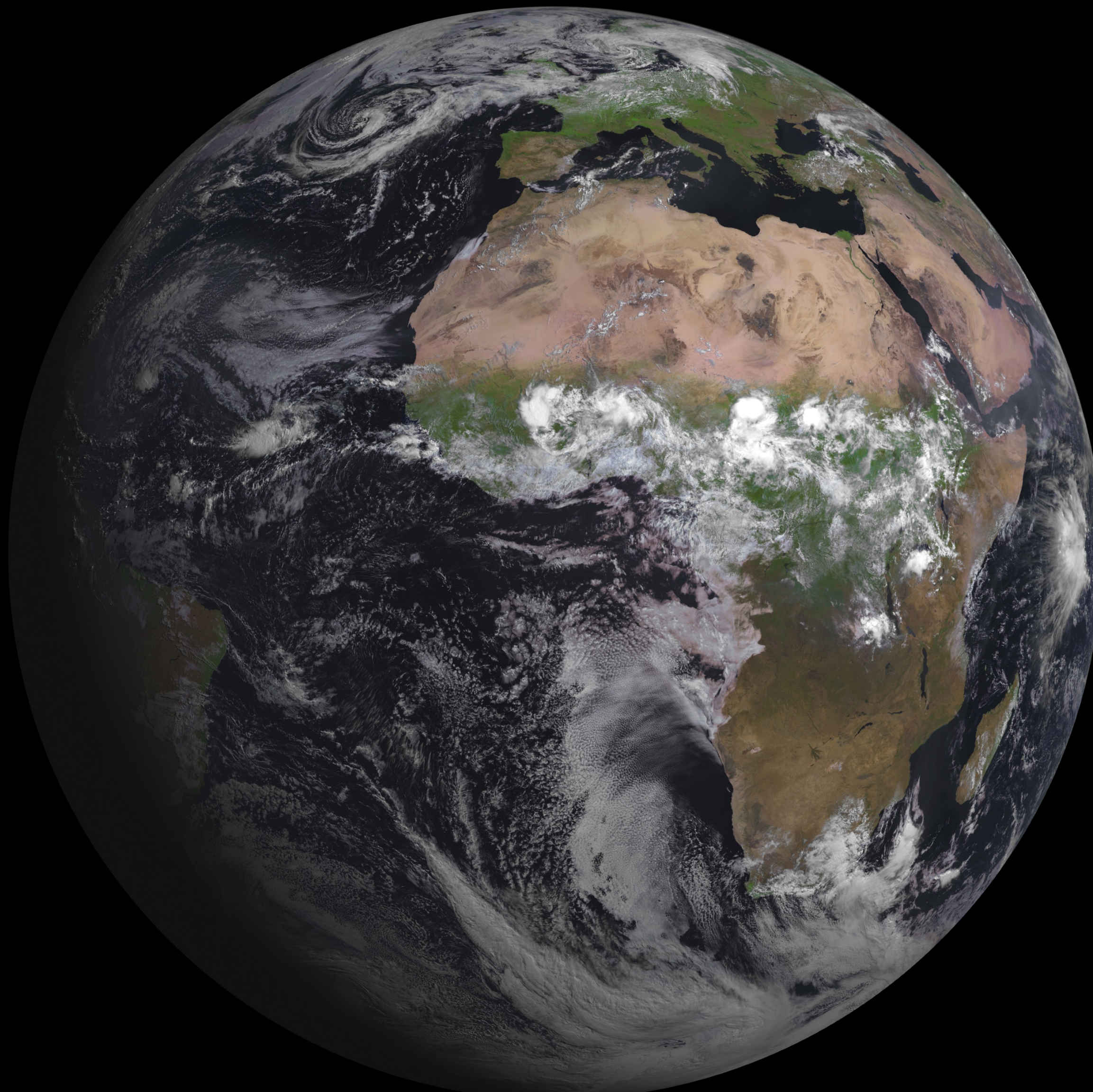
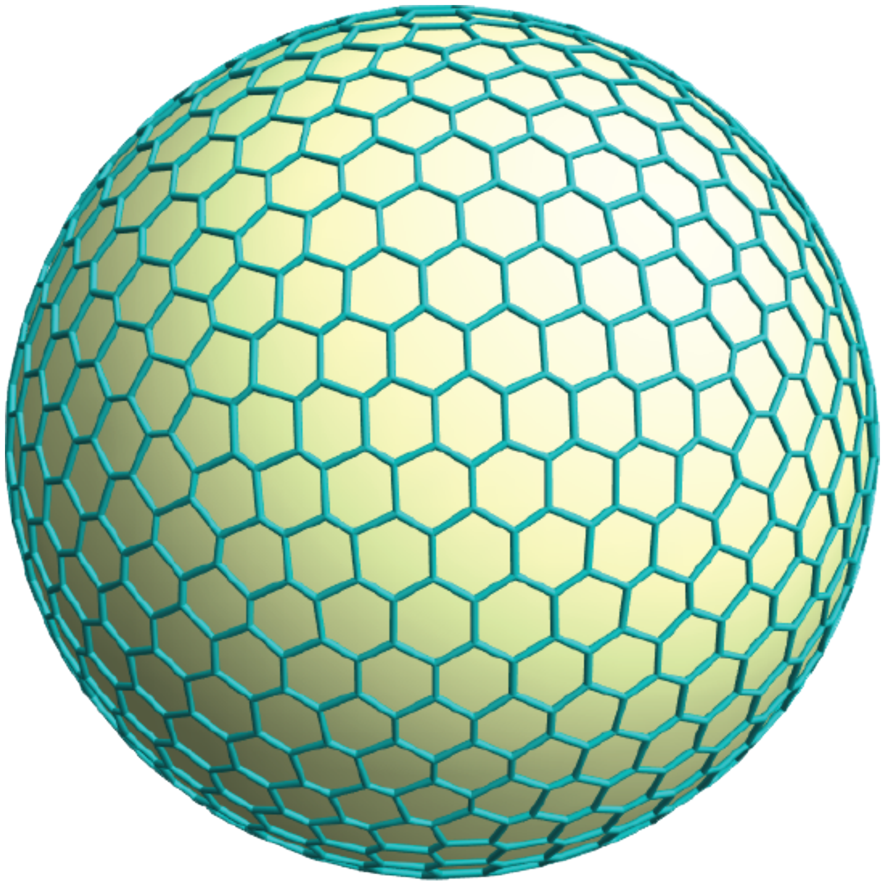
The Last Dynamical Core



David Randall, Celal Konor, Ross Heikes, and Donald Dazlich

Icosahedral (hexagonal-pentagonal) grid



Grid	No. of grid points N	Avg grid distance ℓ (km)
G0	12	6699.1
G1	42	3709.8
G2	162	1908.8
G3	642	961.4
G4	2562	481.6
G5	10 242	240.9
G6	40 962	120.4
G7	163 842	60.2
G8	655 362	30.1
G9	2 621 442	15.0
G10	10 485 762	7.53
G11	41 943 042	3.76
G12	167 772 162	1.88
G13	671 088 642	0.94

Non-hydrostatic regime

We built a non-hydrostatic version of our geodesic model by using the Unified System of equations, which gives accurate solutions for a wide range of scales, and filters vertically propagating sound waves.

