# Using relative buoyancy to refine theoretical expressions for convective updrafts

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# Overview:

- **Improved quantitative and qualitative understanding of the "buoyancy force."**
- **We will define the "absolute buoyancy force" and "relative buoyancy force." How do these forces compare? What are the underlying assumptions involved in their derivations?**
- **Many current qualitative and quantitative analysis frameworks are based on the *absolute buoyancy force*. It is argued here that they should be based on the *relative buoyancy force*.**

# What force causes air to rise in deep moist updrafts?

# Buoyancy:
# The Theory of Archimedes

"Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object."

-Archimedes of Syracuse (c. 250 BC)

# Vertical Momentum equation

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \cancel{F}$$

$$p = p'(x, y, z, t) + p_0(z) \qquad \rho = \rho'(x, y, z, t) + \rho_0(z)$$

$$\frac{dw}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\,\frac{\rho'}{\rho_0}$$

Vertical pressure gradient force

Buoyancy

We'll call this "absolute buoyancy," since its sign and magnitude are dependent on a universal base state

# Parcel Theory

$$\frac{dw}{dt} = \cancel{-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}} - g\frac{\rho'}{\rho_0}$$

Vertical pressure gradient force

Absolute Buoyancy

Neglecting the vertical pressure gradient force: *how valid is this assumption?*

$$\frac{dw}{dt} = \frac{1}{2}\frac{\partial (w^2)}{\partial z} \approx -g\frac{\rho'}{\rho_0}$$

# Parcel Theory

$$\frac{1}{2}\frac{\partial(w^2)}{\partial z} \approx -g\,\frac{\rho'}{\rho_0}$$

$$\frac{\rho'}{\rho_0} \approx \frac{\theta'}{\theta_0}$$

$$\frac{1}{2}\frac{\partial(w^2)}{\partial z} \approx -g\,\frac{\theta'}{\theta_0}$$

# Parcel Theory

$$\frac{1}{2}\frac{\partial(w^2)}{\partial z} \approx -g\frac{\theta'}{\theta_0}$$

$\theta_0 \equiv \theta_{env}$ $\quad \theta' \equiv \theta_{parcel}\text{-}\theta_{env}$

$$\frac{1}{2}w^2 \approx -g\int_{z=z_0}^{z=z_{EL}} \frac{\theta_{parcel}-\theta_{env}}{\theta_{env}}dz$$

$$\text{CIN} \equiv -g\int_{z=z_0}^{z=z_{LFC}} \frac{\theta_{parcel}-\theta_{env}}{\theta_{env}}dz$$

$$\text{CAPE} \equiv -g\int_{z=z_{LFC}}^{z=z_{EL}} \frac{\theta_{parcel}-\theta_{env}}{\theta_{env}}dz$$

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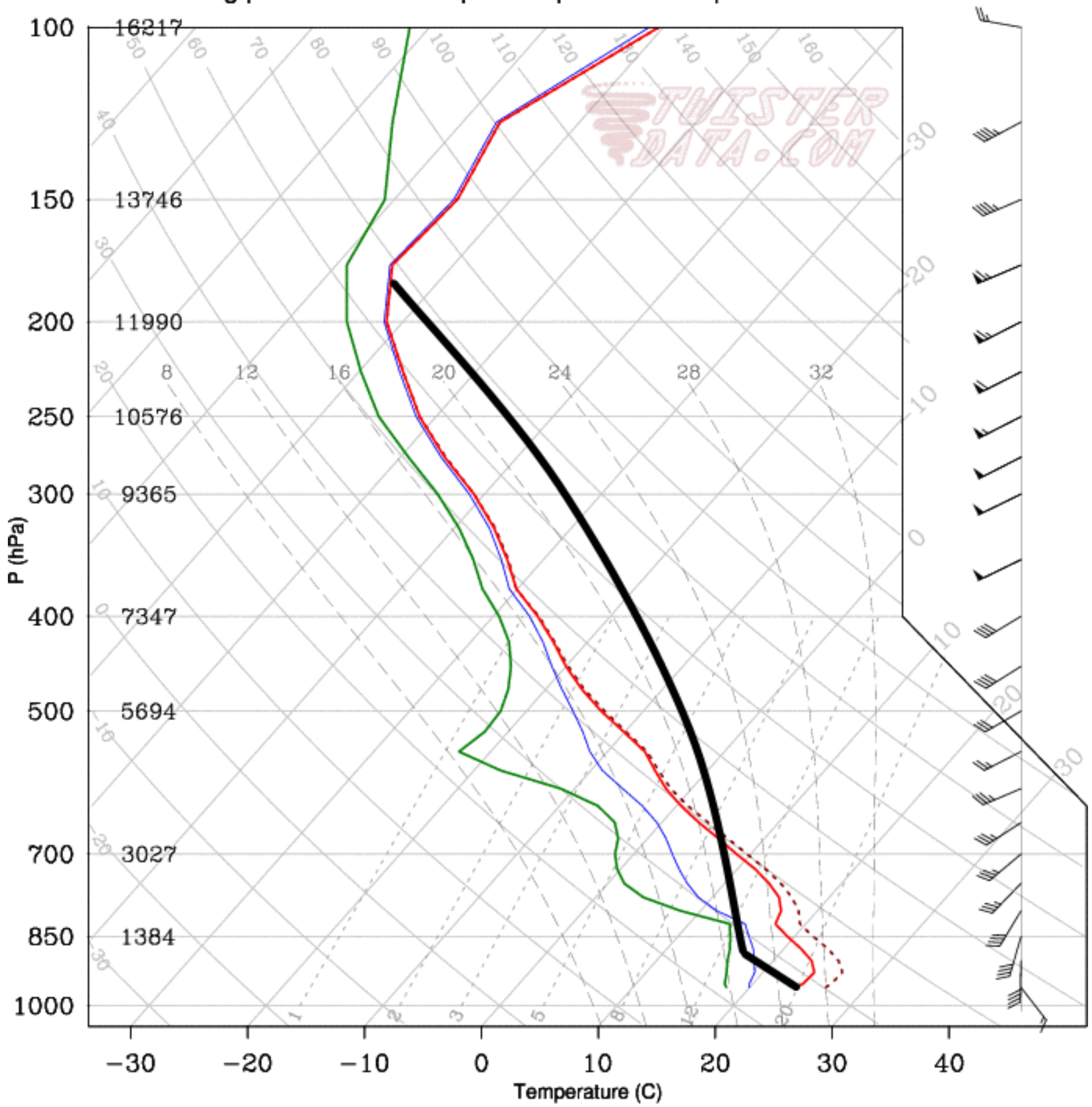

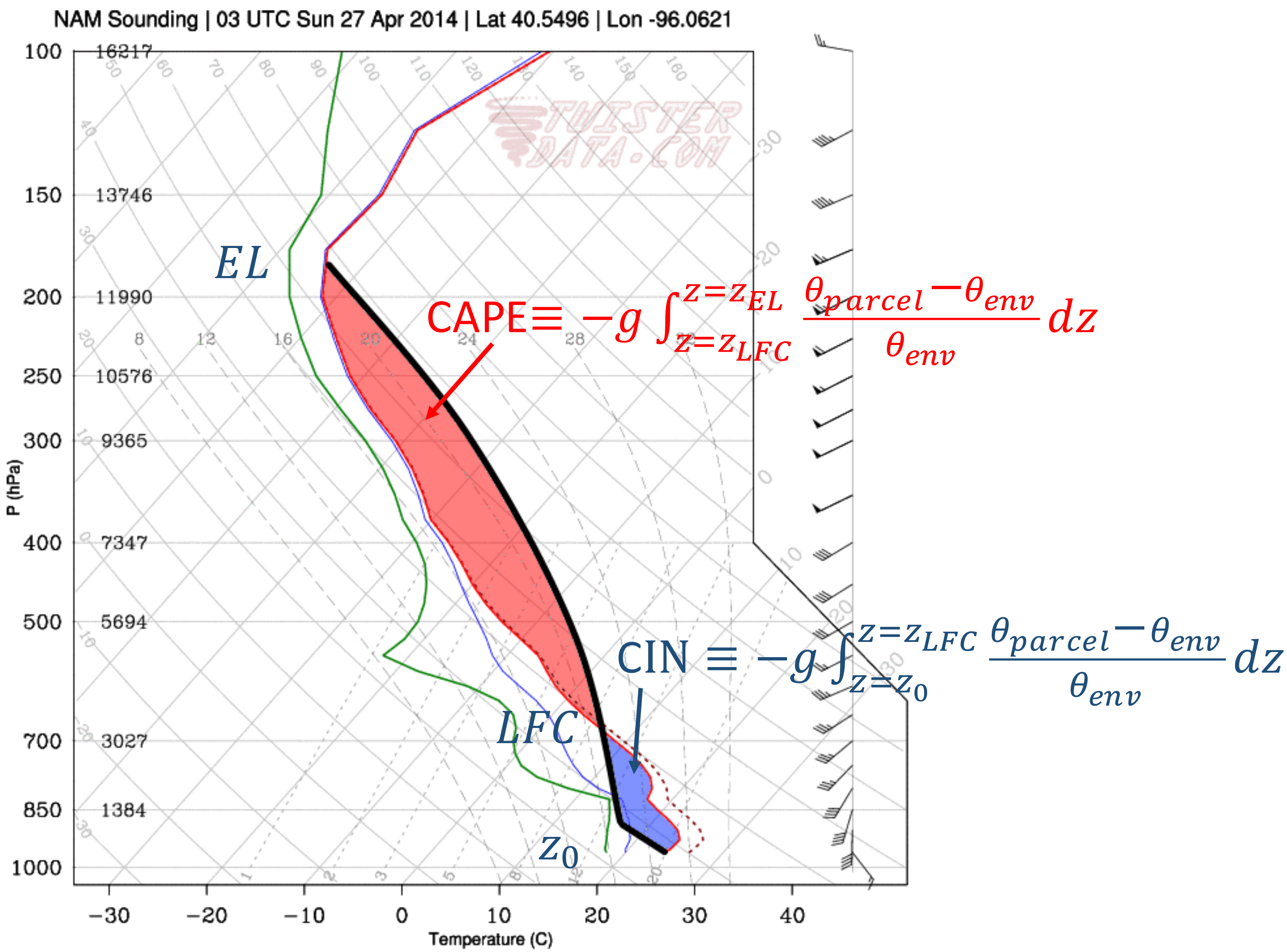
NAM Sounding | 03 UTC Sun 27 Apr 2014 | Lat 40.5496 | Lon -96.0621
P (hPa)
Temperature (C)
EL
LFC
$z_0$
CAPE $\equiv -g\int_{z=z_{LFC}}^{z=z_{EL}} \frac{\theta_{parcel}-\theta_{env}}{\theta_{env}}\,dz$
CIN $\equiv -g\int_{z=z_0}^{z=z_{LFC}} \frac{\theta_{parcel}-\theta_{env}}{\theta_{env}}\,dz$
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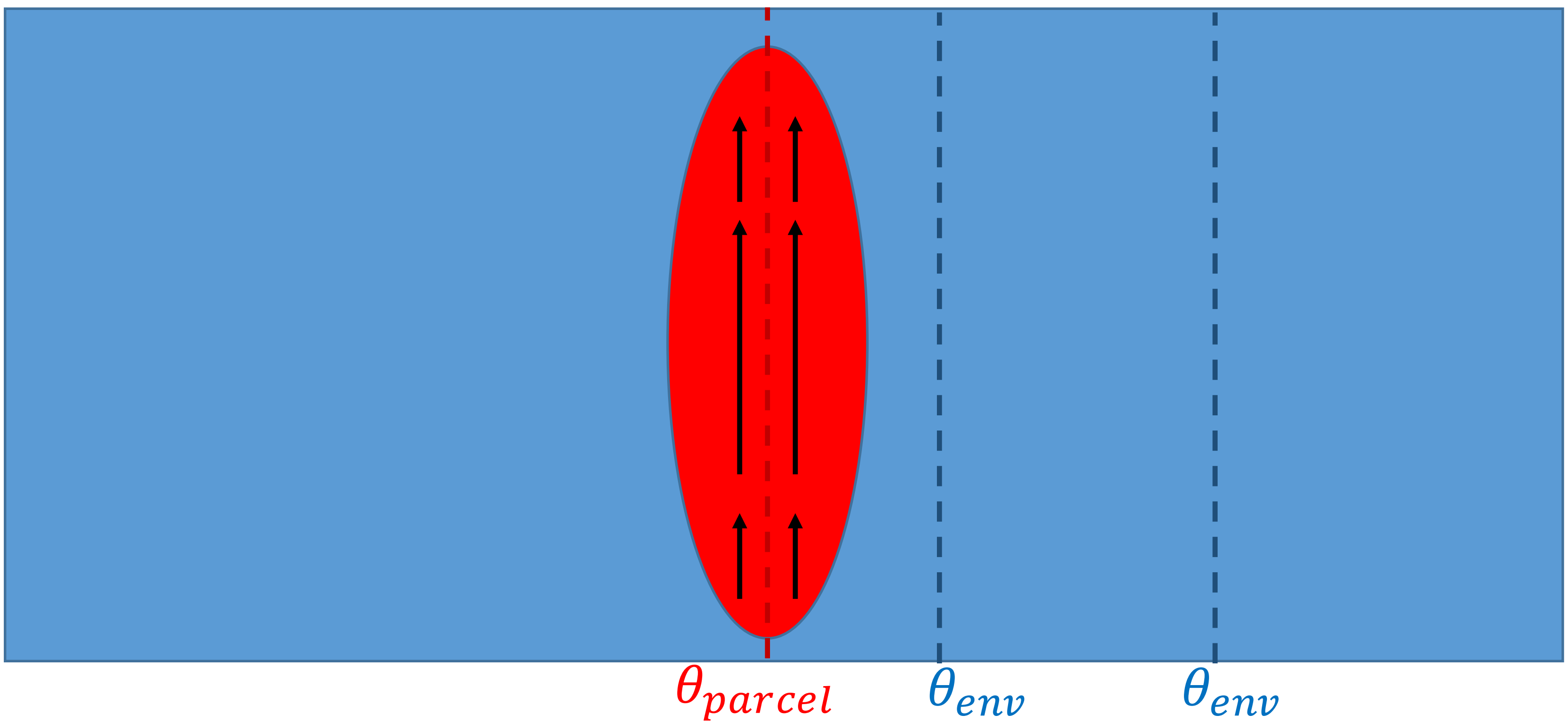
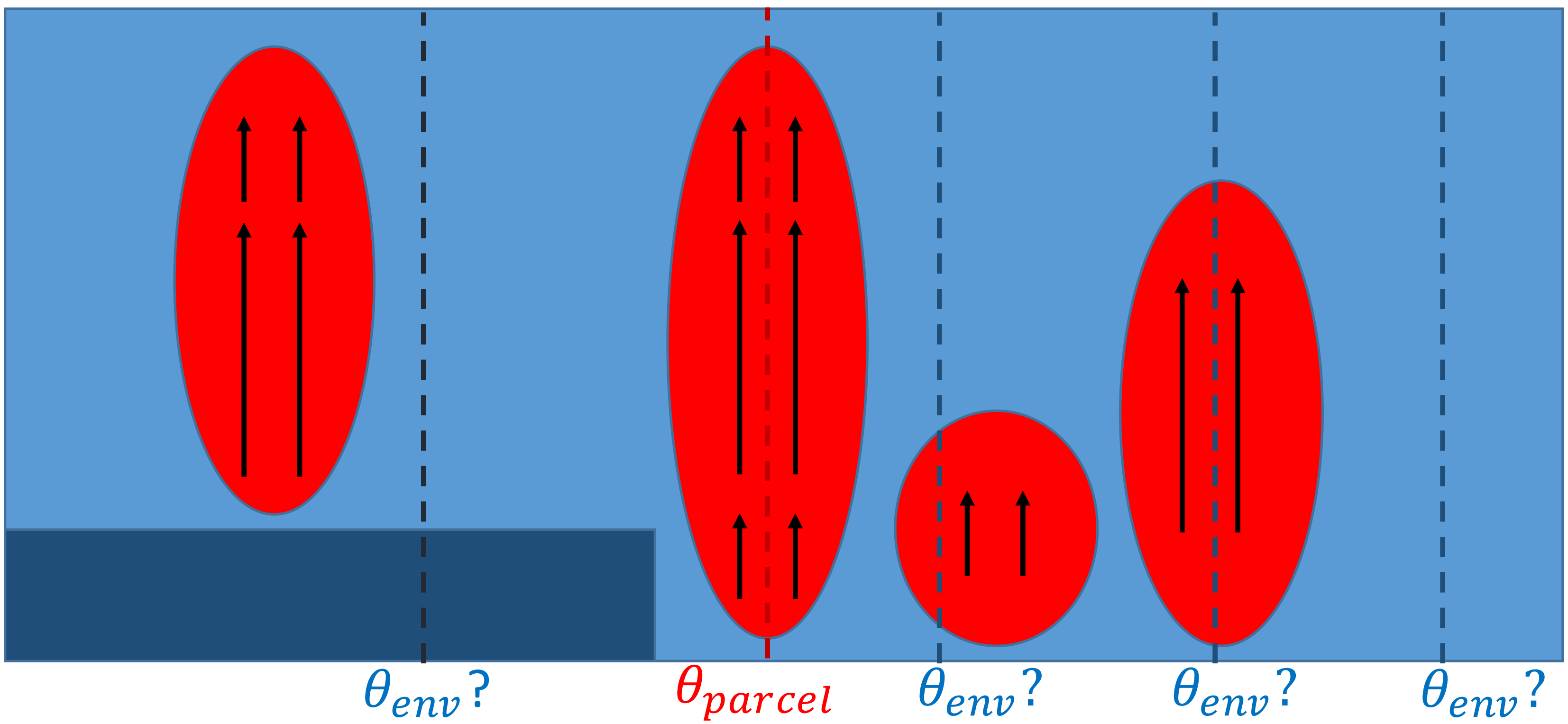
# Which environment "acts" upon an updraft once it has formed?



