kappa+1)}_{i,3/2}\right)\right] = -\Gamma_{i,3/2}$$

$$\frac{1}{A_i}\sum_{i'}\frac{w^{(\kappa)}_{i+i',k+1/2} - w^{(\kappa+1)}_{i,k+1/2}}{L_{i;i+i'}} l_{i;i+i'} + \frac{1}{\delta z_{k+1/2}}\left[\frac{1}{\rho_{k+1} \delta z_{k+1}}\left(\rho_{k+3/2} w^{(\kappa+1)}_{i,k+3/2} - \rho_{k+1/2} w^{(\kappa+1)}_{i,k+1/2}\right) - \frac{1}{\rho_k \delta z_k}\left(\rho_{k+1/2} w^{(\kappa+1)}_{i,k+1/2} - \rho_{k-1/2} w^{(\kappa+1)}_{i,k-1/2}\right)\right] = -\Gamma_{i,k+1/2}$$

$$\frac{1}{A_i}\sum_{i'}\frac{w^{(\kappa)}_{i+i',km-1/2} - w^{(\kappa+1)}_{i,km-1/2}}{L_{i;i+i'}} l_{i;i+i'} - \frac{1}{\delta z_{km-1/2}}\left[\frac{1}{\rho_{km-1} \delta z_{km-1}}\left(\rho_{km-1/2} w^{(\kappa+1)}_{i,km-1/2} - \rho_{km-3/2} w^{(\kappa+1)}_{i,km-3/2}\right)\right] = -\Gamma_{i,km-1/2}$$

where (κ) denotes an iteration index.

3D elliptic solver

This problem is highly anisotropic so we use line relaxation in the vertical direction and standard point wise relaxation in the horizontal

Re-arrange terms to form an implicit tridiagonal system in the vertical:

$$-\left[\frac{1}{A_i}\sum_{i'}\frac{l_{i;i+i'}}{L_{i;i+i'}}+\frac{\rho_{3/2}}{\delta z_{3/2}}\left(\frac{1}{\rho_2\delta z_2}\right)\right]w_{i,3/2}^{(\kappa+1)}+\frac{\rho_{5/2}}{\delta z_{3/2}\rho_2\delta z_2}w_{i,5/2}^{(\kappa+1)}=-\Gamma_{i,3/2}-\frac{1}{A_i}\sum_{i'}\frac{l_{i;i+i'}}{L_{i;i+i'}}w_{i+i',3/2}^{(\kappa)}$$

$$\frac{\rho_{k-1/2}}{\delta z_{k+1/2}\rho_k\delta z_k}w_{i,k-1/2}^{(\kappa+1)}-\left[\frac{1}{A_i}\sum_{i'}\frac{l_{i;i+i'}}{L_{i;i+i'}}+\frac{\rho_{k+1/2}}{\delta z_{k+1/2}}\left(\frac{1}{\rho_k\delta z_k}+\frac{1}{\rho_{k+1}\delta z_{k+1}}\right)\right]w_{i,k+1/2}^{(\kappa+1)}+\frac{\rho_{k+3/2}}{\delta z_{k+1/2}\rho_{k+1}\delta z_{k+1}}w_{i,k+3/2}^{(\kappa+1)}=-\Gamma_{i,k+1/2}-\frac{1}{A_i}\sum_{i'}\frac{l_{i;i+i'}}{L_{i;i+i'}}w_{i+i',k+1/2}^{(\kappa)}$$

$$\frac{\rho_{km-3/2}}{\delta z_{km-1/2}\rho_{km-1}\delta z_{km-1}}w_{i,km-3/2}^{(\kappa+1)}-\left[\frac{1}{A_i}\sum_{i'}\frac{l_{i;i+i'}}{L_{i;i+i'}}+\frac{\rho_{km-1/2}}{\delta z_{km-1/2}}\left(\frac{1}{\rho_{km-1}\delta z_{km-1}}\right)\right]w_{i,km-1/2}^{(\kappa+1)}=-\Gamma_{i,km-1/2}-\frac{1}{A_i}\sum_{i'}\frac{l_{i;i+i'}}{L_{i;i+i'}}w_{i+i',km-1/2}^{(\kappa)}$$

Anelastic test case. Rising warm bubble.