The Unified System

Fully compressible system

$$p \equiv p_{qs} + \delta p \qquad \rho \equiv \rho_{qs} + \delta\rho$$
$$\pi \equiv \pi_{qs} + \delta\pi \qquad T \equiv T_{qs} + \delta T$$

Horizontal momentum equation:

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k} \times \mathbf{v} = -c_p \theta \nabla_H \left(\pi_{qs} + \delta\pi\right) + \mathbf{F}_\mathbf{v}$$

Vertical momentum equation:

$$\frac{Dw}{Dt} = -c_p \theta \frac{\partial \delta\pi}{\partial z} + F_w$$

Thermodynamic equation:

$$\frac{D\theta}{Dt} = \frac{Q}{c_p \pi}$$

Continuity equation:

$$\frac{\partial\left(\rho_{qs} + \delta\rho\right)}{\partial t} = -\nabla_H \cdot \left[\left(\rho_{qs} + \delta\rho\right)\mathbf{v}\right] - \frac{\partial\left[\left(\rho_{qs} + \delta\rho\right)w\right]}{\partial z}$$

Unified System

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Thermodynamic equation:

$$\frac{D\theta}{Dt} = \frac{Q}{c_p \pi}$$

Continuity equation:

$$\frac{\partial \rho_{qs}}{\partial t} = -\nabla_H \cdot \left(\rho_{qs}\mathbf{v}\right) - \frac{\partial\left(\rho_{qs} w\right)}{\partial z}$$

$\rho_{qs} \gg \delta\rho$ *is assumed*

Like the anelastic system, the Unified System includes a 3D elliptic equation, e.g., for the pressure.

Strengths & Weaknesses of the Unified System

Strengths:

- Filters vertically propagating sound waves
- Does not need a basic or reference or mean state
- Is as accurate as the fully compressible system for non-acoustic modes
- Is easy to implement into an existing quasi-static model
- Can easily be "switched" to the quasi-static system
- Conserves energy

Weaknesses:

- Requires solution of a three-dimensional elliptic system

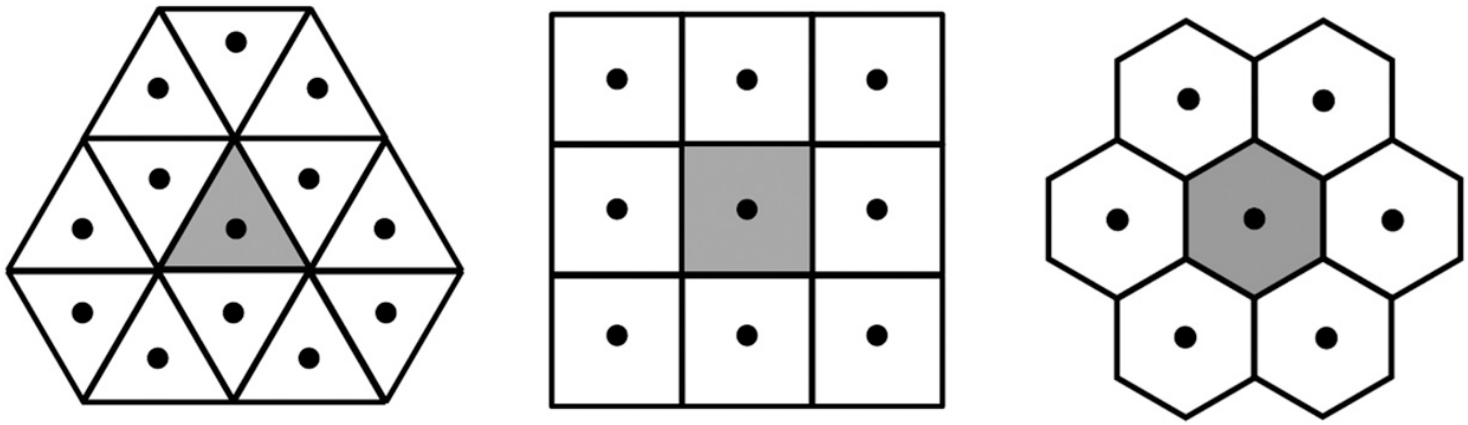
Predicting the winds with the Unified System

With the Unified System, there are only two prognostic degrees of freedom in the wind field.

For example, if the horizontal wind vector is predicted, then the vertical velocity can be diagnosed from the continuity equation.

We built a Unified System model that predicts the curl and divergence of the horizontal wind vector, following the "Z grid" approach.

Too many sheep, not enough dogs



Grid	Triangles	Squares	Hexagons
A	2	2	2
B	1	2	4
C	3/2	2	3
D	3/2	2	3
E	Does not exist	2	Does not exist
Z	2	2	2

The right answer is 2.

Table 8.3: The number of prognostic degrees of freedom in the horizontal wind field, per mass point, on grids A-E and Z, and for triangular, square, and hexagonal meshes. For the Z-grid, the vorticity and divergence carry the information about the wind field.

Z Grid

Predict the curl and divergence of the horizontal wind vector.

