Design of an elastic, non-hydrostatic, multi-coordinate model:

Vertical staggering -- Normal mode analysis -- 2D linear model results

Michael D. Toy and David A. Randall -- *Department of Atmospheric Science, Colorado State University*

Introduction

We present the development of a new cloud model with an elastic, non-hydrostatic dynamical core for use in the multi-scale modeling framework. The model uses multiple vertical coordinates:

- A height-based terrain-following coordinate, similar to that of Gal-Chen and Somerville (1975), near the surface
- A hybrid isentropic/height-based vertical coordinate similar to that of Konor and Arakawa (1997), to take advantage of the semi-Lagrangian properties of the isentropic coordinate throughout most of the free atmosphere
- An isentropic coordinate (which is tested in the linear analysis but will most likely be omitted in the nonlinear model)

We have chosen the continuous form of the equations to be used and worked out the vertical staggering. Use of the "Charney-Phillips" grid removes the possibility of a stationary vertical computational mode in the thermal field. Here we present an analysis of the linearized form of the equations and the vertical discretization.

Generalized Vertical Coordinate (η)

Domain III: Isentropic

$$\eta \equiv \frac{\theta - \theta_I}{\tau_3} + 2, \quad \text{where} \quad \tau_3 \equiv \theta_T - \theta_I = \text{constant}.$$

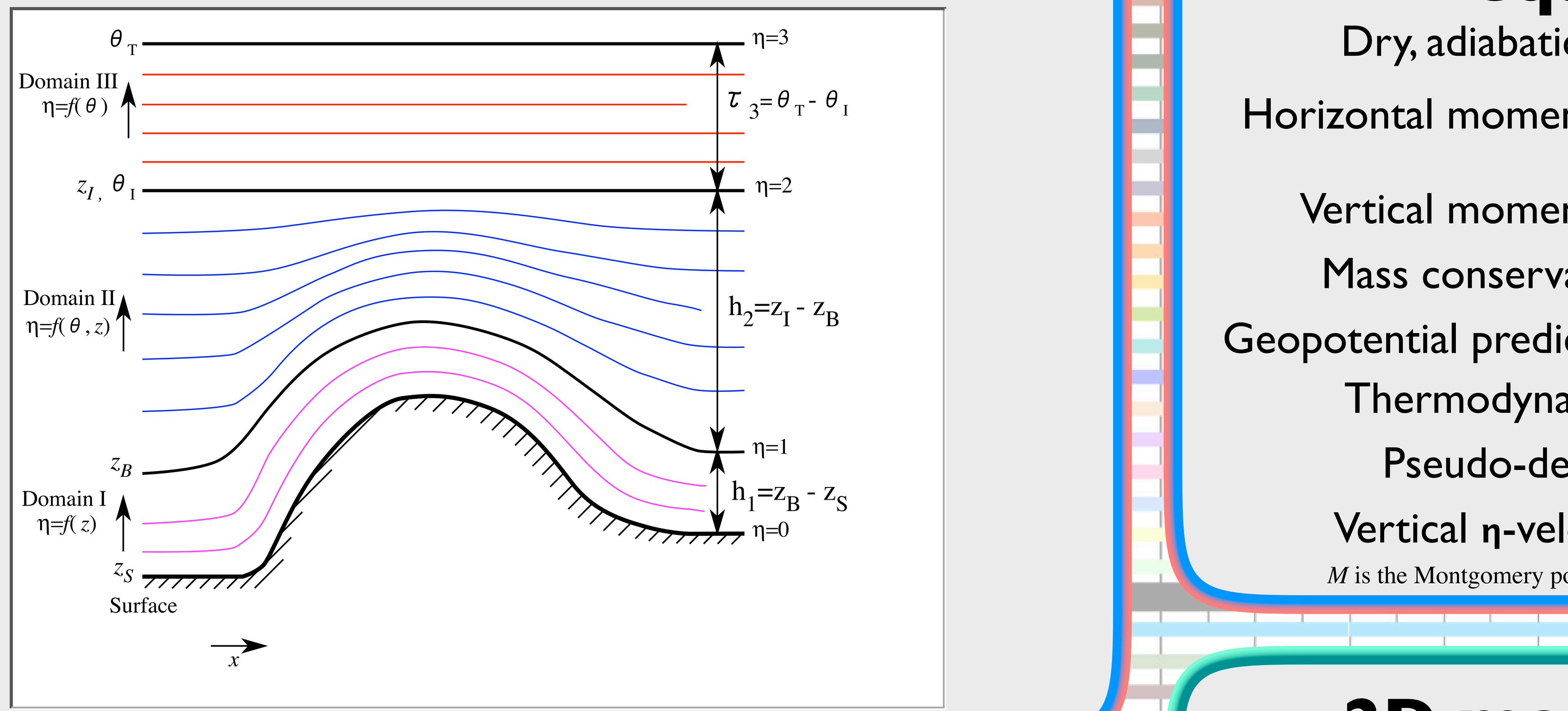
Domain II: Hybrid isentropic/height-based terrain-following