# Updraft in *horizontally and temporally* heterogeneous environment

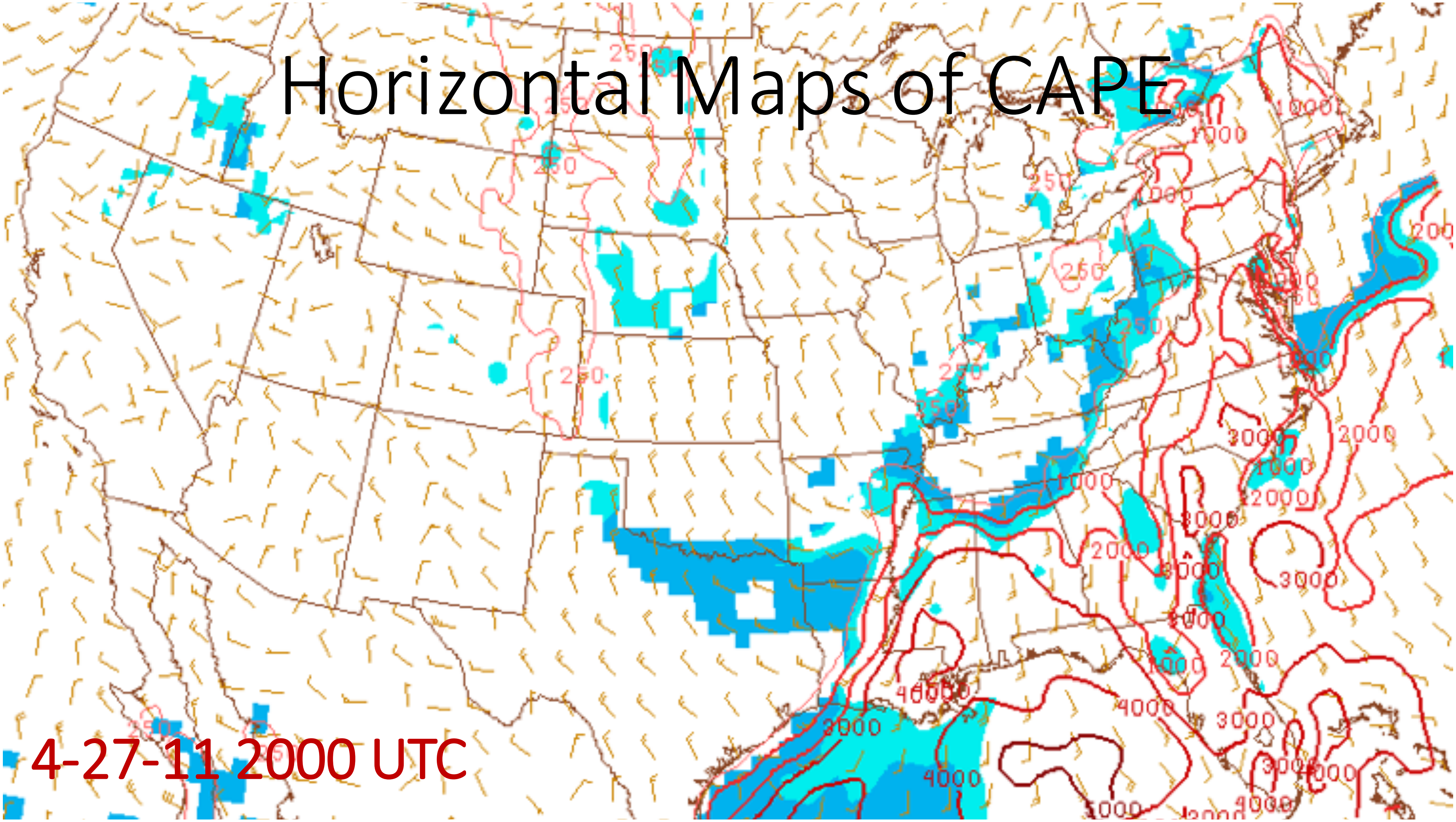


# What do we choose as the base state?

"Do parcels within the WER <referring to the core of an updraft> respond to their immediate neighbors with similar properties or do they respond to more distant parcels within the cloud that have distinctly different properties? Or do they respond to parcels *outside* the cloud?"

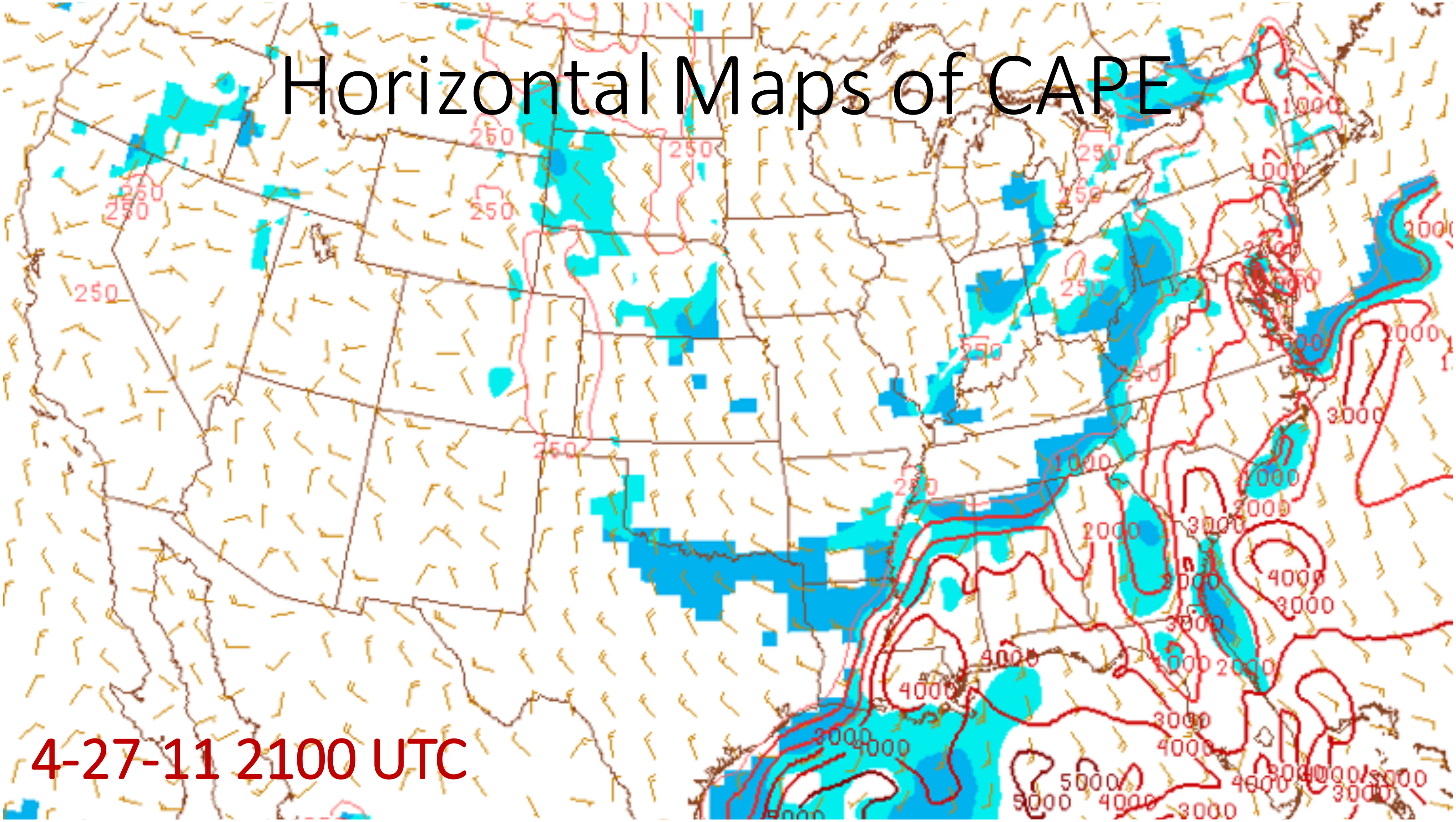
-Doswell and Markowski (2004)