$$\omega_z \equiv \mathbf{k} \cdot \nabla_{\mathrm{H}} \times \mathbf{v}$$

$$D \equiv \nabla_{\mathrm{H}} \cdot \mathbf{v}$$

Diagnose the winds

$$\nabla^2_{\mathrm{H}} \psi = \omega_z$$

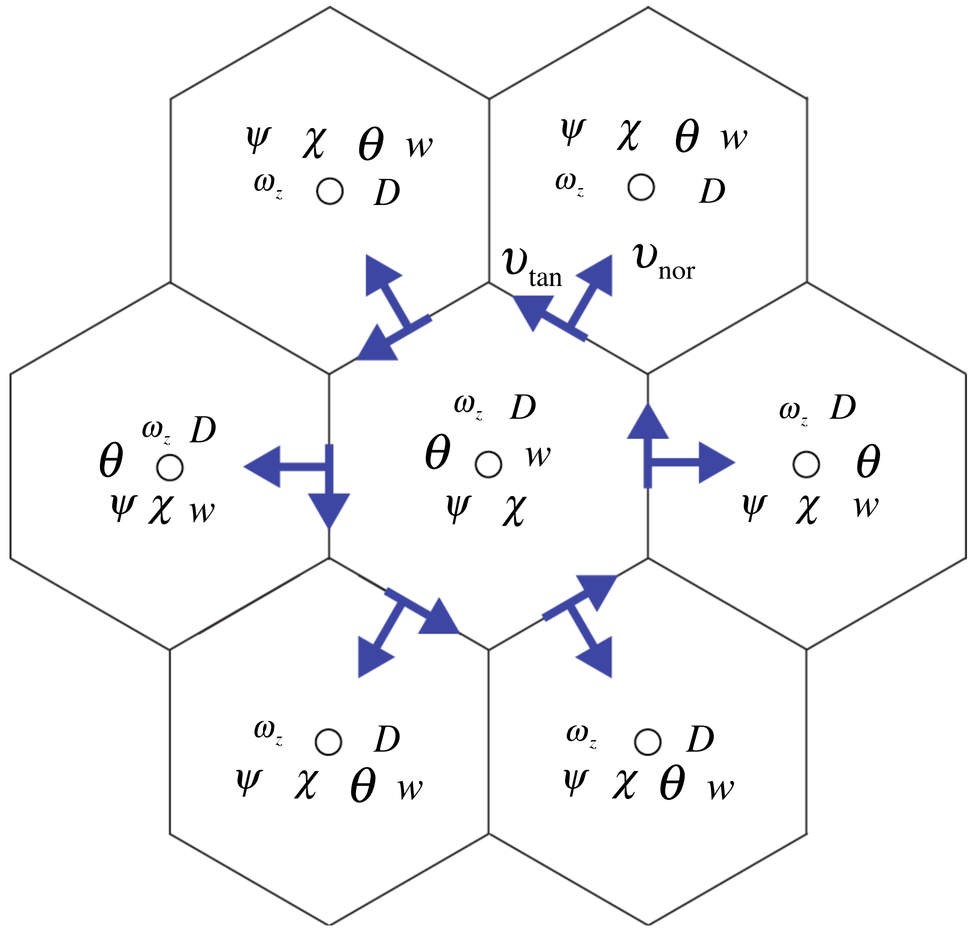
$$\nabla^2_{\mathrm{H}} \chi = D$$

Elliptic equations

$$\mathbf{v}_\psi \equiv \mathbf{k} \times \nabla_{\mathrm{H}} \psi$$

$$\mathbf{v}_\chi \equiv \nabla_{\mathrm{H}} \chi$$

$$\mathbf{v} \equiv \mathbf{v}_\psi + \mathbf{v}_\chi$$



The Z grid has no computational modes.

Strengths & Weaknesses of the Z grid

Strengths:

- No computational modes
- Excellent dispersion properties for inertia-gravity waves
- Direct prediction of the vertical component of the vorticity
- Predicts pseudo-scalars, rather than vectors

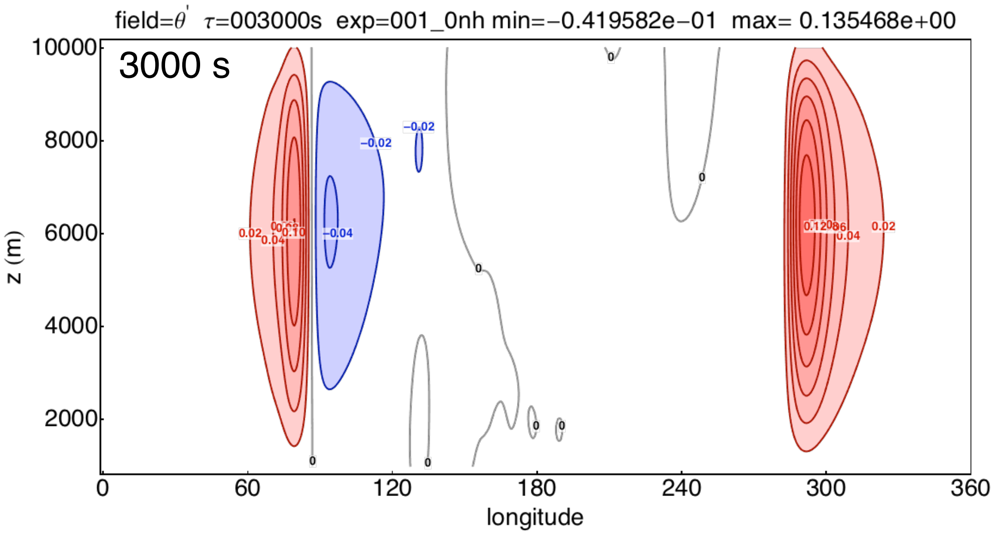
Weaknesses:

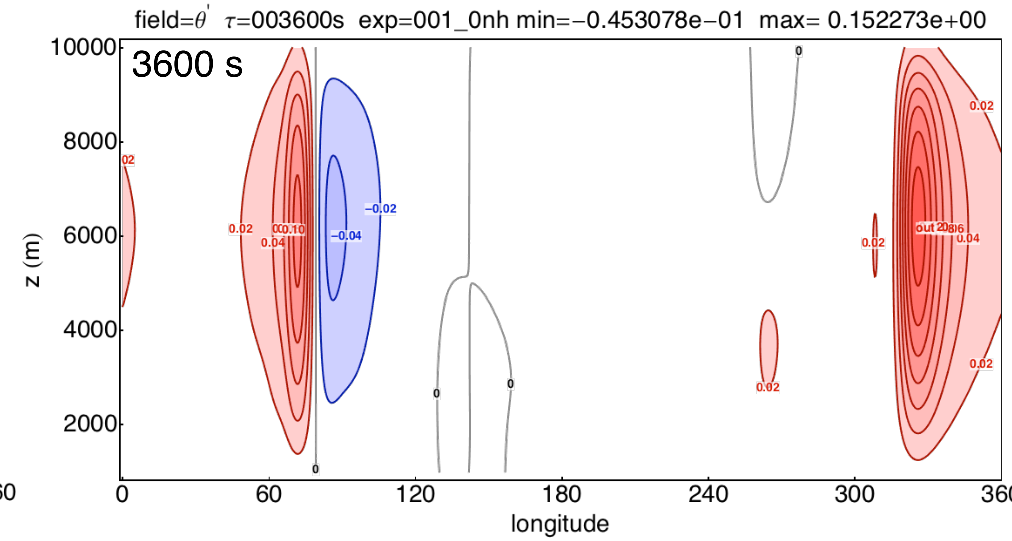
- Requires solution of a pair of two-dimensional elliptic equations at each level on each time step
- Uses the ugly, hard-to-interpret divergence equation