- The earth is smaller

$$a = 6366\ m$$

- The model top is

$$z_{top} = 12000\ m$$

- With 40962 cells and 80 layers, the typical cell is about 100×150 m
- The earth is not rotating
- The potential temperature is given by

$$\theta = \theta_{00} + \theta'$$

where

$$\theta_{00} = 300K$$

and

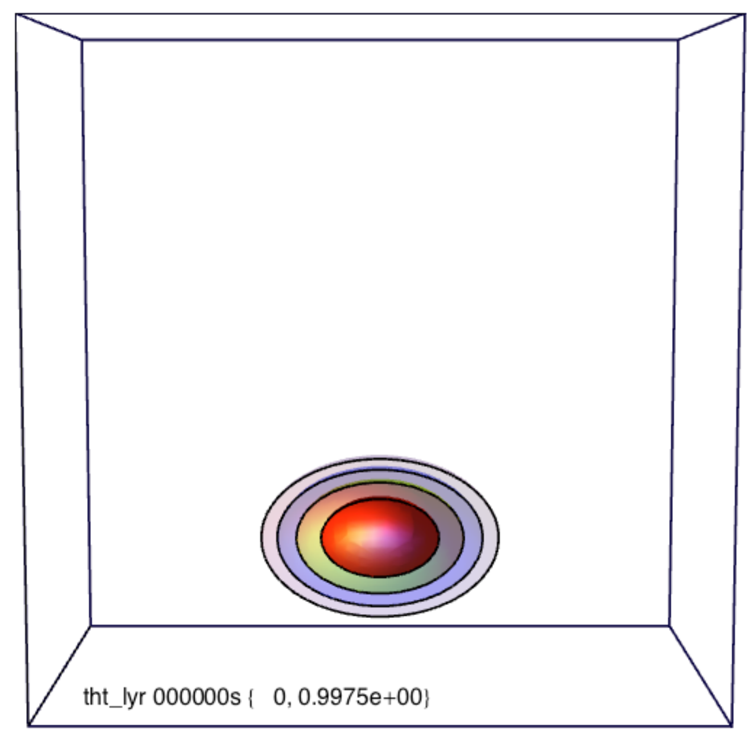
$$\theta' = \theta_0 \exp\left[-\left(\frac{r}{r_0} \right)^6 \right]$$

where $\theta_0 = 6.6$ K, $r_0 = 2500$ m

- The initial gamma and vorticity is given by

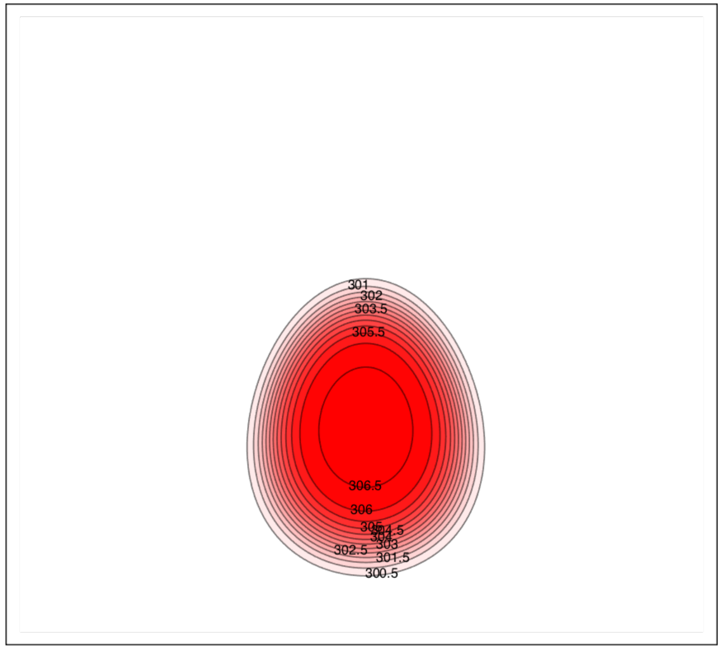
$$\Gamma \equiv 0 \text{ and } \zeta \equiv 0$$