# Horizontal Maps of CAPE



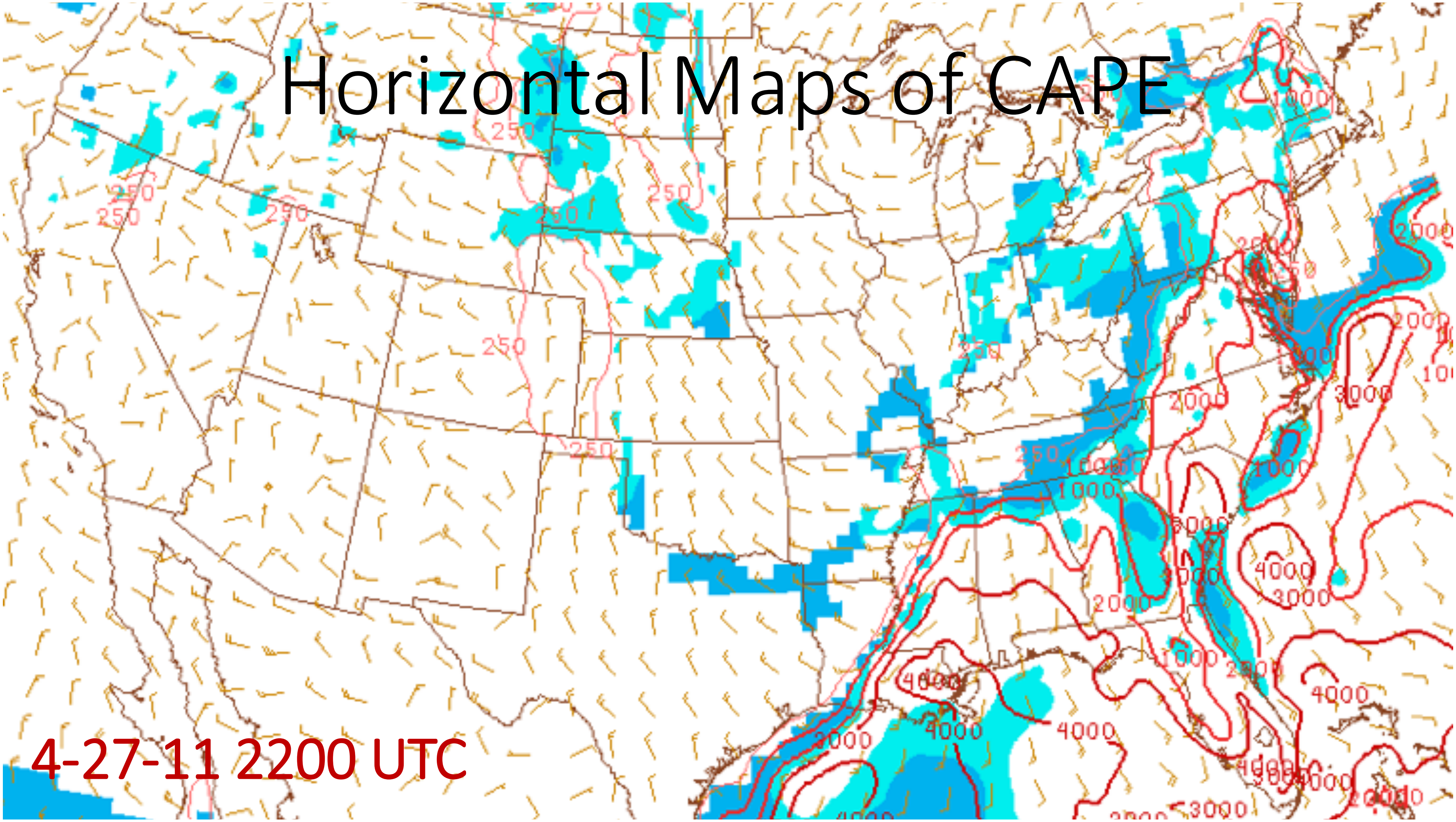

4-27-11 2000 UTC

# Horizontal Maps of CAPE



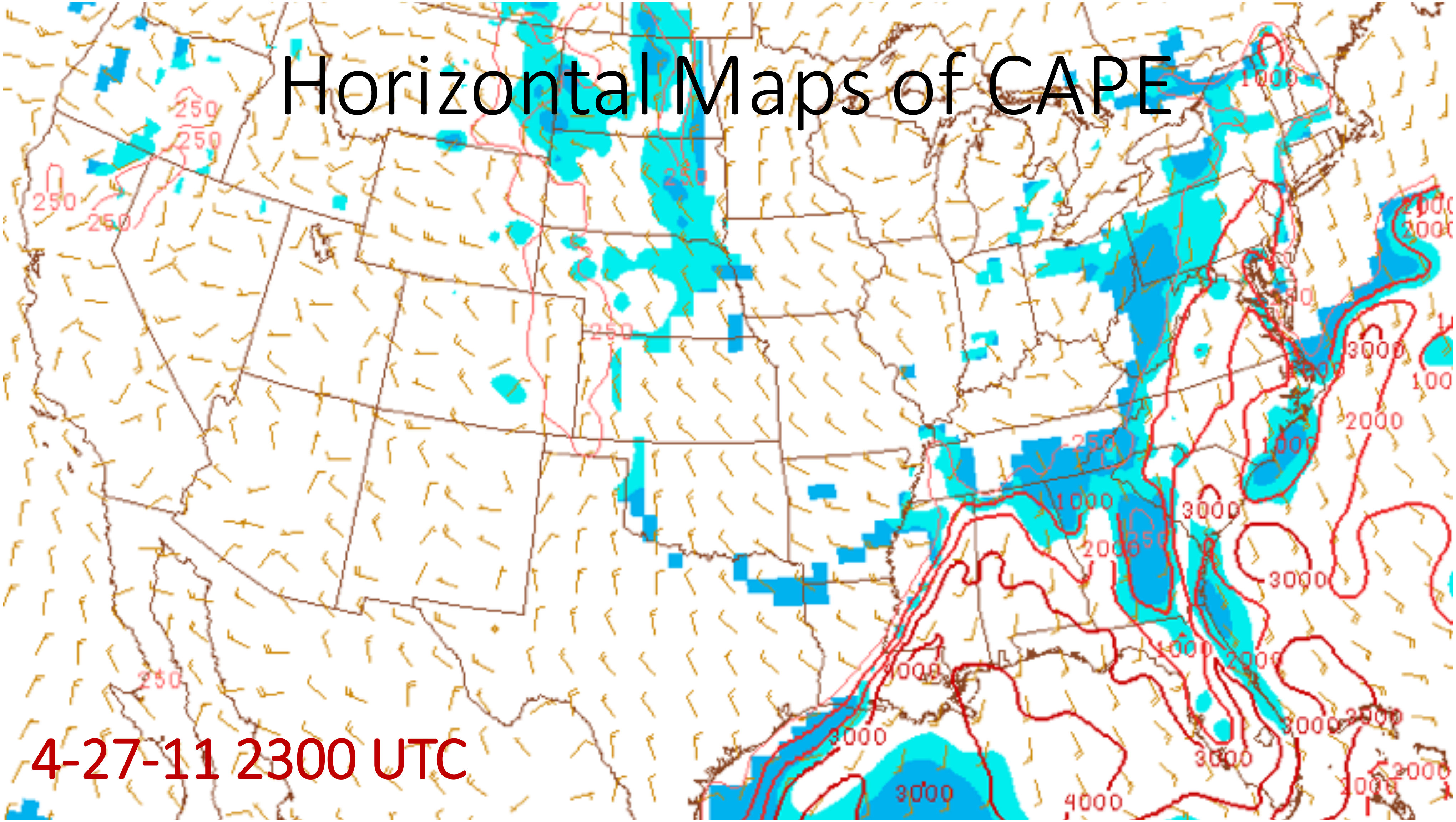
4-27-11 2100 UTC

# Horizontal Maps of CAPE



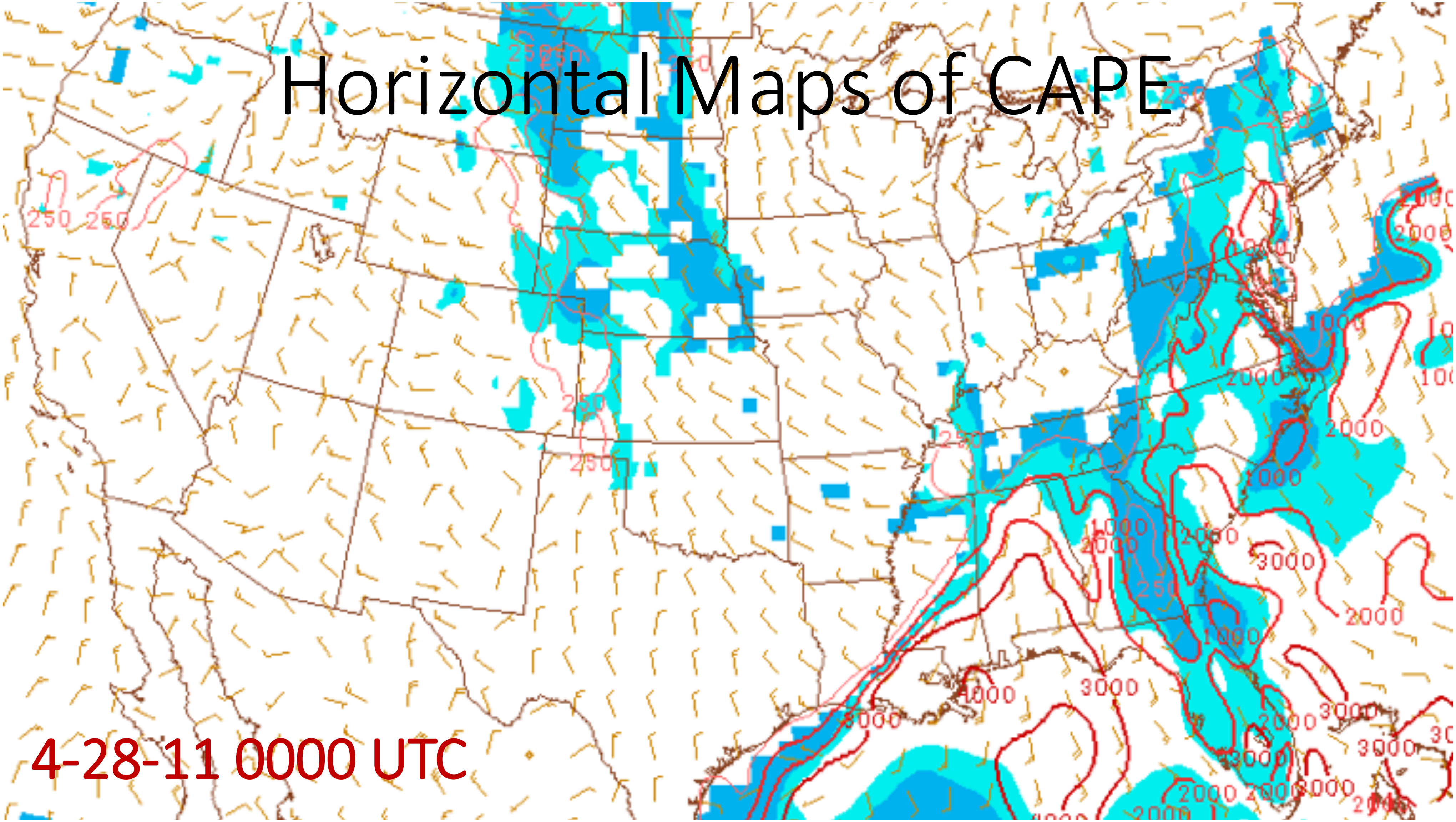
4-27-11 2200 UTC

# Horizontal Maps of CAPE



4-27-11 2300 UTC

# Horizontal Maps of CAPE



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# What do we choose as the base state?

$$\theta = \theta'(x, y, z, t) + \theta_0(z) \qquad \rightarrow \qquad \theta = \theta'(x, y, z, t) + \theta_0(x, y, z, t)$$

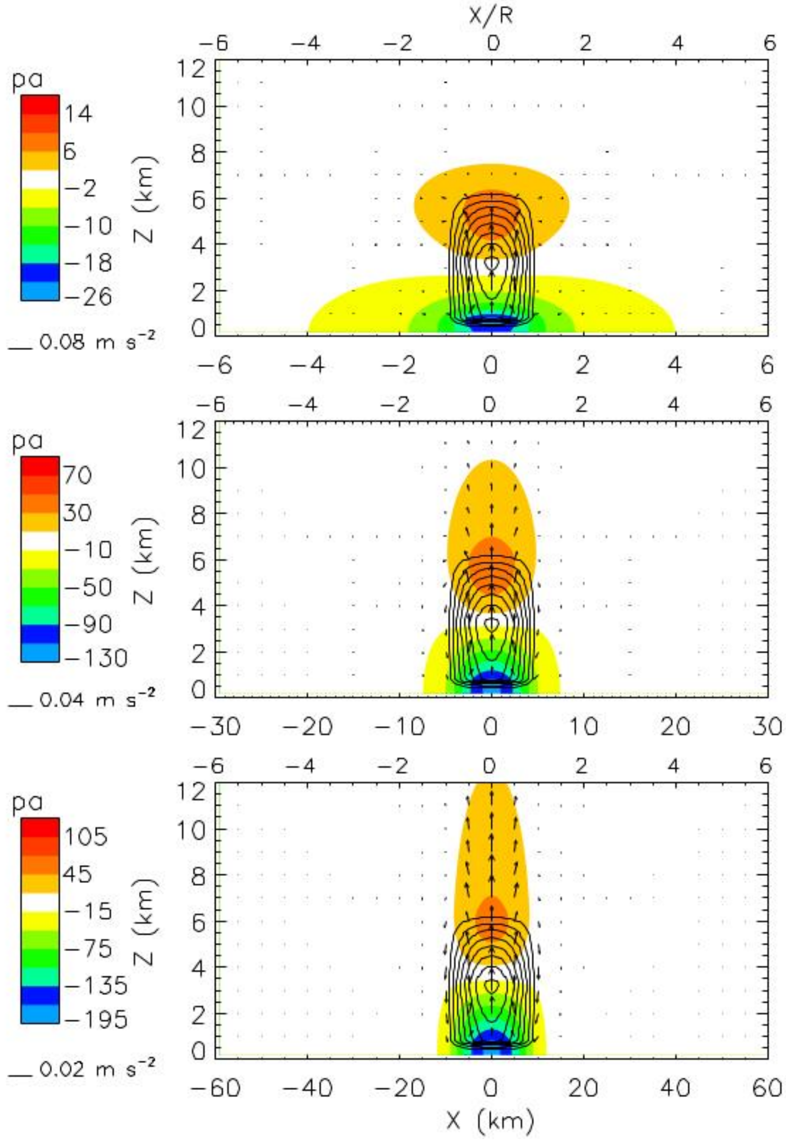
# What do we choose as the base state?

- Base state re-defined for every horizontal grid location
- This is inconsistent with the original base state definition (and the derivation of absolute buoyancy), where the base state is horizontally and temporally invariant

# Lingering questions

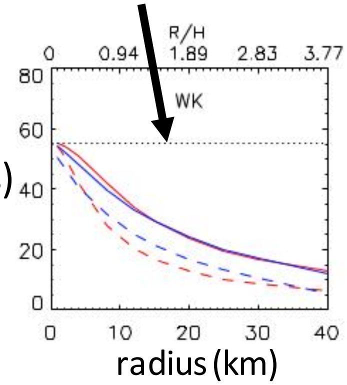
1. Can we neglect the vertical pressure gradient force?
2. Horizontal maps of CAPE are useful in forecasting and research. How is this so, if the assumptions involved in their computation are insufficiently justified?

Can we neglect the vertical pressure gradient force?




Parcel theory

w (m/s)



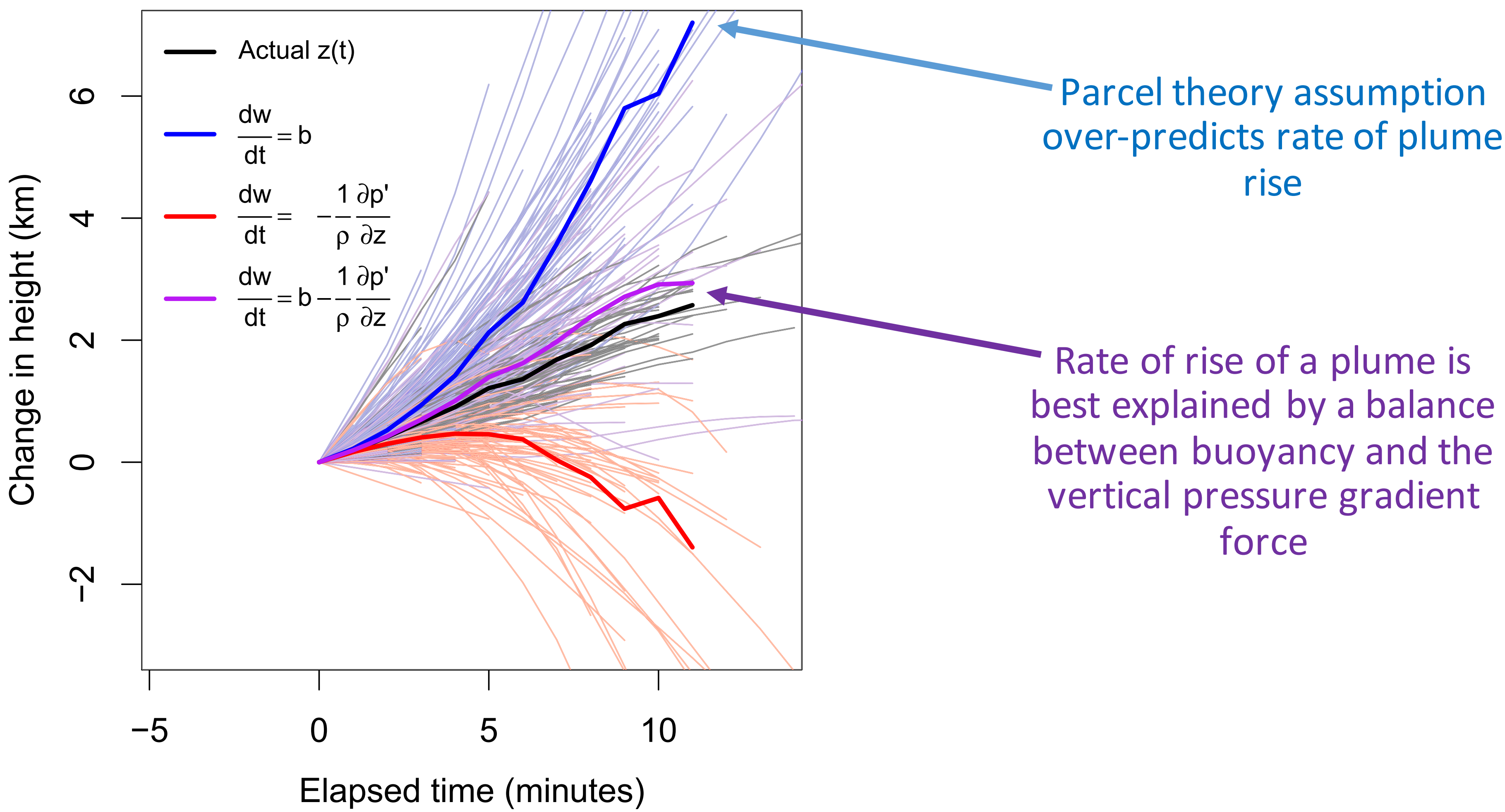

From Morrison (2015b)

Theta perturbation (black contours)

perturbation pressure field (color shading)

Downward oriented perturbation pressure force partially offsets upward oriented absolute buoyancy force

As width of updraft increases, the magnitude of “pressure offset” increases




From Romps and Charn (2015) – the “sticky hypothesis”

# Vertical Pressure gradient Force

$$\frac{dw}{dt} = -\underline{\frac{1}{\rho_0}\frac{\partial p'}{\partial z}} - g\frac{\rho'}{\rho_0}$$

How do we understand this term?