Unified, Z grid, small planet

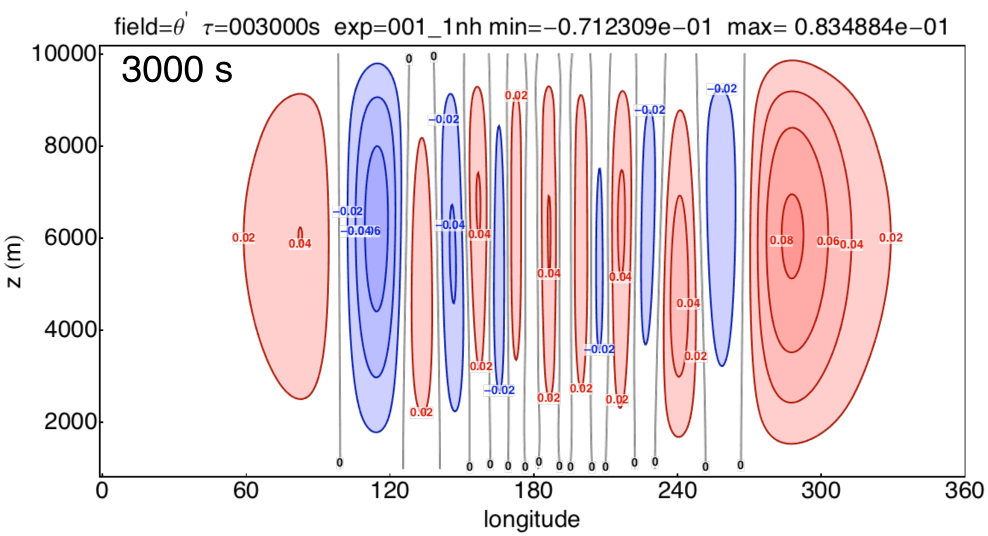
DCMIP test case 3.1

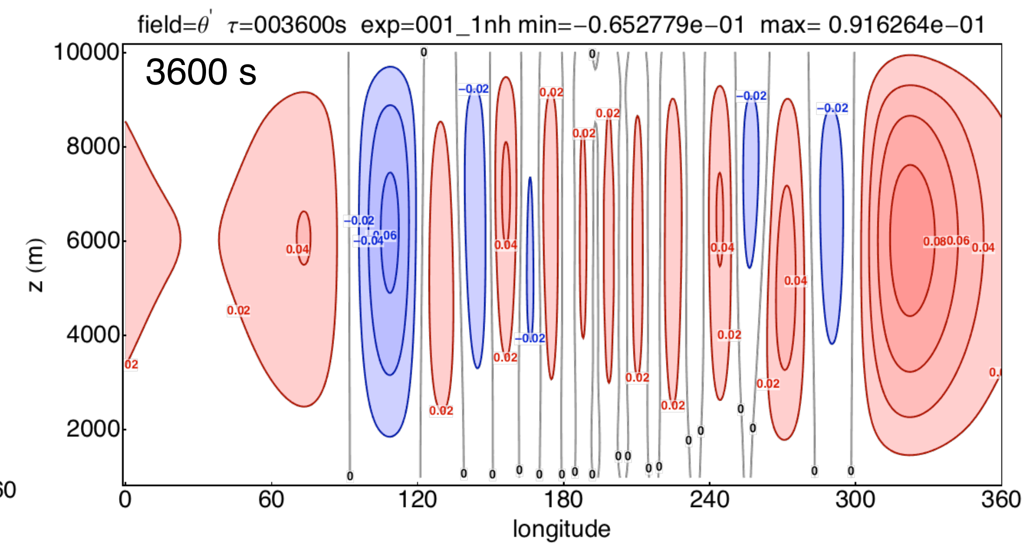
Quasi-hydrostatic run





Nonhydrostatic run





G7 (~2.5 km), 32 L (dz=500 m), a=300 km

Elliptic equation for the pressure

$$\nabla_H \cdot \left(\rho_{qs} c_p \theta \nabla_H \delta\pi\right) + \frac{\partial}{\partial z}\left(\rho_{qs} c_p \theta \frac{\partial \delta\pi}{\partial z}\right) = forcing$$

$$\left(\frac{\partial \delta\pi}{\partial z}\right)_S = \left(\frac{\partial \delta\pi}{\partial z}\right)_T = 0$$

Neumann boundary conditions

Convergence is good with the Neumann boundary conditions in simulations with a miniature Earth.

Convergence is poor with a full size Earth.

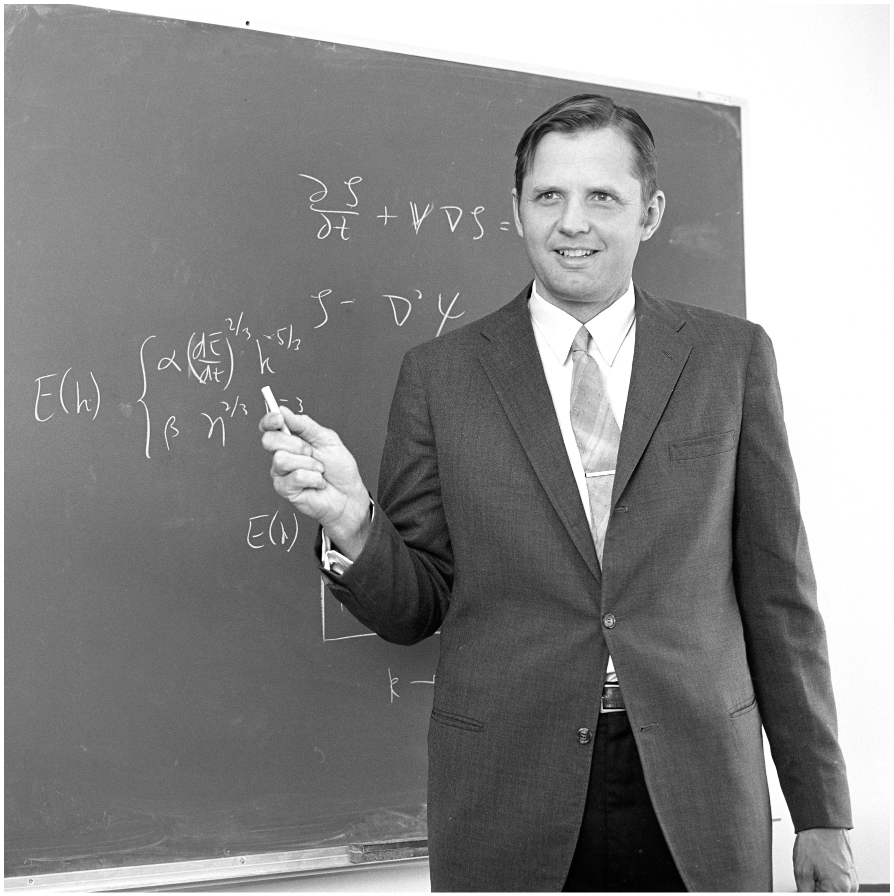
There is a way to escape the Neumann boundary conditions…

Vorticity across scales

Large-scale motions are controlled by the vertical component of the vorticity.