$$\eta \equiv f(z) + g(z)\,\tau(\theta), \quad \text{where} \quad g(z) \to 0; \quad z \to z_B$$
$$f(z) \to 0,\ g(z) \to 1; \quad z \to z_I$$

A suitable choice for $g(z)$ is $g(z) = 1 - \left(\frac{z_I - z}{h_2}\right)^r$, where r is a constant and $h_2 \equiv z_I - z_B$. Following Konor and Arakawa (1997), we can ensure that η is a monotonically increasing function of height by defining $\tau_{\min}$ and $(\partial\theta/\partial z)_{\min}$ as lower bounds of $\tau(\theta)$ and static stability, respectively, and choosing $f(z)$ from $\frac{df}{dz} + \frac{dg}{dz}\tau_{\min} + g\frac{d\tau}{d\theta}\left(\frac{\partial\theta}{\partial z}\right)_{\min} = 0$.

Domain I: Height-based terrain-following

$$\eta \equiv \frac{z - z_S}{h_1}, \quad \text{where} \quad h_1(x, y, t) \equiv z_B(x, y, t) - z_S(x, y).$$



Continuous nonlinear equations

Dry, adiabatic, frictionless motion

Horizontal momentum: $\frac{D\mathbf{V}}{Dt} + f\hat{k}\times\mathbf{V} = -\theta\left(\nabla_\eta \Pi - \frac{\partial\Pi}{\partial\phi}\nabla_\eta\phi\right) = -\nabla_\eta M + \Pi\nabla_\eta\theta + \left(\frac{\partial M}{\partial\theta} - \Pi\right)\frac{\partial\theta}{\partial\phi}\nabla_\eta\phi$

Vertical momentum: $\frac{Dw}{Dt} = -\frac{\rho}{m}\theta\frac{\partial\Pi}{\partial\eta} - g$

Mass conservation: $\frac{\partial m}{\partial t} + \nabla_\eta\cdot(m\mathbf{V}) + \frac{\partial}{\partial\eta}(m\dot\eta) = 0$

Geopotential prediction: $\frac{D\phi}{Dt} = wg$

Thermodynamics: $\frac{D\theta}{Dt} = 0$

Pseudo-density: $m \equiv \frac{\rho}{g}\frac{\partial\phi}{\partial\eta}$

Vertical η-velocity: $\dot\eta = \frac{D\eta}{Dt}$

M is the Montgomery potential and Π is the Exner function.

Continuous 2D linearized equations for each domain

Linearized about a resting, isothermal, hydrostatic, horizontally homogeneous basic state