# Vertical Pressure gradient Force

$$p' = p'_{hyd} + p'_{nh}$$

$$\frac{dw}{dt} = -\frac{1}{\rho_0}\frac{\partial p'_{nh}}{\partial z} - \frac{1}{\rho_0}\frac{\partial p'_{hyd}}{\partial z} - g\frac{\rho'}{\rho_0}$$

$$\frac{\partial p'_{hyd}}{\partial z} = -g\rho'$$

$$\frac{dw}{dt} = -\frac{1}{\rho_0}\frac{\partial p'_{nh}}{\partial z} \cancel{+ \frac{1}{\rho_0} g\rho'} \cancel{- g\frac{\rho'}{\rho_0}}$$

# Vertical Pressure gradient Force

$$\frac{dw}{dt} = -\frac{1}{\rho_0}\frac{\partial p'_{nh}}{\partial z}$$

Wait… The absolute buoyancy term drops out? Seems like we *really* can't neglect the vertical pressure gradient force

# Nonhydrostatic pressure field

$$\nabla \cdot \left(\rho_0 \frac{d\mathbf{V}}{dt} = -\mathbf{V} \cdot \nabla \mathbf{V} - \frac{1}{\rho_0} \nabla p'_{hyd} - \frac{1}{\rho_0} \nabla p'_{nh} \right)$$

$$\nabla \cdot (\rho_0 \mathbf{V}) = 0$$

$$\nabla^2 p'_{nh} = -\nabla \cdot \rho_0 (\mathbf{V} \cdot \nabla \mathbf{V}) - \nabla_H^2 p'_{hyd}$$

$$\nabla^2 p'_{nh} = \underbrace{-\nabla \cdot \rho_0 (\mathbf{V} \cdot \nabla \mathbf{V})}_{\text{“dynamic” pressure field, } F_{dynam}} \underbrace{- \nabla_H^2 \int_{z*=z}^{z*=\infty} g \rho_0 \frac{\theta'}{\theta_0} dz*}_{\text{“static” pressure field}}$$

# Vertical gradient in $p_{nh}$

$$\nabla^2 \frac{\partial p'_{nh}}{\partial z} = \underbrace{-\frac{\partial}{\partial z}\left[\nabla \cdot \rho_0 (\mathbf{V} \cdot \nabla \mathbf{V})\right]}_{\text{“dynamic” pressure field}} \underbrace{- \rho_0 \nabla_H^2 \left[g \frac{\theta'}{\theta_0}\right]}_{\text{“static” pressure field}}$$

$$\frac{dw}{dt} = \frac{1}{\rho_0} L_{\nabla^2} \left\{\frac{\partial}{\partial z}\left[\nabla \cdot \rho_0 (\mathbf{V} \cdot \nabla \mathbf{V})\right]\right\} + \frac{1}{\rho_0} L_{\nabla^2} \left\{\rho_0 \nabla_H^2 \left[g \frac{\theta'}{\theta_0}\right]\right\}$$

$$\frac{dw}{dt} = \frac{1}{\rho_0} L_{\nabla^2} \xcancel{\left\{\frac{\partial}{\partial z}[\nabla\cdot\rho_0(\mathbf{V}\cdot\nabla\mathbf{V})]\right\}} + \frac{1}{\rho_0} L_{\nabla^2}\left\{\rho_0 \nabla_H^2\left[g\frac{\theta'}{\theta_0}\right]\right\}$$

"dynamic" pressure *forcing*
(ignoring for time being)

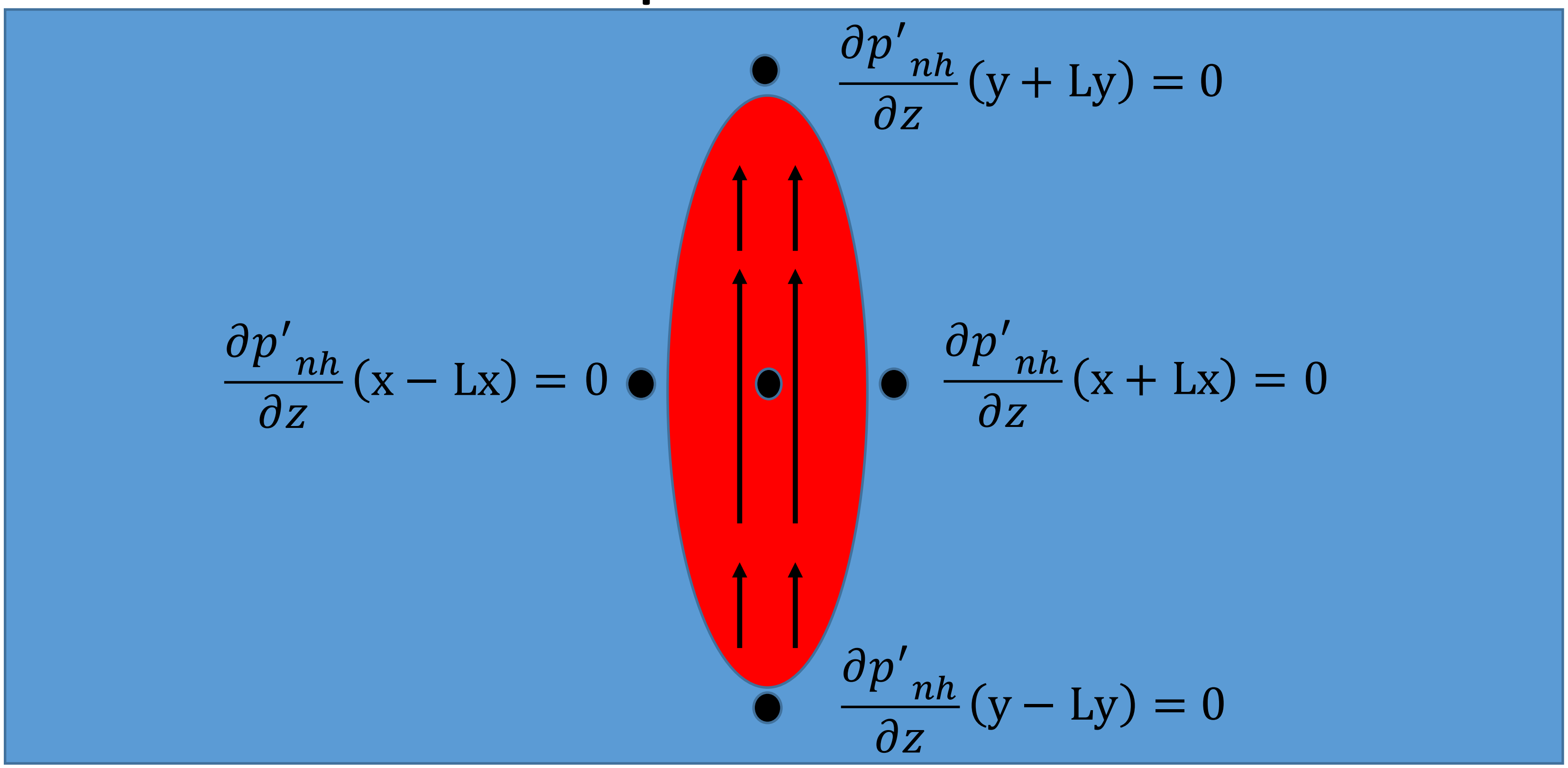
"static" pressure
*forcing*

$$\frac{dw}{dt} \approx \frac{1}{\rho_0} L_{\nabla^2}\left\{\rho_0 \nabla_H^2\left[g\frac{\theta'}{\theta_0}\right]\right\}$$

This term looks
"buoyancy-like"

We'll call the RHS the "relative buoyancy force." It isn't quite clear that it's embodies Archimedes' principle yet…

# Finite difference grid centered on 2-D updraft



$$\frac{dw}{dt} \approx \frac{1}{\rho_0} L_{\nabla^2} \left\{ \rho_0 \nabla_H^2 \left[ g \frac{\theta'}{\theta_0} \right] \right\}$$

$$\Downarrow$$

$$\frac{dw}{dt} \approx g \frac{1}{1 + \left(\frac{L_x}{L_z}\right)^2} \frac{\delta\theta'}{\theta_0}$$

$$\delta\theta' \equiv \theta'(x, z, t) - \hat{\theta}'(x, z, t)$$

$$\hat{\theta}'(x, z, t) \equiv \frac{1}{2} [\theta'(x + L_x/2, z, t) + \theta'(x - L_x/2, z, t)]$$

# Simplified expression

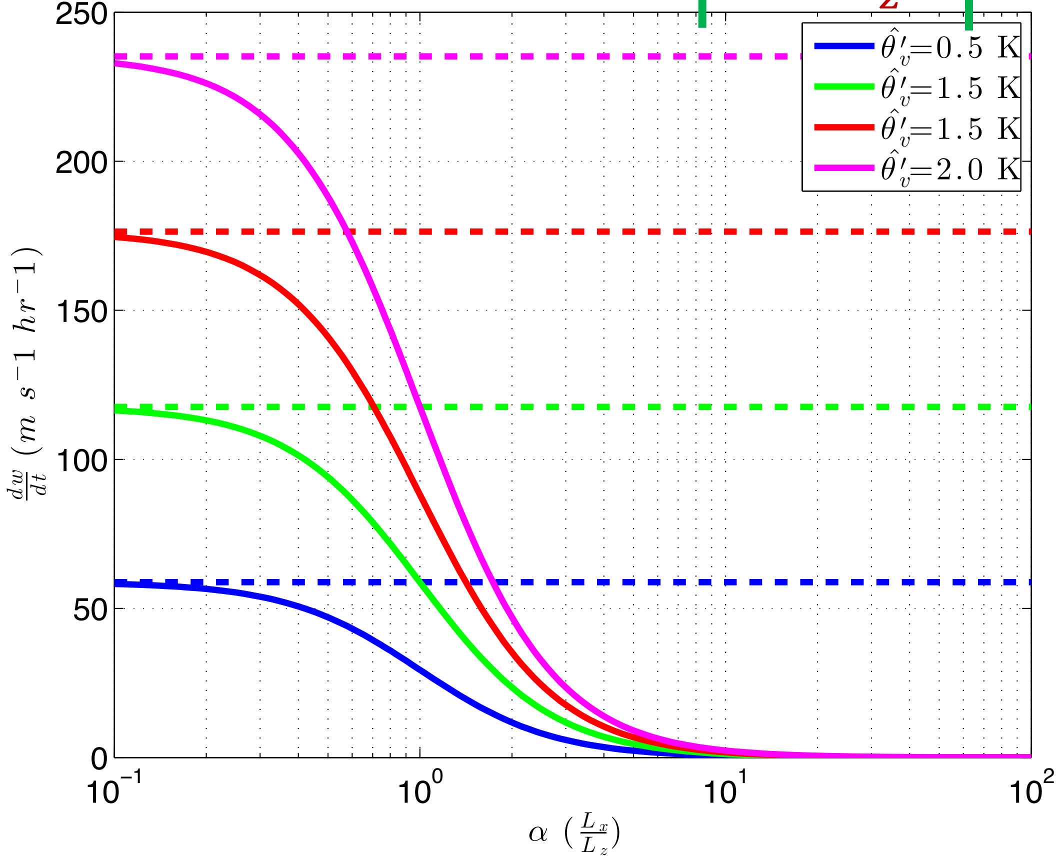
$$\frac{dw}{dt} \approx g \frac{1}{1+\left(\frac{L_x}{L_z}\right)^2} \frac{\delta\theta'}{\theta_0}$$

We'll call this a "relative temperature difference term"

We'll call this a "scaling term" (e.g. Weisman et al. 1997; Morrison 2015a).

$$\frac{dw}{dt} \approx g \frac{1}{1+\left(\frac{L_x}{L_z}\right)^2} \frac{\delta\theta'}{\theta_0}$$

Scaling term




Acceleration at center of updraft falls off quickly with $L_x > L_z$

Wider updrafts are weaker than narrower updrafts with the same absolute buoyancy. In other words, two updrafts with different shapes can have the same absolute buoyancy, but *different relative buoyancy*

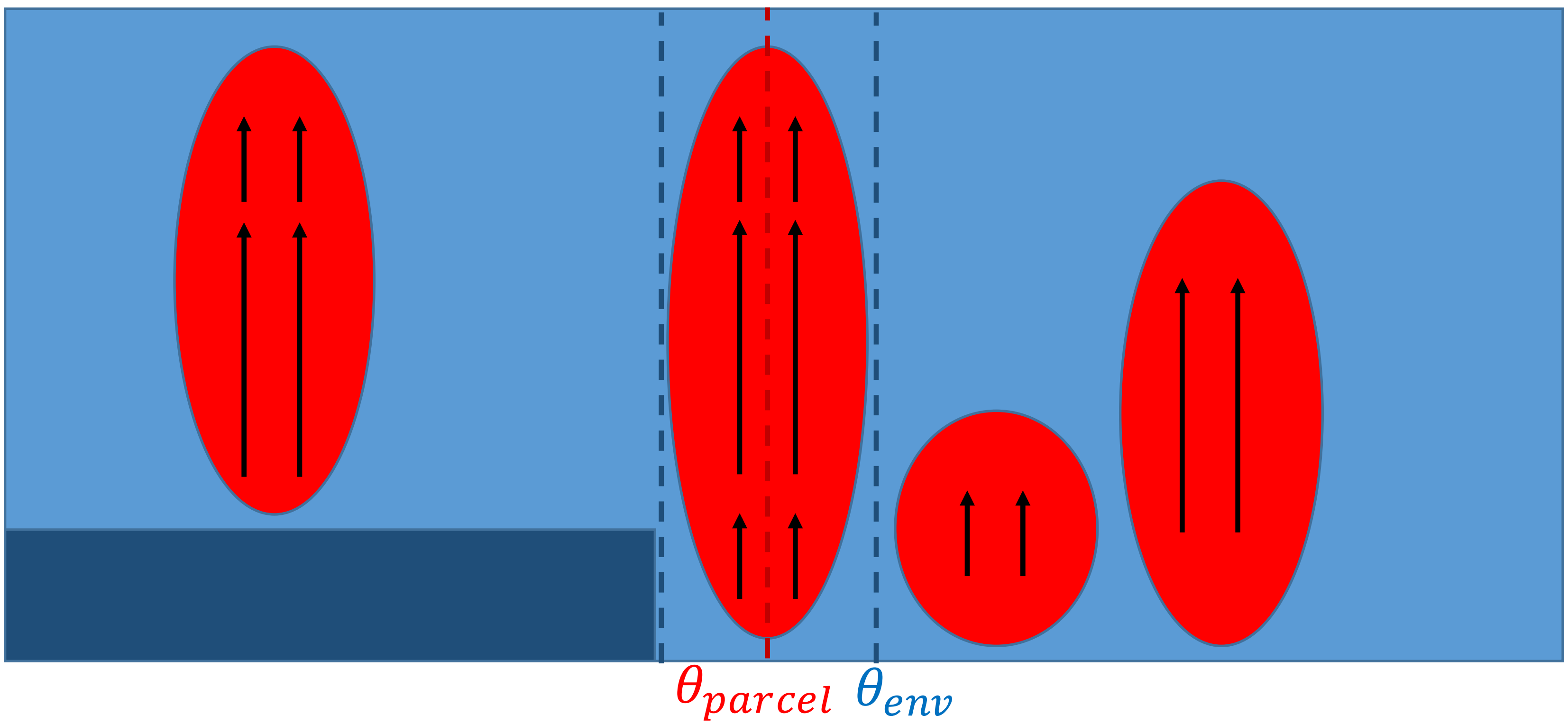
See Morrison 2015a,b

$$\frac{dw}{dt} \approx g \frac{1}{1 + \left(\frac{L_x}{L_z}\right)^2} \frac{\delta\theta'}{\theta_0}$$

relative
temperature
difference term

$\delta\theta'$ is different than $\theta'$ in that it is the temperature difference between the center of an updraft and its immediate surroundings, rather than an arbitrary base state