Small-scale motions are controlled by the horizontal vorticity vector.

A distinguished history



On the numerical simulation of buoyant convection

By D. K. LILLY, *General Circulation Research Laboratory, U.S. Weather Bureau, Washington*

(Manuscript received October 21, 1961, revised version March 6, 1962)

ABSTRACT

The two-dimensional turbulent vortex generated by release of buoyant fluid from an instantaneous thermal line source has been simulated by machine numerical solution of a complete set of Eulerian gas equations. The equations included turbulent eddy exchange terms, similar to those used by Smagorinsky, which made possible the generation of computationally stable solutions qualitatively and quantitatively resembling the convective "thermals" studied and described by Scorer and Richards. The results of a number of numerical experiments, performed with varying computational approximations, lead to conclusions as to the importance of various sources of numerical errors and the validity of the eddy exchange formulation. The formulation leads to qualitatively good results with the resolution provided by about 1500 grid points, but it has not yet been possible to exhibit the shape-preserving stage assumed in theoretical treatments and found approximately by laboratory experiments. This is probably due in large part to the neglect of the effects of eddies in the third dimension.

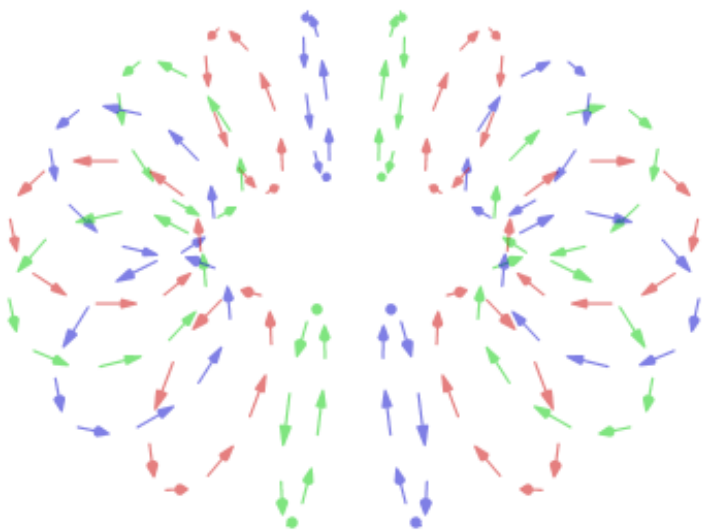
1. Goals and general procedure

In this article we present some results of a theoretical investigation of turbulent thermal convection in a compressible fluid (dry air) by means of direct numerical time-integration of a complete set of dynamic equations. Before discussing in detail the methods and results of this investigation we will briefly discuss the position of this type of study in relation to that of more conventional analytic methods. Due to the sequential and initially uncertain nature of the results of this type of investigation we conveniently call it a numerical experiment. One should always keep in mind, however, that (barring code errors) the results are purely logical consequences of the various theoretical approximations and simplifications initially assumed, difficult though it may be to trace through the effects of a particular assumption.

The application of numerical experimentation to physical theory is generally justifiable only when more concise analytic methods have been unproductive or have reached apparent limits of usefulness, but these conditions seem to prevail in the field of turbulent fluid mechanics. Although it would be possible to formulate and numerically integrate sets of differential equations, initial and boundary values, appropriate to a broad range of fluid dynamics phenomena, there would be little merit in going to this considerable labor for cases where general analytic solutions are available. This occurs under various conditions but most generally when the ratio of viscous and diffusive terms to those connected with inertial, advective, and gravitational forces is large, i.e. when the Reynolds and Rayleigh numbers are small. For thermal convective motions the linear solutions have some qualitative significance even for moderately large Rayleigh numbers, several times the critical value for onset of unstable motion. In addition there are some non-linear steady-state analytic solutions, or asymptotic approaches to solutions, available for this range (MALKUS & VERONIS, 1958; KUO, 1960). Thus it is doubtful whether in that regime a numerical initial-value approach would be justified, except perhaps for specific engineering purposes. When the scale and energy

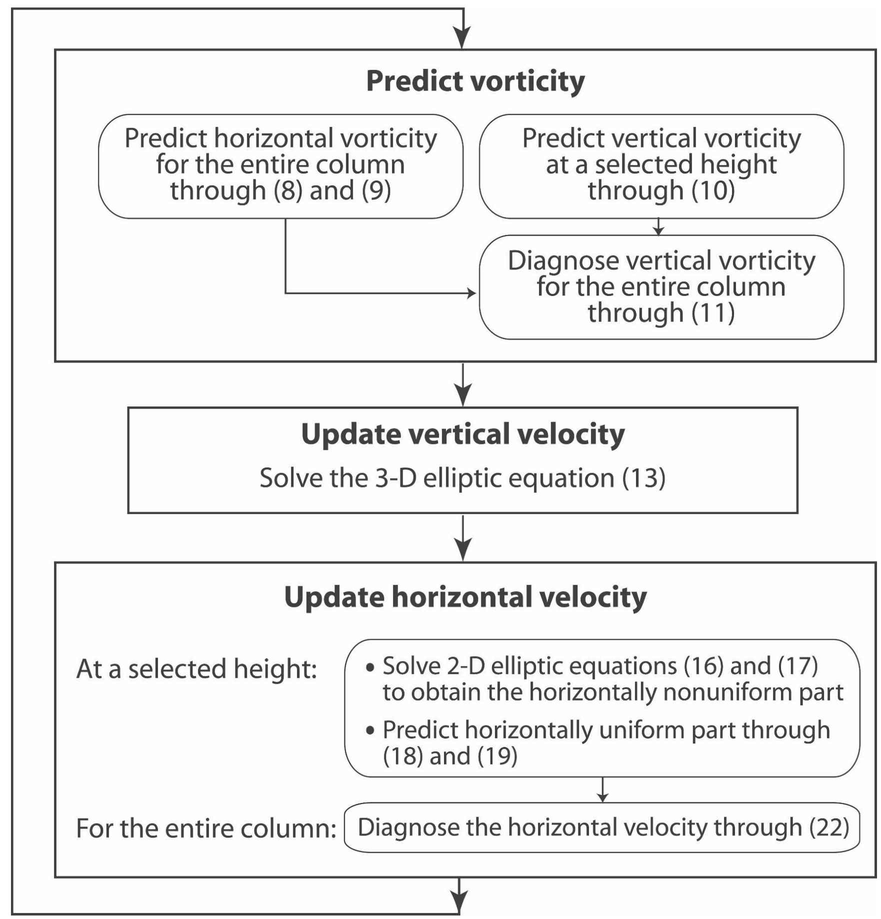
Vector Vorticity Model (VVM)