- Bubble centered on the equator

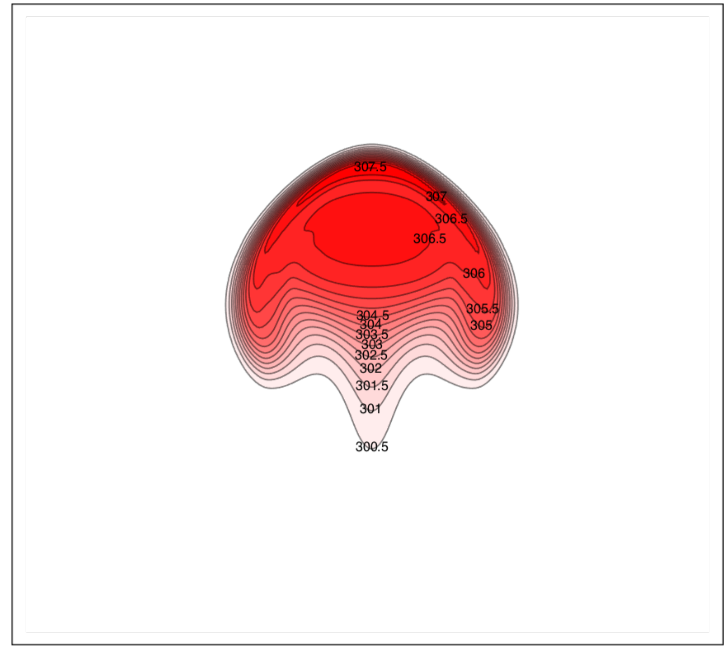


Anelastic test case. Rising warm bubble.

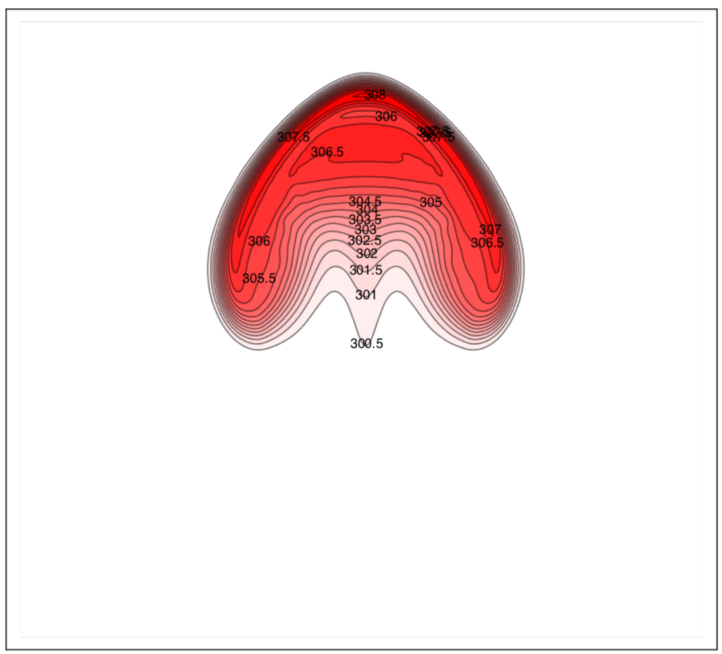
- Cross sections of potential temperature
- 0s, 240s, 300s and 360s



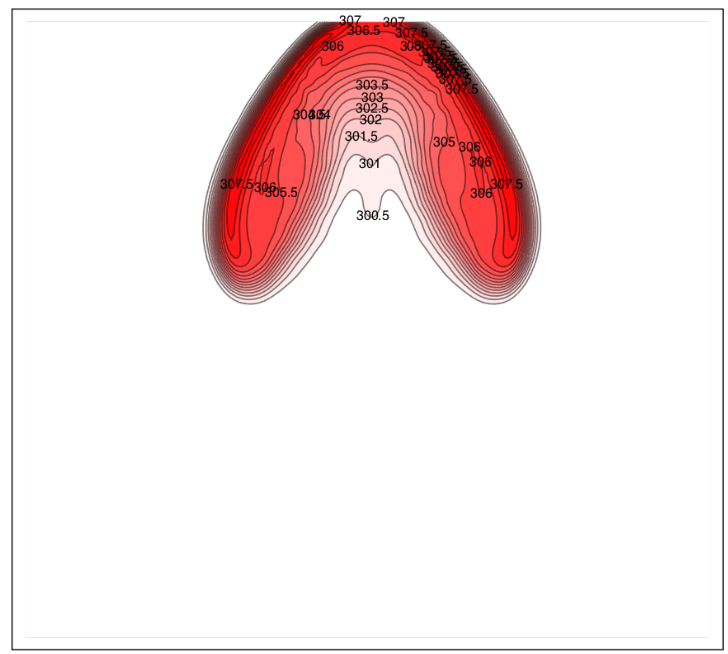
tht 000000s min= 0.300000e+03 max= 0.306599e+03



tht 000240s min= 0.299920e+03 max= 0.307446e+03



tht 000300s min= 0.299893e+03 max= 0.307938e+03



tht 000360s min= 0.299892e+03 max= 0.307795e+03