# We now have an unambiguous "background environment"



$$\mathrm{w}(z) = \sqrt{g \frac{2}{1+\left(\frac{L_x}{L_z}\right)^2} \int_{z*=0}^{z*=z} \frac{\delta\theta'}{\theta_0} dz*}$$

If we assume that the scaling term applies evenly through the depth of an updraft, we can estimate the profile of vertical velocity through the depth of an updraft

$$CIN_{rel} = g \frac{1}{1+\left(\frac{L_x}{L_z}\right)^2} \int_{z*=z}^{z*=z_{LFC}} \frac{\delta\theta'}{\theta_0} dz*$$

$$CAPE_{rel} = g \frac{1}{1+\left(\frac{L_x}{L_z}\right)^2} \int_{z*=z_{LFC}}^{z*=z_{EL}} \frac{\delta\theta'}{\theta_0} dz*$$

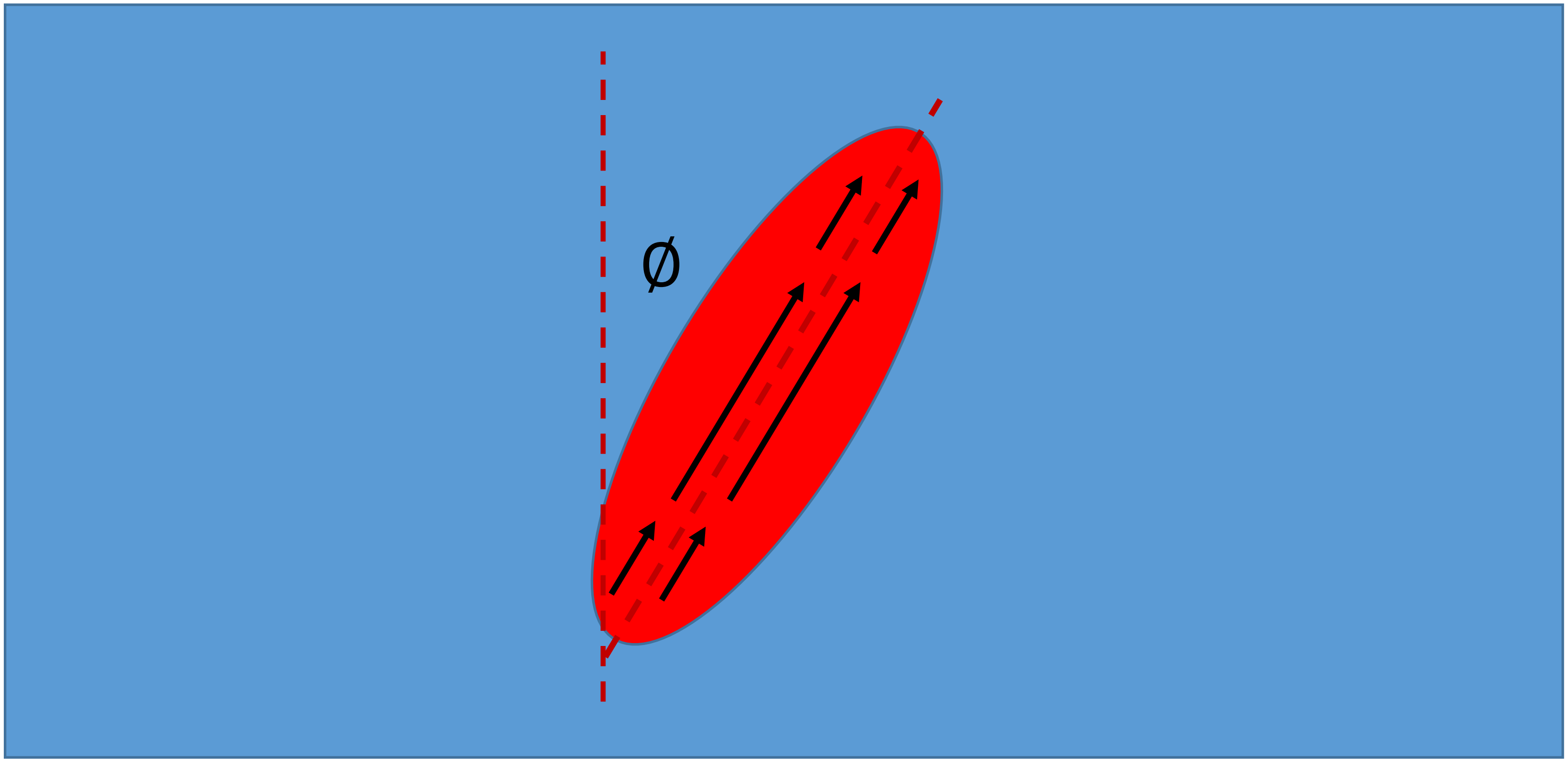
We can also derive new formulas for CIN and CAPE

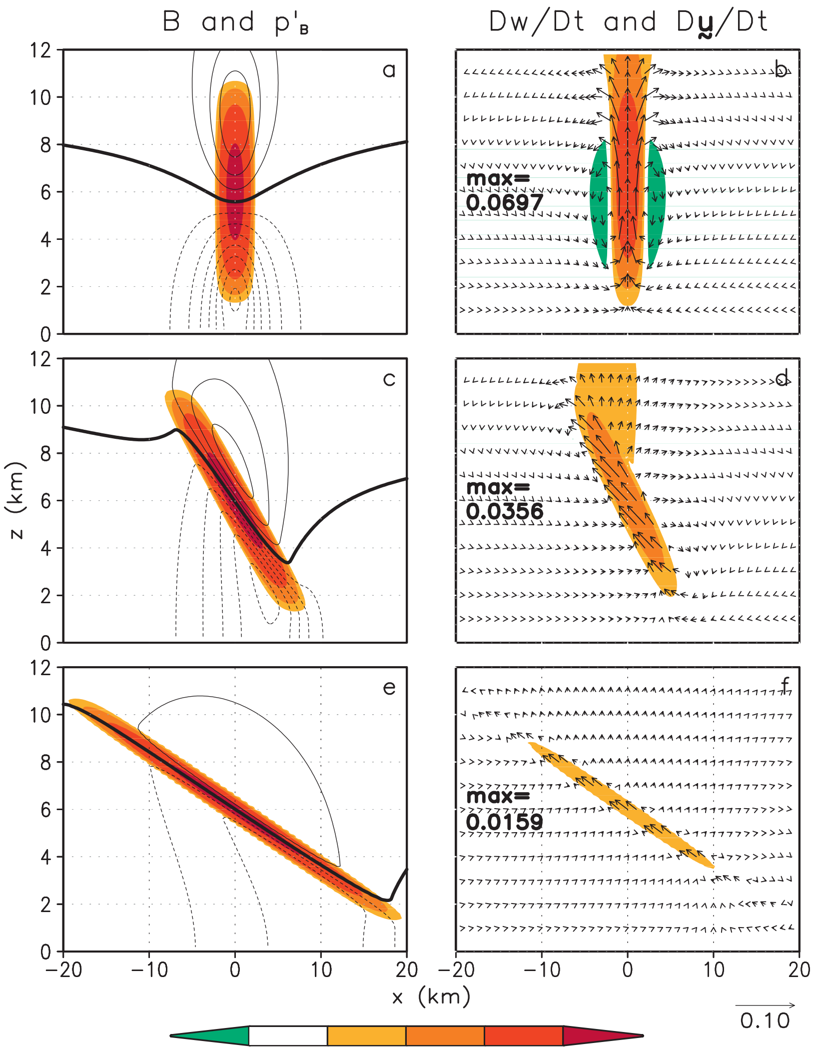
We will further discuss the potential usefulness of the expressions for w, CIN, and CAPE a bit later

What happens when an updraft becomes slanted?

(most updrafts in the mid latitudes are somewhat slanted, due to vertical wind shear)

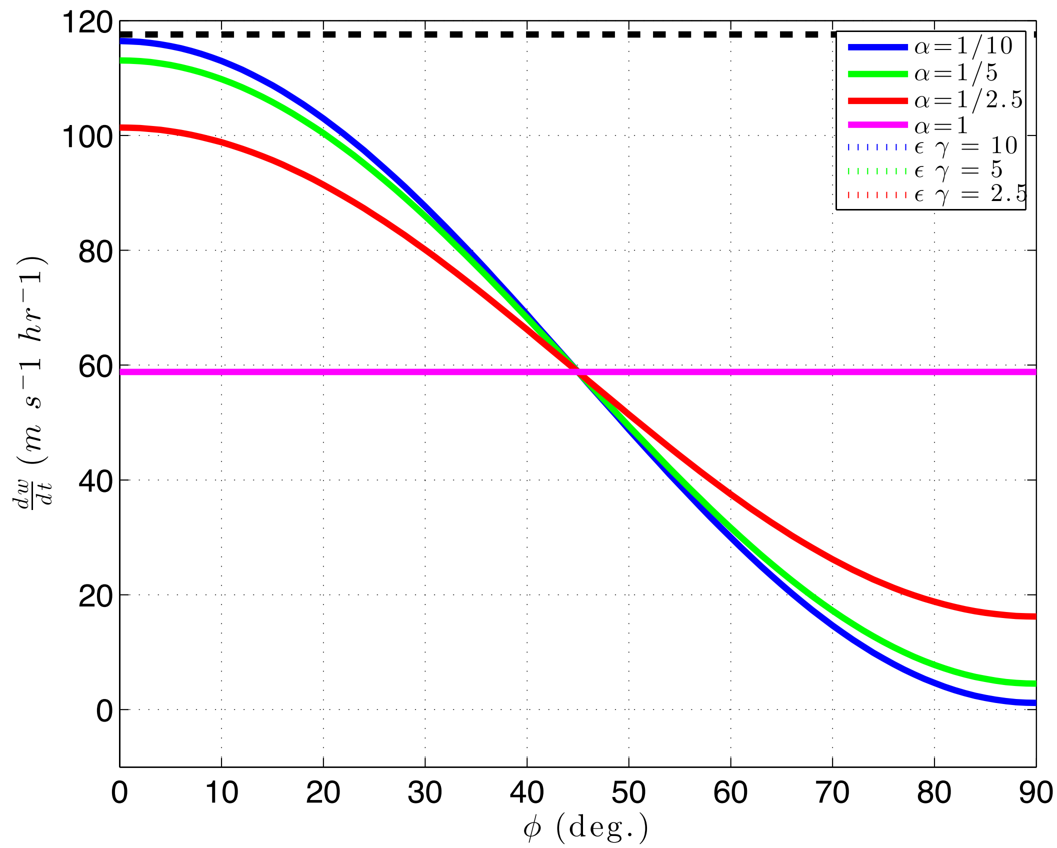
# Slanted updraft






From Parker (2010)

$$\frac{dw}{dt} \approx g\left[\frac{\cos^2\emptyset}{1+\left(\frac{L_x^*}{L_z^*}\right)^2}+\frac{\sin^2\emptyset}{1+\left(\frac{L_z^*}{L_x^*}\right)^2}\right]\frac{\delta\theta'}{\theta_0}$$



Answer to our question: We can’t neglect the pressure gradient force for a comprehensive understanding of convective updrafts

Updraft strength (e.g. max dw/dt, w) depends on the updraft's:

Aspect ratio: taller skinnier updrafts are stronger (neglecting entrainment)