Jung & Arakawa, 2008



Predict the
horizontal vorticity vector
on a C grid

$$\boldsymbol{\omega}_{\mathrm{H}} = \frac{\partial \mathbf{V}_{\mathrm{H}}}{\partial z} - \nabla_{\mathrm{H}} w$$



Three-dimensional vorticity vector is nondivergent : $\nabla_{\mathrm{H}} \cdot \boldsymbol{\omega}_{\mathrm{H}} + \frac{\partial \omega_{z}}{\partial z} = 0$

Elliptic equation for the vertical velocity

In the VVM, the three-dimensional elliptic equation determines the vertical velocity, rather than the pressure:

$$\nabla_{\mathrm{H}}^2 w + \frac{\partial}{\partial z}\left[\frac{1}{\rho_{qs}}\frac{\partial}{\partial z}\left(\rho_{qs} w\right)\right] = -\Gamma \equiv \mathbf{k}\cdot\nabla_{\mathrm{H}}\times\boldsymbol{\omega}_{\mathrm{H}}$$

$$w_S = w_T = 0$$ Dirichlet boundary conditions

The "forcing" on the right-hand side is the curl of the horizontal vorticity.

Reconstructing the wind field

We have now solved for w , and we have predicted $\boldsymbol{\omega}_{\mathrm{H}}$. Then we can use

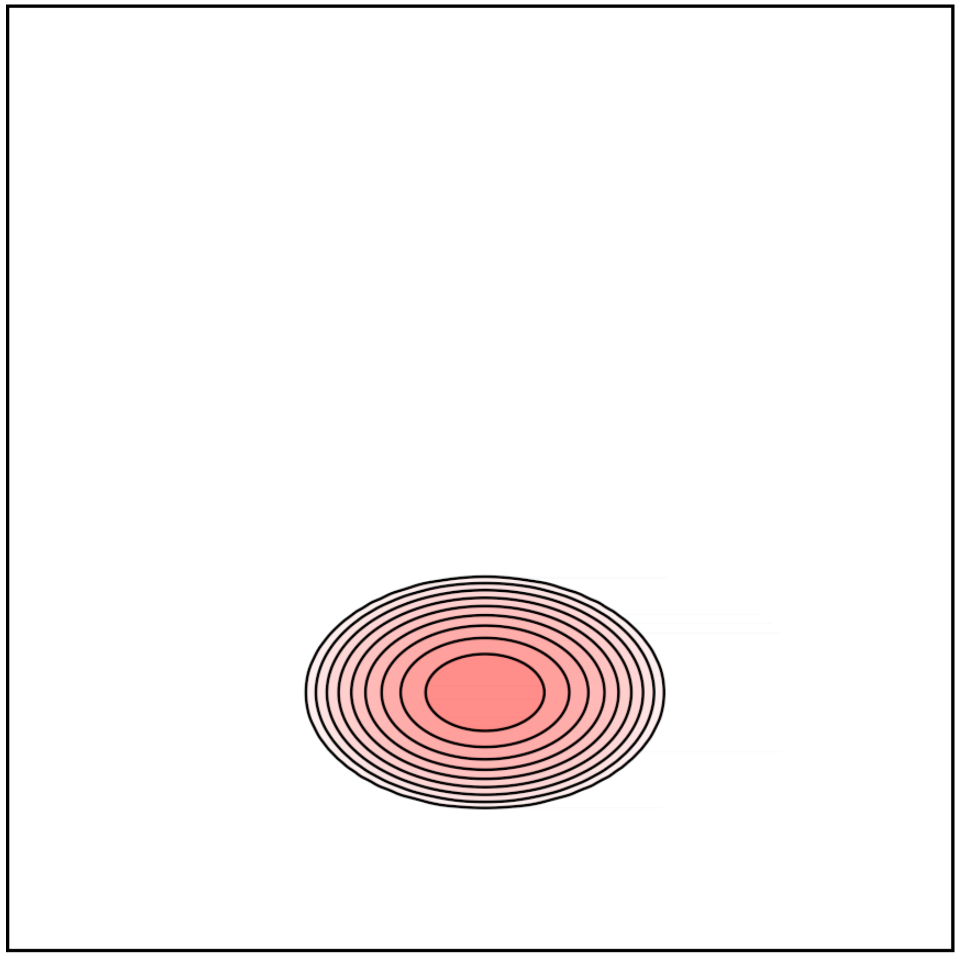
$$\boldsymbol{\omega}_{\mathrm{H}} = \frac{\partial \mathbf{V}_{\mathrm{H}}}{\partial z} - \nabla_{\mathrm{H}} w$$

to determine $\mathbf{V}_{\mathrm{H}}$, if we have a starting value of $\mathbf{V}_{\mathrm{H}}$ at the model's top.

For this purpose, we use the method of the "old" Z grid model,
which is to diagnose $\mathbf{V}_{\mathrm{H}}$ from $\boldsymbol{\omega}_{\mathrm{z}}$ and the divergence of the horizontal wind.

We predict $\boldsymbol{\omega}_{\mathrm{z}}$, and use the upper boundary condition $w = 0$ to determine $\frac{\partial}{\partial z}\left(\rho_{qs} w\right)$. Then the divergence can be obtained from the continuity equation.

We built a geodesic VVM that solves a 3D elliptic equation for the vertical velocity.



000000s

But the VVM is a C grid model, and C staggering leads to computational modes on a geodesic grid, so we made one more change…

Curl Curl

Vorticity & Divergence		Curl Curl
$\omega_z \equiv \mathbf{k} \cdot \left(\nabla_{\mathrm{H}} \times \mathbf{V}_{\mathrm{H}} \right)$	⇒	$\Gamma \equiv \mathbf{k} \cdot \left(\nabla_{\mathrm{H}} \times \boldsymbol{\omega}_{\mathrm{H}} \right)$
$\nabla_{\mathrm{H}} \cdot \mathbf{V}_{\mathrm{H}}$		$\nabla_{\mathrm{H}} \cdot \boldsymbol{\omega}_{\mathrm{H}}$

Since $\nabla_3 \cdot \boldsymbol{\omega}_3 = 0$ we can write $\nabla_{\mathrm{H}} \cdot \boldsymbol{\omega}_{\mathrm{H}} = -\dfrac{\partial \omega_z}{\partial z}$.