	Domain I	*Domain II*	*Domain III*
Zonal momentum:	$\frac{\partial u'}{\partial t} - fv' = -\bar\theta\frac{\partial\Pi'}{\partial x} - \frac{\partial\phi'}{\partial x} + \frac{1}{g\bar h_1}\frac{\partial\bar\phi}{\partial x}\left(\bar\theta\frac{\partial\Pi'}{\partial\eta} + g z_B'\right)$	$\frac{\partial u'}{\partial t} - fv' = -\frac{\partial M'}{\partial x} + \bar\Pi\frac{\partial\theta'}{\partial x} + \frac{\partial\bar\theta}{\partial x}\left[\left(\frac{\partial\bar\theta}{\partial\eta}\right)^{-1}\frac{\partial M'}{\partial\eta} - \frac{\partial\bar M}{\partial\eta}\left(\frac{\partial\bar\theta}{\partial\eta}\right)^{-2}\frac{\partial\theta'}{\partial\eta}\right]$	$\frac{\partial u'}{\partial t} - fv' = -\frac{\partial M'}{\partial x}$
Vertical momentum:	$\frac{\partial w'}{\partial t} = -\frac{1}{\bar h_1}\bar\theta\frac{\partial\Pi'}{\partial\eta} + g\frac{\theta'}{\bar\theta} - \frac{g}{\bar h_1}z_B'$	$\frac{\partial w'}{\partial t} = -g\,\bar\theta\left(\frac{\partial\bar\phi}{\partial\eta}\right)^{-1}\frac{\partial\Pi'}{\partial\eta} + g\frac{\theta'}{\bar\theta} - g\left(\frac{\partial\bar\phi}{\partial\eta}\right)^{-1}\frac{\partial\phi'}{\partial\eta}$	$\frac{\partial w'}{\partial t} = -g\theta\left(\frac{\partial\bar\phi}{\partial\eta}\right)^{-1}\left[\frac{\partial\Pi'}{\partial\eta} - \frac{d\bar\Pi}{d\eta}\left(\frac{d\bar\phi}{d\eta}\right)^{-1}\frac{\partial\phi'}{\partial\eta}\right]$
Mass conservation:	$\frac{\partial m'}{\partial t} + \bar m\left(\frac{\partial u'}{\partial x} + \frac{\partial\dot\eta'}{\partial\eta}\right) + \frac{\partial\bar m}{\partial x}u' + \frac{\partial\bar m}{\partial\eta}\dot\eta' = 0$	$\frac{\partial m'}{\partial t} + \bar m\left(\frac{\partial u'}{\partial x} + \frac{\partial\dot\eta'}{\partial\eta}\right) + \frac{\partial\bar m}{\partial x}u' + \frac{\partial\bar m}{\partial\eta}\dot\eta' = 0$	$\frac{\partial m'}{\partial t} + \bar m\frac{\partial u'}{\partial x} = 0$
Geopotential prediction:	---	$\frac{\partial\phi'}{\partial t} + \frac{\partial\bar\phi}{\partial x}u' + \frac{\partial\bar\phi}{\partial\eta}\dot\eta' = g w'$	$\frac{\partial\phi'}{\partial t} = g w'$
Thermodynamics:	$\frac{\partial\theta'}{\partial t} + \frac{\partial\bar\theta}{\partial x}u' + \frac{\partial\bar\theta}{\partial\eta}\dot\eta' = 0$	$\frac{\partial\theta'}{\partial t} + \frac{\partial\bar\theta}{\partial x}u' + \frac{\partial\bar\theta}{\partial\eta}\dot\eta' = 0$	---
Pseudo-density:	$m' = \bar\rho z_B' + \bar h_1\rho'$	$m' = \frac{\bar\rho}{g}\frac{\partial\phi'}{\partial\eta} + \frac{1}{g}\frac{\partial\bar\phi}{\partial\eta}\rho'$	$m' = \frac{\bar\rho}{g}\frac{\partial\phi'}{\partial\eta} + \frac{1}{g}\frac{\partial\bar\phi}{\partial\eta}\rho'$
Vertical η-velocity:	$\dot\eta' = -\frac{\eta}{\bar h_1}\frac{\partial z_B'}{\partial t} - \frac{1}{\bar h_1}\left(\frac{dz_S}{dx} + \eta\frac{d\bar h_1}{dx}\right)u' + \frac{w'}{\bar h_1}$	$\dot\eta' = -r\left(\frac{\bar z_I - \bar z}{\bar h_2}\right)^{r-1}\left(\frac{\bar\theta - \theta_I}{\tau_3} + 1\right)\left[\frac{1}{\bar h_2}\left(\frac{\partial z_I'}{\partial t} - w'\right) - \left(\frac{1}{\bar h_2}\right)^2(\bar z_I - \bar z)\left(\frac{\partial z_I'}{\partial t} - \frac{\partial z_B'}{\partial t} - \frac{d\bar z_B}{dx}u'\right)\right]$	---

Vertical staggering and vertically discrete linearized equations each domain

No bottom topography (for simplicity)

Domain I:

Vertical Index	Prognostic variables	Diagnostic variables
$l + 3/2$		
$l + 1$		
$l + 1/2$	w', θ'	η', Φ'
l	u', v', m'	Π', p', ρ', T'
$l - 1/2$	w', θ'	η', Φ'
$l - 1$		
$l - 3/2$		

Domain II:

Vertical Index	Prognostic variables	Diagnostic variables
$l + 3/2$		
$l + 1$		
$l + 1/2$	w', θ', Φ'	η'
l	u', v', m'	Π', M', p', ρ', T'
$l - 1/2$	w', θ', Φ'	η'
$l - 1$		
$l - 3/2$		

Domain III:

Vertical Index	Prognostic variables	Diagnostic variables
$l + 3/2$		
$l + 1$		
$l + 1/2$	w', Φ'	
l	u', v', m'	Π', M', p', ρ', T'
$l - 1/2$	w', Φ'	
$l - 1$		
$l - 3/2$		