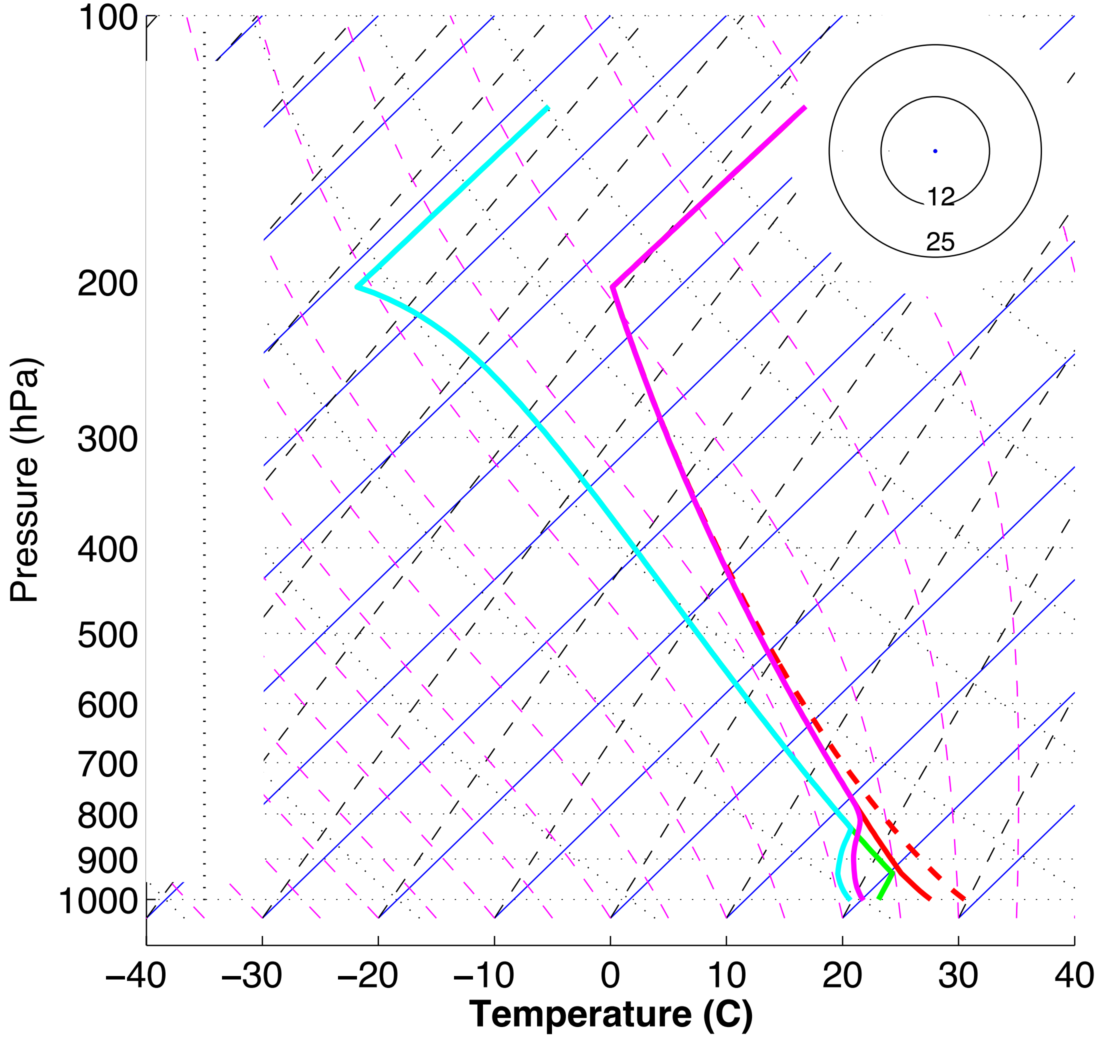
Slant: more slant, weaker updraft

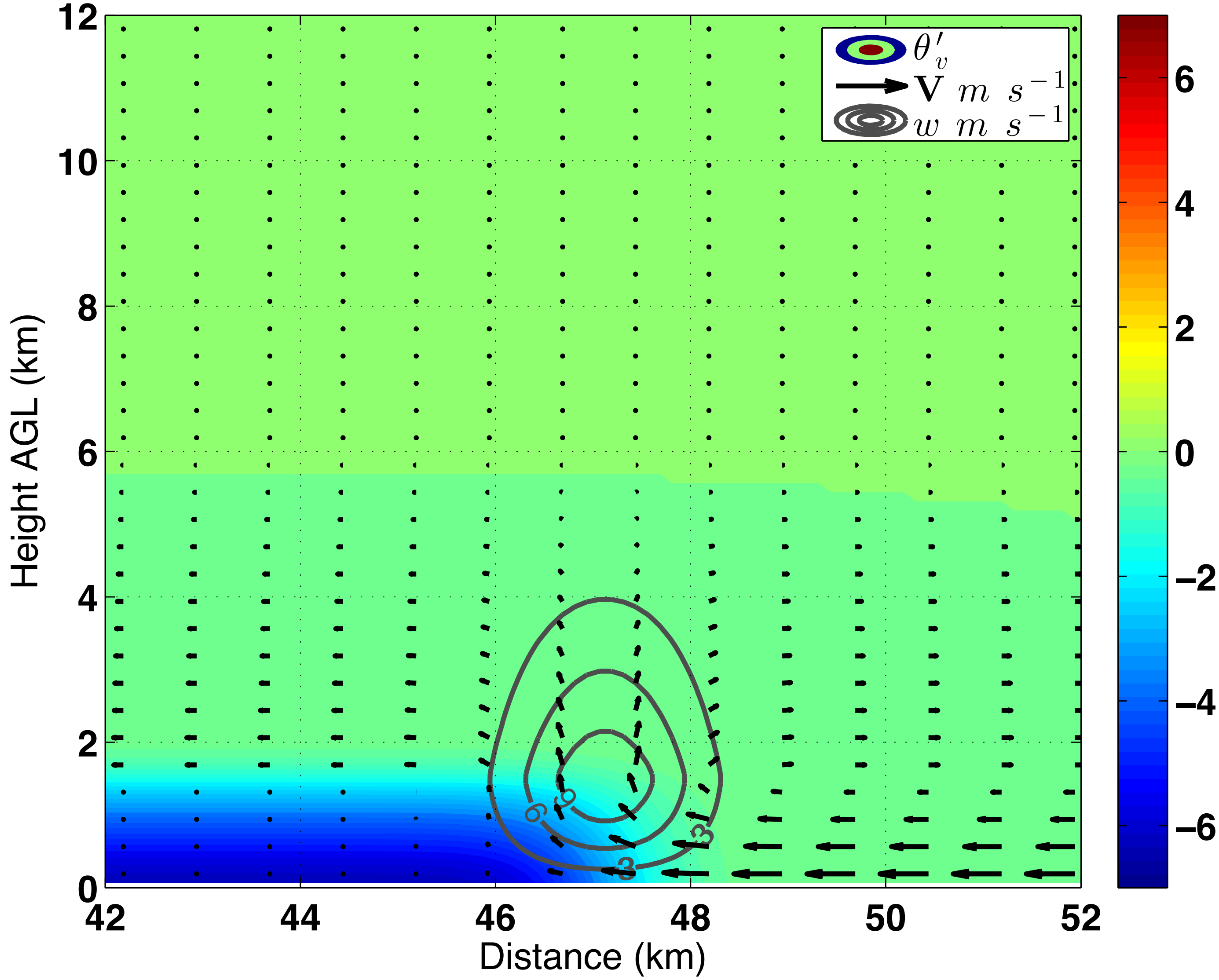
Surroundings. Put cold (warm) air next to an updraft, you make it more (less) buoyant.

These effects are neglected by parcel theory/absolute buoyancy frameworks

Now lets compare the theoretical expressions based on relative buoyancy to some characteristics of simulated updrafts

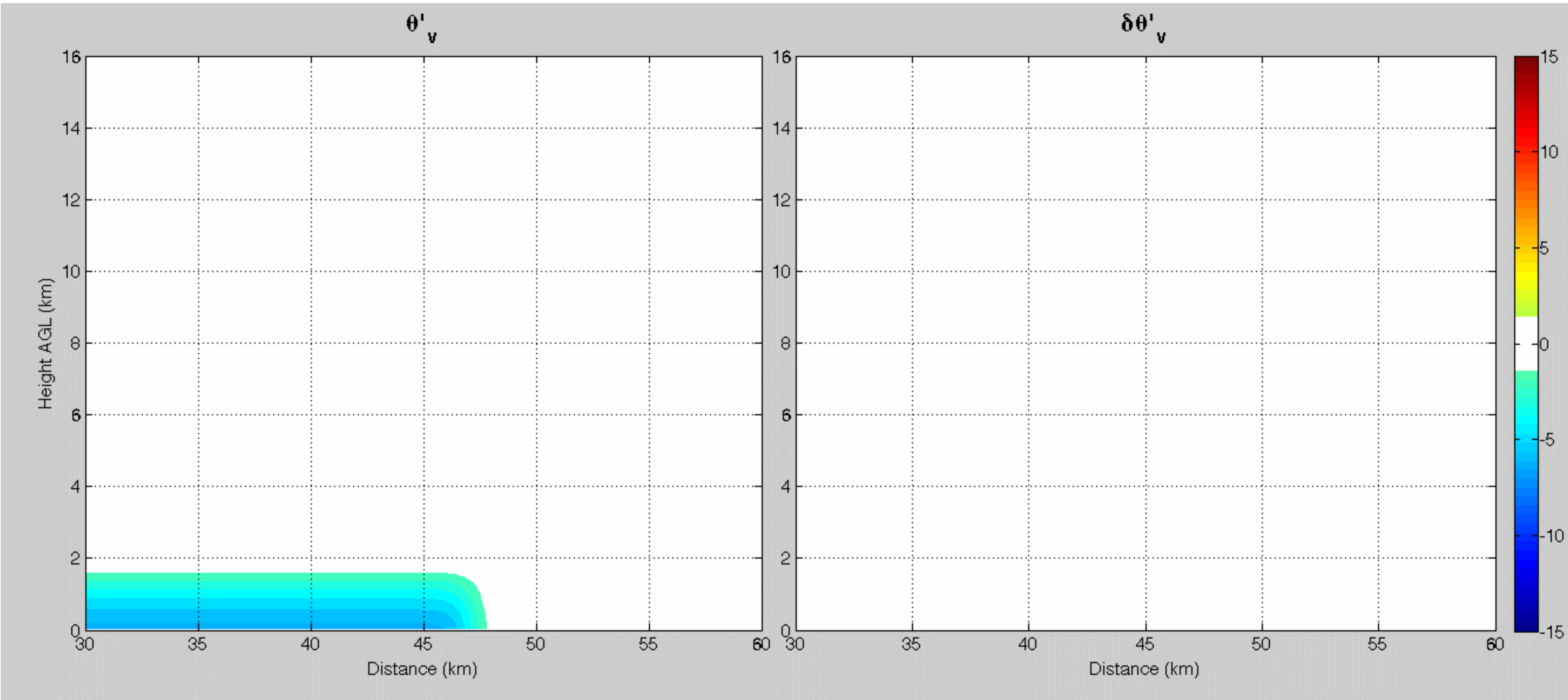
- CM 1 simulation
- 2-D
- 125 m grid spacing
- Initialized with low-level cold pool
- Wind profile outside of cold pool favorable for lifting along cold pool edge
- 18.5 m/s low-level shear, c=26 m/s
- WK88 sounding


100
200
300
400
500
600
700
800
900
1000
Pressure (hPa)
−40
−30
−20
−10
0
10
20
30
40
Temperature (C)
12
25

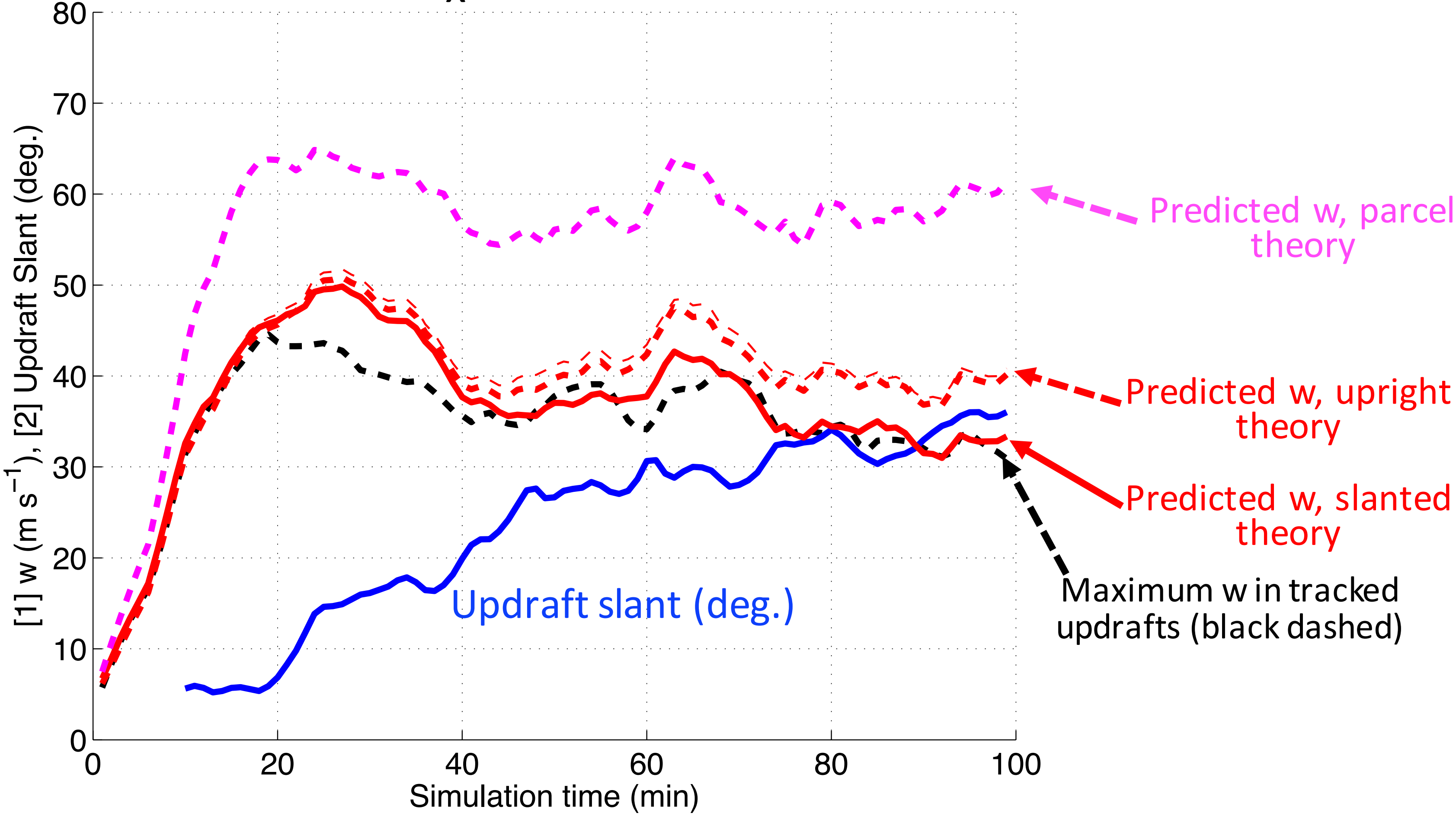


$\theta'_v$
$\mathbf{V}\ m\ s^{-1}$
$w\ m\ s^{-1}$
Height AGL (km)
Distance (km)
12
10
8
6
4
2
0
42
44
46
48
50
52
6
4
2
0
−2
−4
−6
3
6
9

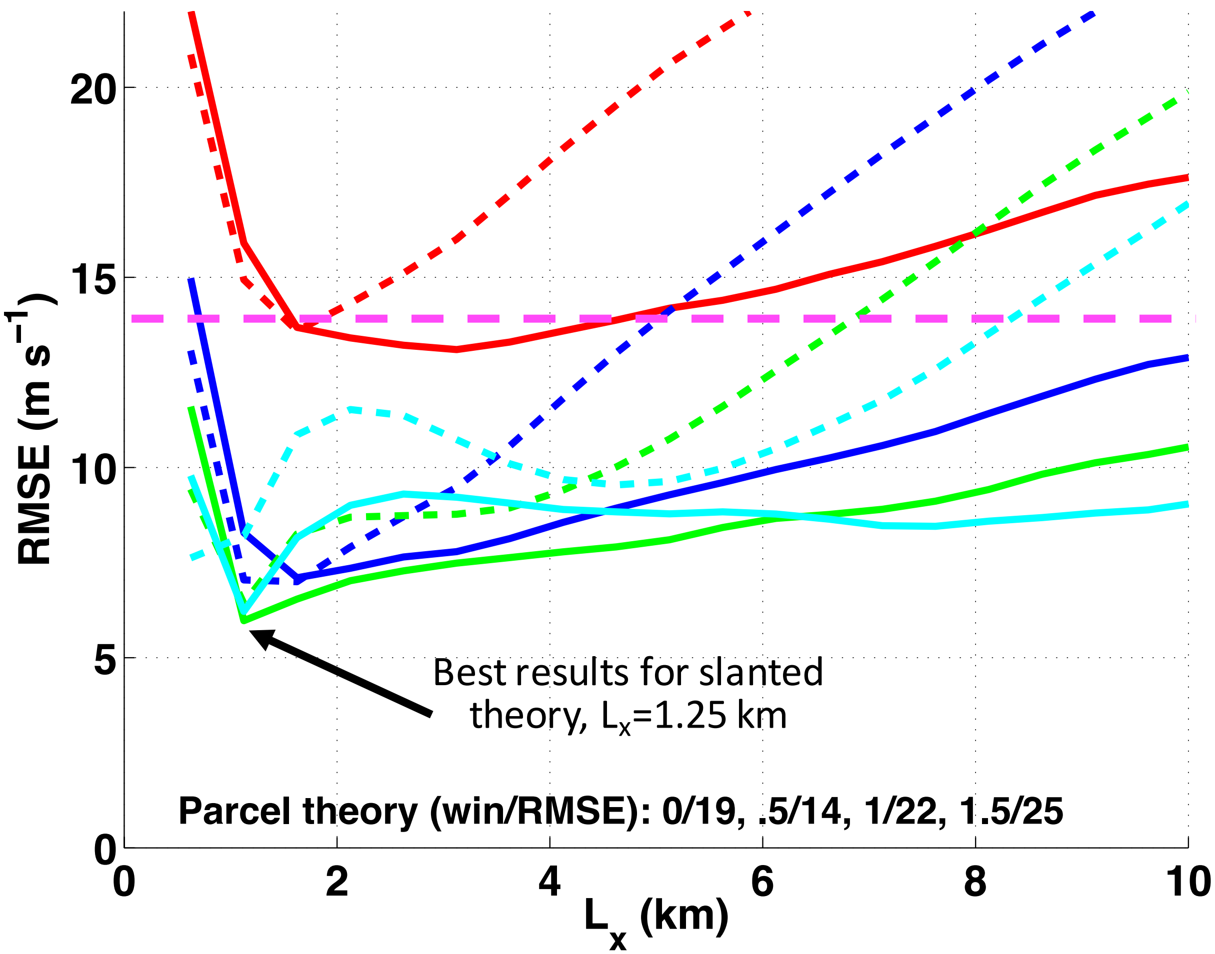
# HISHR CM1 simulation



# $L_x$ assumed to be 1.25 km



Pay attention to blue and green lines




Theory based on relative buoyancy performs better than parcel theory for wide range of assumed $L_x$ values