So, what we actually predict are ω_z and $\Gamma \equiv \mathbf{k} \cdot \left(\nabla_{\mathrm{H}} \times \boldsymbol{\omega}_{\mathrm{H}} \right)$.

What is the curl of the vorticity?

The vorticity has a curl when the vortex lines make loops or rings, analogous to the circular structures sometimes formed by the wind field when the velocity has a curl.

Vortex loops or rings surround jets, plumes and thermals.

The horizontal vorticity also has a curl in a field of cloud streets or "rolls," because in that situation the horizontal vorticity changes sign in the direction perpendicular to the vorticity vector.

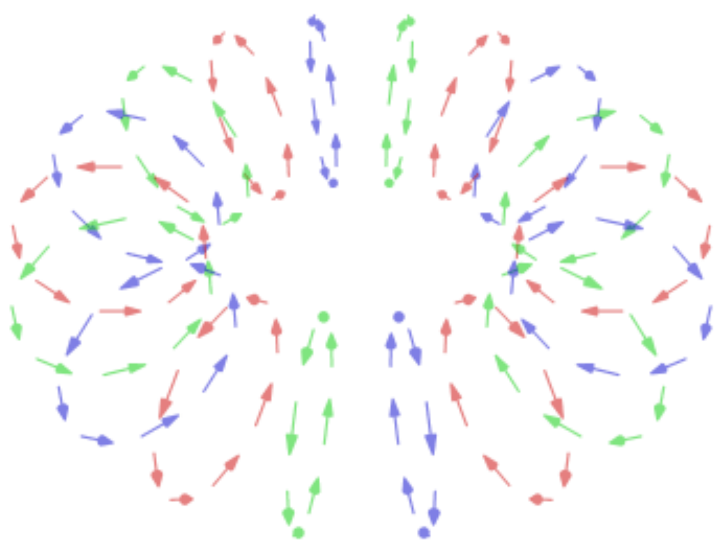
The Gamma Equation

$$\frac{\partial \Gamma}{\partial t} - \nabla_r^2 d - \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\mathbf{e}_r \cdot \nabla_r \times \left[\mathbf{V}_h \left(\zeta + 2\Omega_r \right) - w\left(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h \right) \right] \right\}$$

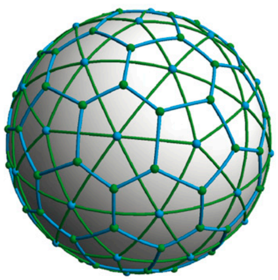
$$= \nabla_r \cdot \left(-\frac{\partial \alpha}{\partial r} \nabla_r p + \frac{\partial p}{\partial r} \nabla_r \alpha \right) - \mathbf{e}_r \cdot \nabla_r \times \left(\nabla \times \mathbf{F} \right)_h$$

where

$$d\mathbf{e}_r \equiv \left(\boldsymbol{\eta} + 2\boldsymbol{\Omega}_h \right) \times \mathbf{V}_h$$



Curl Curl is a geodesic Z-grid version of the VVM, and uses the Unified System of equations.



Geodesic grid

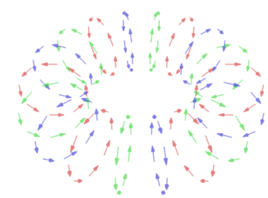
+

Unified system

+

Z staggering

+



Vector vorticity

= Curl Curl

Strengths & Weaknesses of Curl Curl

Strengths:

- No computational modes (because Z grid)
- Excellent dispersion properties for inertia-gravity waves (because Z grid)
- Direct prediction of the vertical component of the vorticity, which controls large-scale dynamics
- Direct prediction of the curl of the horizontal vorticity, which controls small-scale dynamics
- Predicts pseudo-scalars, rather than vectors (because Z grid)
- Guarantees the non-divergence of the three-dimensional vorticity vector (because VVM)

Weaknesses:

- Requires solution of a pair of 2D elliptic equations at each level on each time step
- Requires solution of a 3d elliptic equation, but with “friendly” Dirichlet boundary conditions

Closing Remarks

- This talk has been about structural issues.
- Structural design comes first. The ideas discussed in this talk form the concrete and steel of the model, on which everything else depends.
- The forms of the various operators also have to be specified, of course, and it's important to do a good job with that. Ross has developed some very accurate and flexible operators for use with Curl Curl.
- Computational performance and scaling are also important. Curl Curl scales well.

Extra slides

Relevant publications

Konor, C. S., and A. Arakawa, 1997: Design of an atmospheric model based on a generalized vertical coordinate. Mon. Wea. Rev., 125, 1649–1673.

Konor, C. S., 2014: Design of a Dynamical Core Based on the Nonhydrostatic "Unified System" of Equations*. *Mon. Wea. Rev.*, **142**, 364–385.

Jung, J.-H., and A. Arakawa, 2008: A Three-Dimensional Anelastic Model Based on the Vorticity Equation. *Mon. Wea. Rev.*, **136**, 276–294.

Heikes, R. P., D. A. Randall, and C. S. Konor, 2013: Optimized icosahedral grids: Performance of finite-difference operators and multigrid solver. *Mon. Wea. Rev.*, **141**, 4450-4469.

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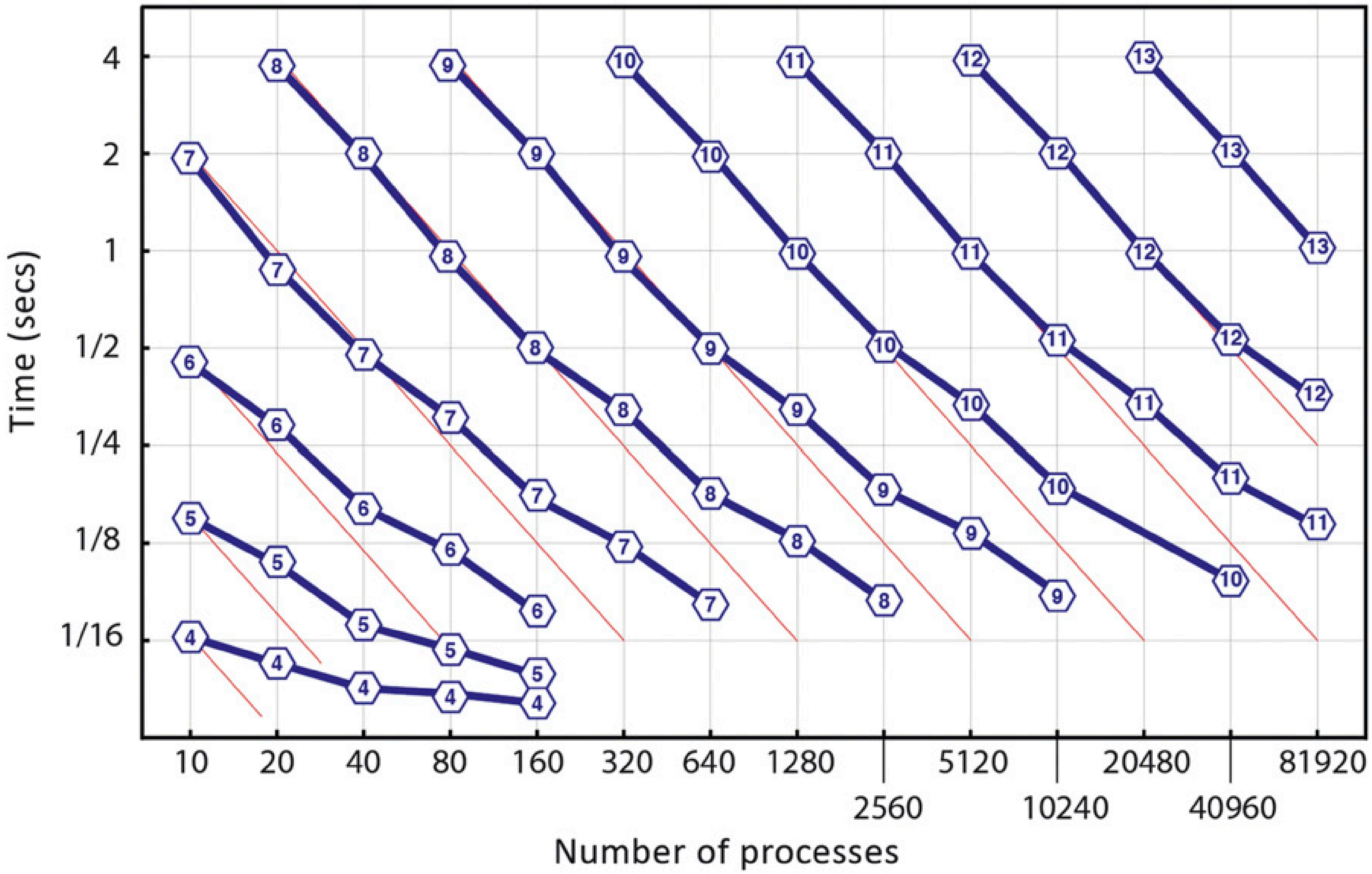
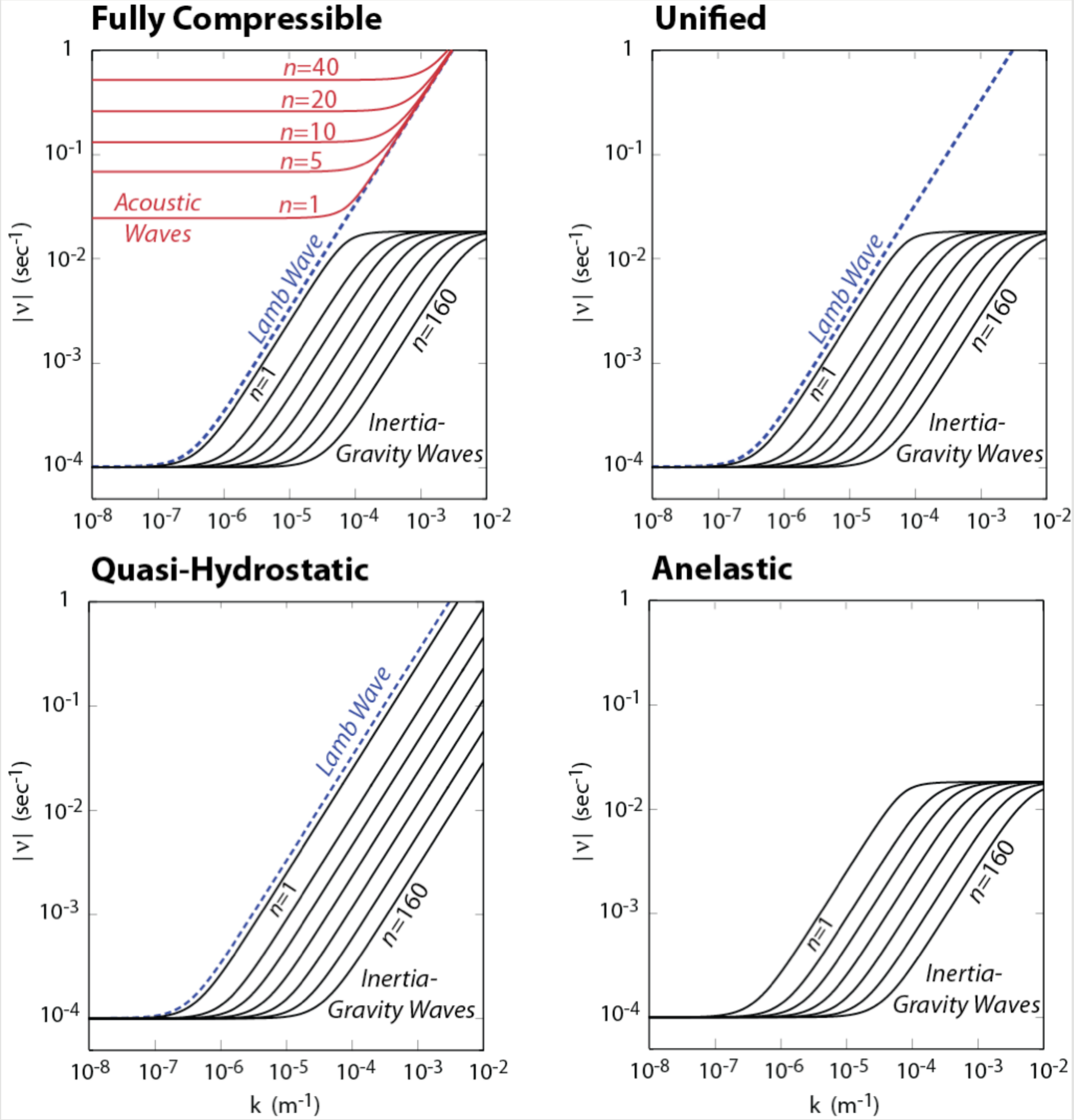
Multigrid scaling



Dispersion of Inertia-Gravity Waves



Dispersion of Rossby Waves