# Returning to our new CAPE and CIN expressions

$$CIN_{rel} = g\frac{1}{1+\left(\frac{L_x}{L_z}\right)^2}\int_{z*=z}^{z*=z_{LFC}}\frac{\delta\theta'}{\theta_0}dz*$$

$$CAPE_{rel} = g\frac{1}{1+\left(\frac{L_x}{L_z}\right)^2}\int_{z*=z_{LFC}}^{z*=z_{EL}}\frac{\delta\theta'}{\theta_0}dz*$$

Lx=1.25 and Lz=6 are reasonable parameter values from the CM1 simulation. This yields a scaling value of .96… so the scaling parameter minimally affects CAPE/CIN values.

$$CIN_{rel} = g\int_{z*=z}^{z*=z_{LFC}} \frac{\delta\theta'}{\theta_0} dz *$$

$$CAPE_{rel} = g\int_{z*=z_{LFC}}^{z*=z_{EL}} \frac{\delta\theta'}{\theta_0} dz *$$

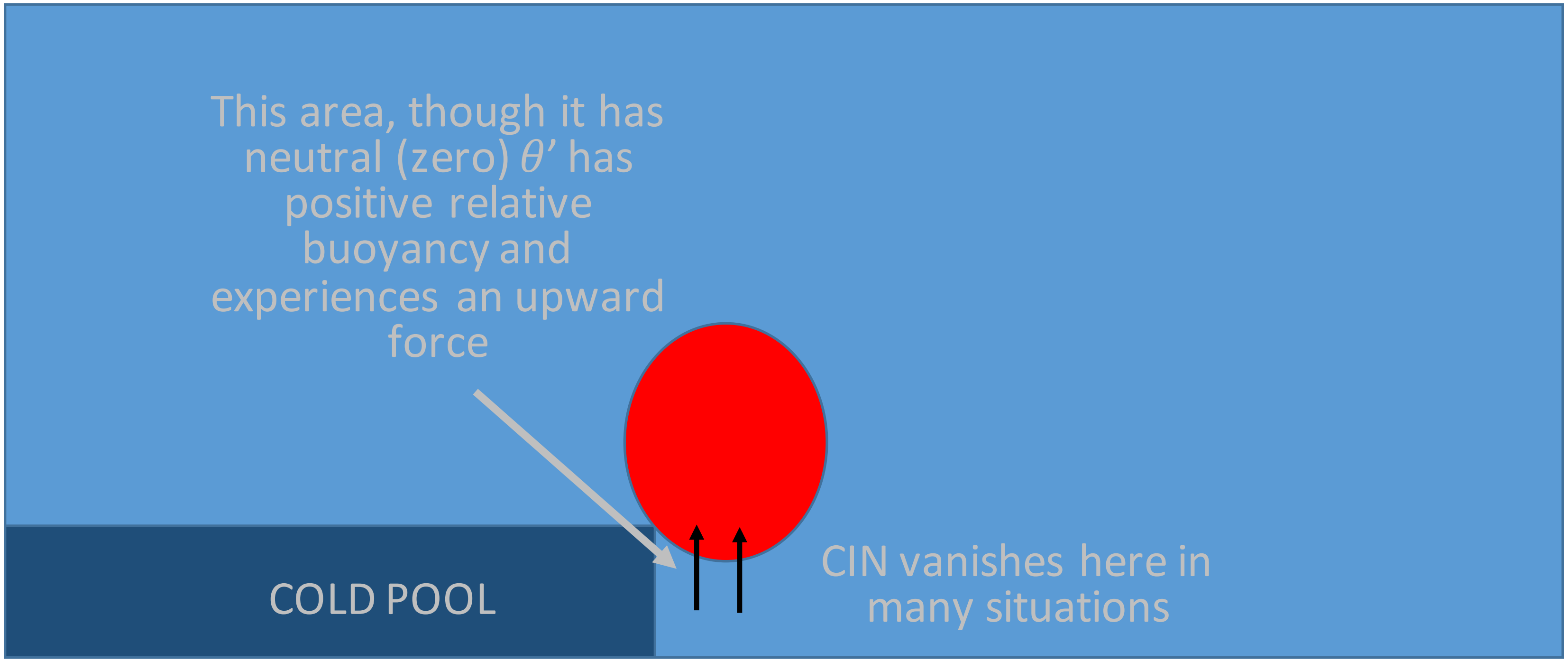
So how do these expressions differ from traditional CAPE and CIN?

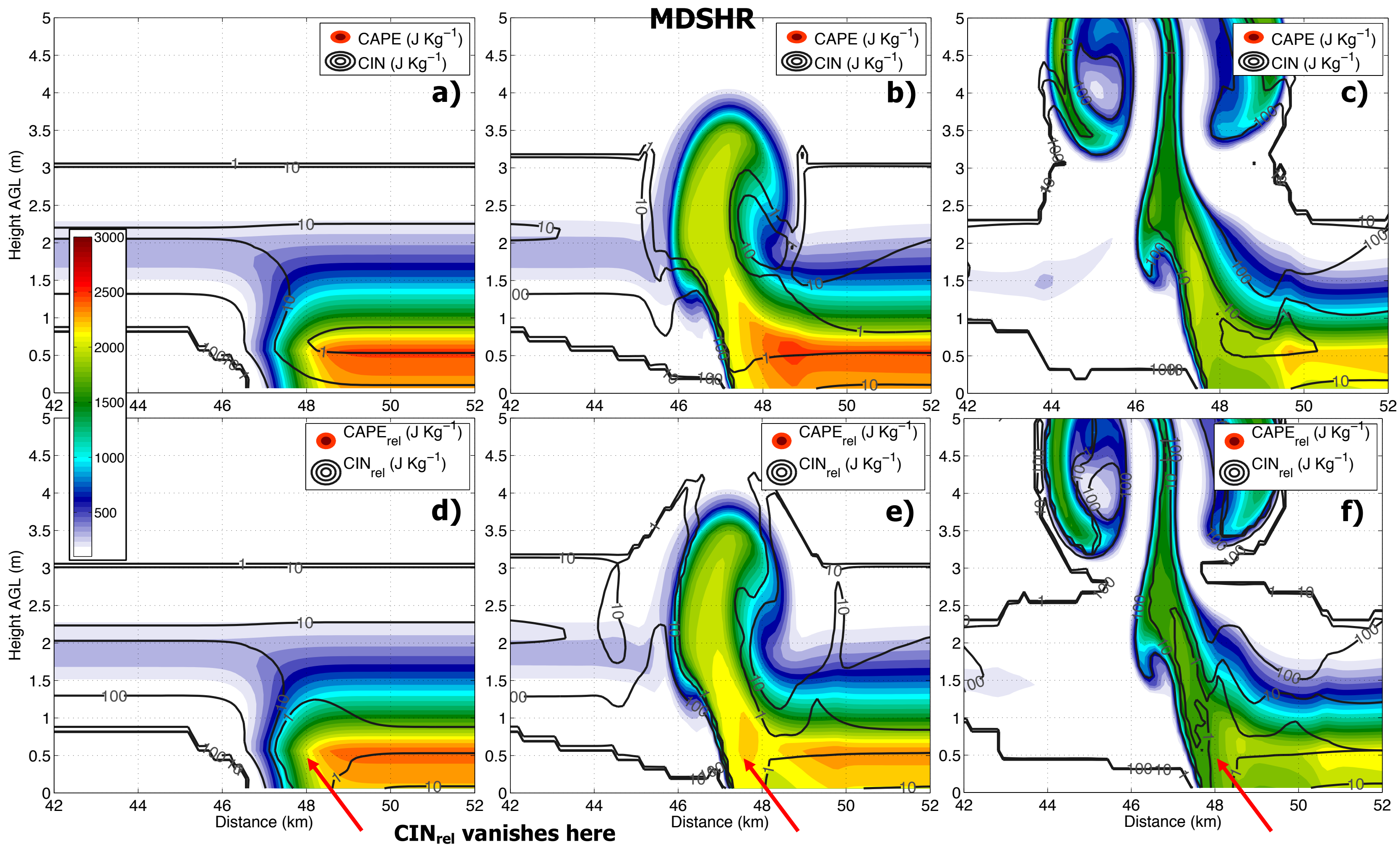
$$\delta\theta' \equiv \theta'(x,z,t) - \hat{\theta}'(x,z,t)$$

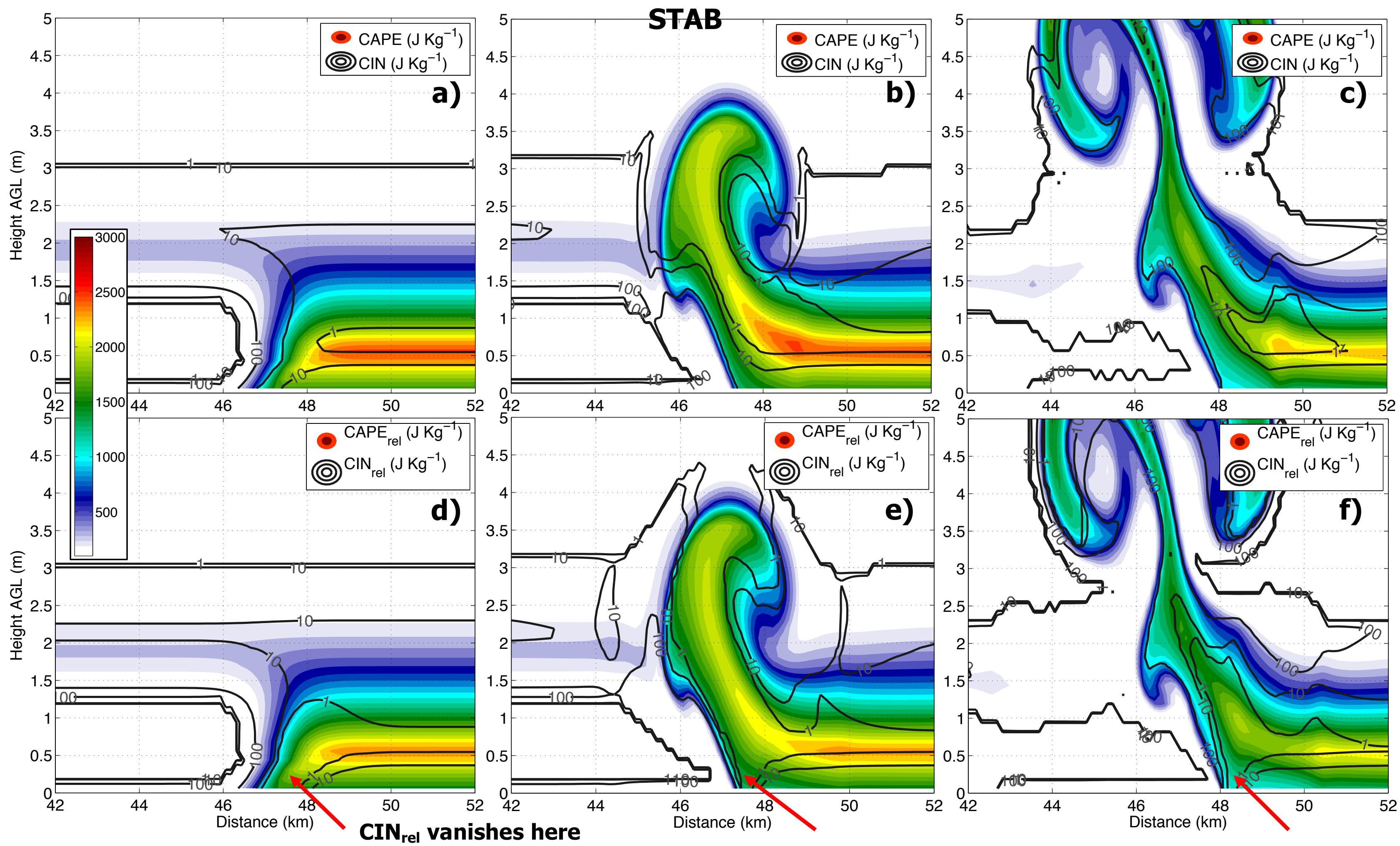
$$\hat{\theta}'(x,z,t) \equiv \frac{1}{2}[\theta'(x + L_x/2, z, t) + \theta'(x - L_x/2, z, t)]$$

If you put anomalously cool or warm air next to an air parcel, you change $\hat{\theta}'$, which in turn affects the parcel's relative buoyancy, and it's $CIN_{rel}$ and $CAPE_{rel}$

# Application: Lifting along outflow boundaries




MDSHR
CAPE (J Kg$^{-1}$)
CIN (J Kg$^{-1}$)
a)
b)
c)
CAPE$_{rel}$ (J Kg$^{-1}$)
CIN$_{rel}$ (J Kg$^{-1}$)
d)
e)
f)
Height AGL (m)
Distance (km)
CIN$_{rel}$ vanishes here



STAB
CAPE (J Kg$^{-1}$)
CIN (J Kg$^{-1}$)
CAPE$_{rel}$ (J Kg$^{-1}$)
CIN$_{rel}$ (J Kg$^{-1}$)
a)
b)
c)
d)
e)
f)
Height AGL (m)
Distance (km)
CIN$_{rel}$ vanishes here

In the mid-latitudes, CIN often increases for parcels near the surface at night

It has been argued that “mechanical lifting” (e.g. dynamic pressure force) is needed to overcome this CIN and lift these parcels into updrafts

I argue here that as the flow approaches a cold pool, CIN vanishes in many instances. There any “resistance” to upward motion.

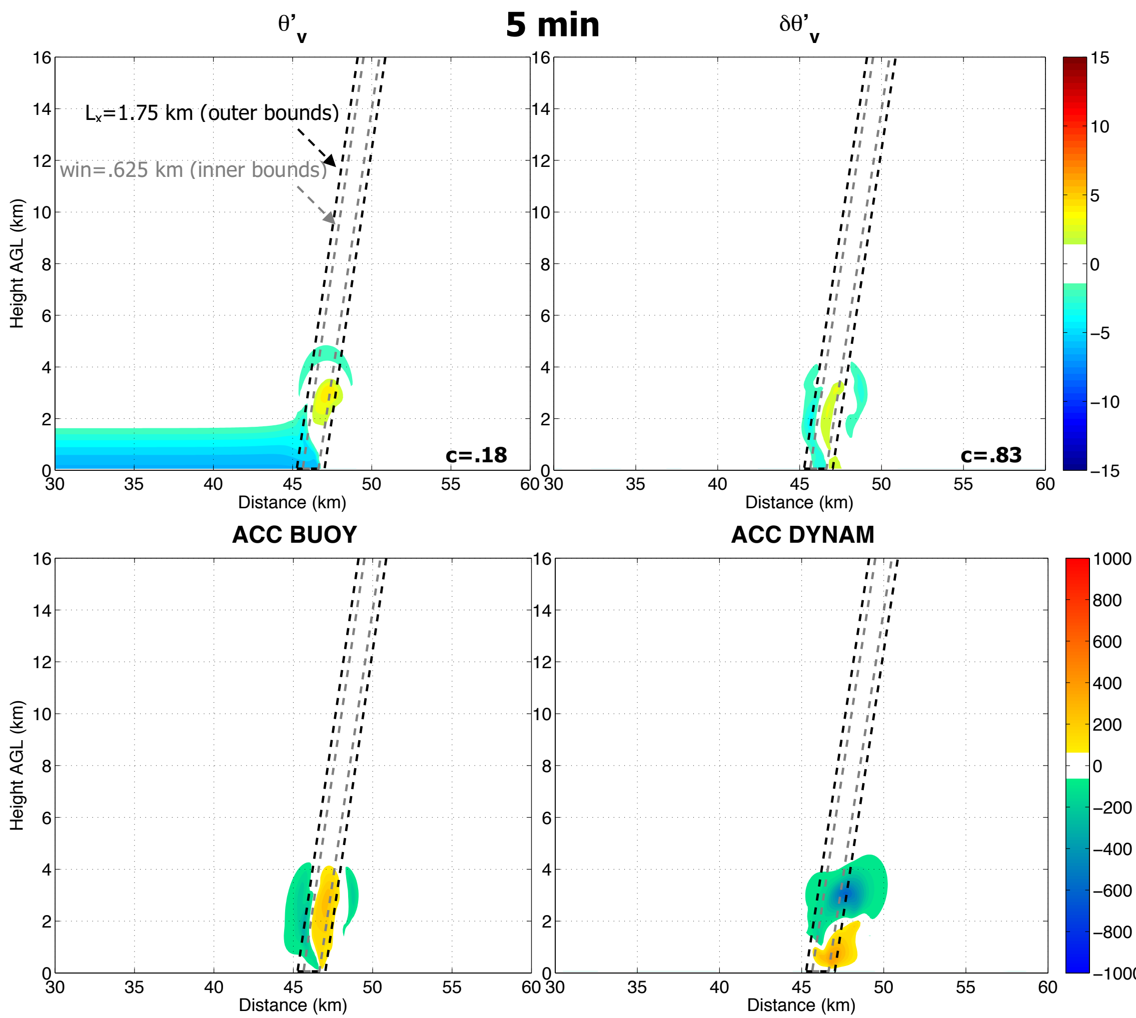
# Overall Conclusions

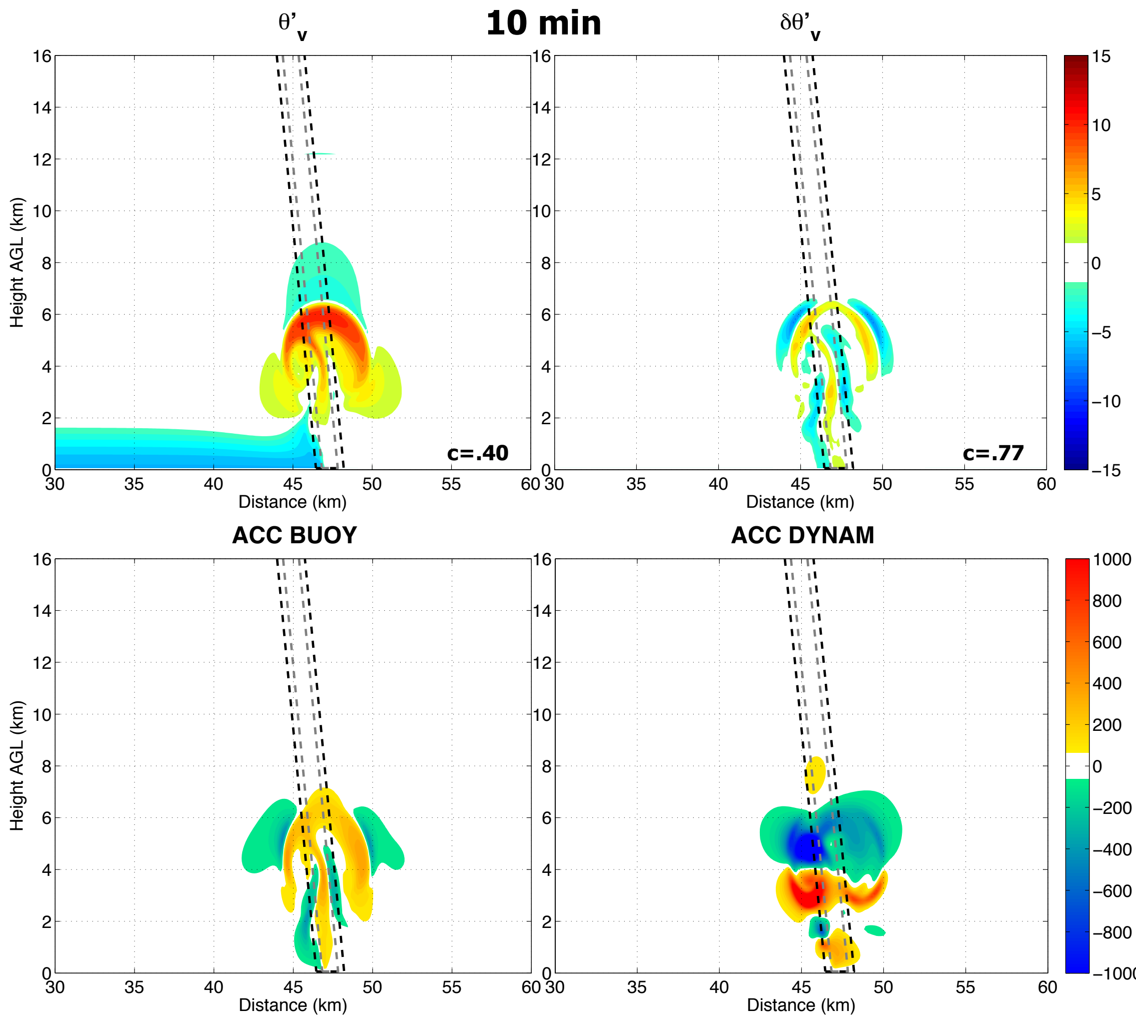
For a complete conceptual picture of updraft dynamics, theory must be based on relative buoyancy

Relative buoyancy (which includes pressure effects) is probably a better interpretation of Archimedes' theory, in that buoyancy is a response of parcels to local properties of a fluid, rather than properties of an arbitrary base state

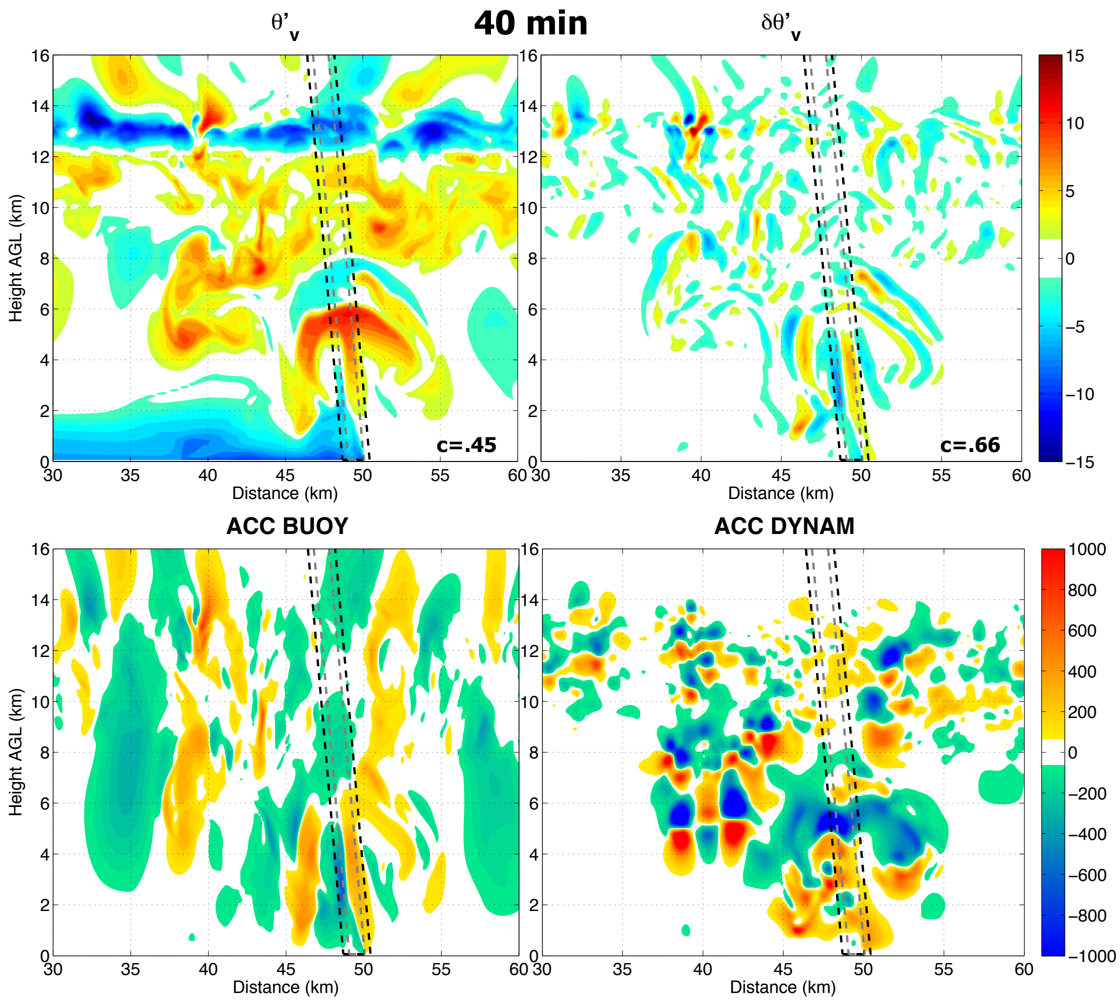
$CIN_{rel}$ may be useful in storm scale processes analysis, in that it is a more correct representation of a parcel's inhibition to upward motion that traditional CIN

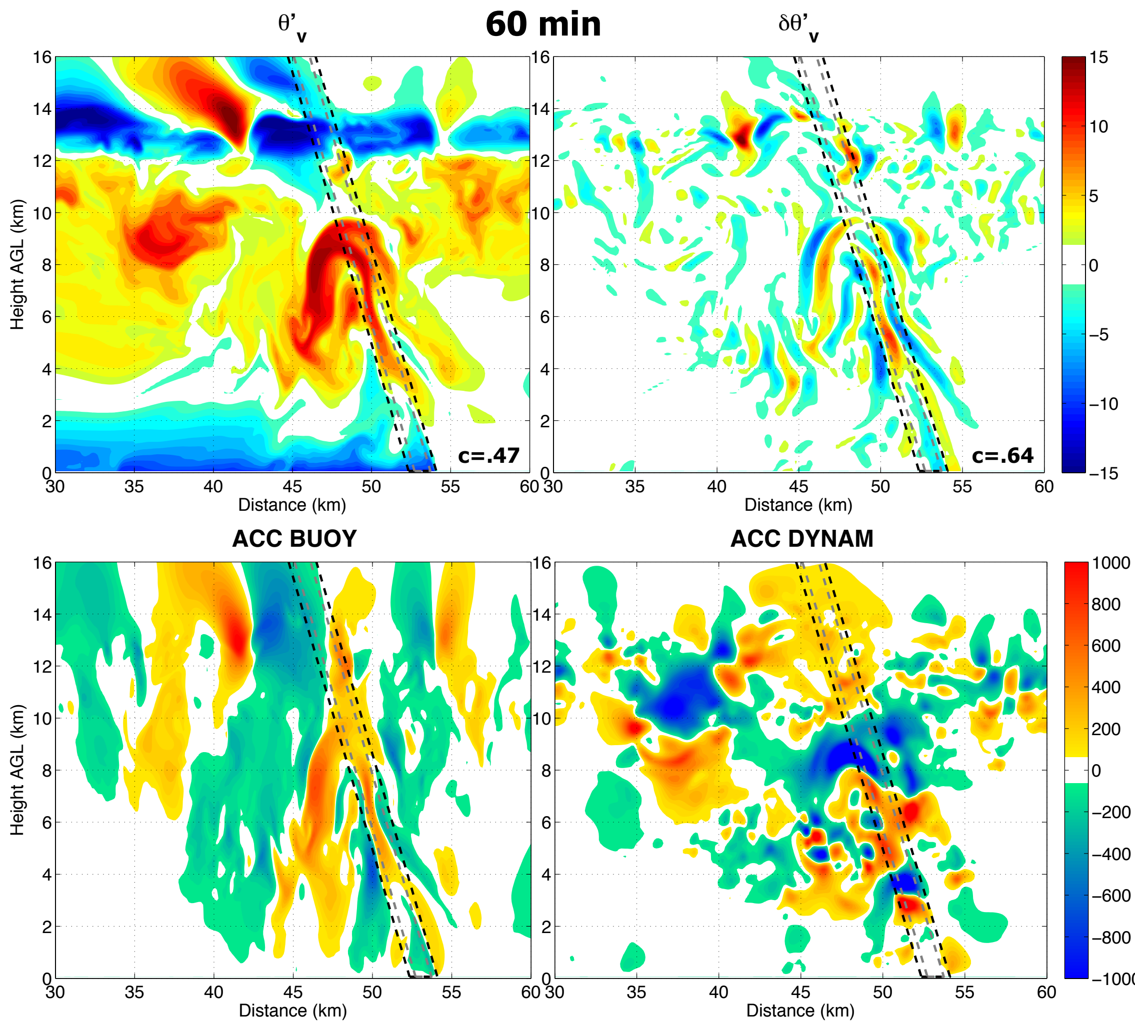
$$\frac{dw}{dt} = \frac{1}{\rho_0} L_{\nabla^2} \left\{ \underbrace{\frac{\partial}{\partial z} [\nabla \cdot \rho_0 (\mathbf{V} \cdot \nabla \mathbf{V})]}_{\text{ACC DYNAM}} \right\} + \frac{1}{\rho_0} L_{\nabla^2} \left\{ \underbrace{\rho_0 \nabla_H^2 \left[ g \frac{\theta'}{\theta_0} \right.}_{\text{ACC BUOY}} \left. \vphantom{\frac{\theta'}{\theta_0}} \right] \right\}$$


5 min
θ'v
δθ'v
Lx=1.75 km (outer bounds)
win=.625 km (inner bounds)
c=.18
c=.83
ACC BUOY
ACC DYNAM
Height AGL (km)
Distance (km)



10 min
θ'v
δθ'v
c=.40
c=.77
ACC BUOY
ACC DYNAM
Height AGL (km)
Distance (km)


40 min
θ'v
δθ'v
c=.45
c=.66
ACC BUOY
ACC DYNAM
Height AGL (km)
Distance (km)


60 min
θ'v
δθ'v
c=.47
c=.64
ACC BUOY
ACC DYNAM
Height AGL (km)
Distance (km)