Domain I	*Domain II*	*Domain III*
$\frac{\partial u_l}{\partial t} - fv_l = -\bar\theta_l\frac{\partial\Pi_l}{\partial x} - g\eta_l\frac{\partial z_B}{\partial x}$	$\frac{\partial u_l}{\partial t} - fv_l = -\frac{\partial M_l}{\partial x} + \bar\Pi_l\frac{1}{2}\left(\frac{\partial\theta_{l+1/2}}{\partial x} + \frac{\partial\theta_{l-1/2}}{\partial x}\right)$	$\frac{\partial u_l}{\partial t} - fv_l = -\frac{\partial M_l}{\partial x}$
$\frac{\partial w_{l+1/2}}{\partial t} = -\frac{1}{\bar h_1}\bar\theta_{l+1/2}\frac{\Pi_{l+1} - \Pi_l}{(\delta\eta)_{l+1/2}} + g\frac{\theta_{l+1/2}}{\bar\theta_{l+1/2}} - \frac{g}{\bar h_1}z_B$	$\frac{\partial w_{l+1/2}}{\partial t} = -g\left[\bar\theta\left(\frac{\partial\bar\phi}{\partial\eta}\right)^{-1}\right]_{l+1/2}\frac{\Pi_{l+1} - \Pi_l}{(\delta\eta)_{l+1/2}} + g\frac{\theta_{l+1/2}}{\bar\theta_{l+1/2}} - g\left[\left(\frac{\partial\bar\phi}{\partial\eta}\right)^{-1}\right]_{l+1/2}\frac{1}{2}\left[\frac{\phi_{l+3/2} - \phi_{l+1/2}}{(\delta\eta)_{l+1}} + \frac{\phi_{l+1/2} - \phi_{l-1/2}}{(\delta\eta)_l}\right]$	$\frac{\partial w_{l+1/2}}{\partial t} = -g\left[\theta\left(\frac{d\bar\phi}{d\eta}\right)^{-1}\right]_{l+1/2}\left\{\frac{\Pi_{l+1} - \Pi_l}{(\delta\eta)_{l+1/2}} - \left[\frac{d\bar\Pi}{d\eta}\left(\frac{d\bar\phi}{d\eta}\right)^{-1}\right]_{l+1/2}\frac{1}{2}\left[\frac{\phi_{l+3/2} - \phi_{l+1/2}}{(\delta\eta)_{l+1}} + \frac{\phi_{l+1/2} - \phi_{l-1/2}}{(\delta\eta)_l}\right]\right\}$
$\frac{\partial m_l}{\partial t} + \bar m_l\left[\frac{\partial u_l}{\partial x} + \frac{\dot\eta_{l+1/2} - \dot\eta_{l-1/2}}{(\delta\eta)_l}\right] + \left(\frac{\partial\bar m}{\partial\eta}\right)_l\frac{1}{2}\left(\dot\eta_{l+1/2} + \dot\eta_{l-1/2}\right) = 0$	$\frac{\partial m_l}{\partial t} + \bar m_l\left[\frac{\partial u_l}{\partial x} + \frac{\dot\eta_{l+1/2} - \dot\eta_{l-1/2}}{(\delta\eta)_l}\right] + \left(\frac{\partial\bar m}{\partial\eta}\right)_l\frac{1}{2}\left(\dot\eta_{l+1/2} + \dot\eta_{l-1/2}\right) = 0$	$\frac{\partial m_l}{\partial t} + \bar m_l\frac{\partial u_l}{\partial x} = 0$
---	$\frac{\partial\phi_{l+1/2}}{\partial t} + \left(\frac{\partial\bar\phi}{\partial\eta}\right)_{l+1/2}\dot\eta_{l+1/2} = g w_{l+1/2}$	$\frac{\partial\phi_{l+1/2}}{\partial t} = g w_{l+1/2}$
$\frac{\partial\theta_{l+1/2}}{\partial t} + \left(\frac{\partial\bar\theta}{\partial\eta}\right)_{l+1/2}\dot\eta_{l+1/2} = 0$	$\frac{\partial\theta_{l+1/2}}{\partial t} + \left(\frac{\partial\bar\theta}{\partial\eta}\right)_{l+1/2}\dot\eta_{l+1/2} = 0$	---
$m_l = \bar\rho_l z_B + \bar h_1\rho_l$	$m_l = \frac{\bar\rho_l}{g}\frac{\phi_{l+1/2} - \phi_{l-1/2}}{(\delta\eta)_l} + \frac{1}{g}\left(\frac{\partial\bar\phi}{\partial\eta}\right)_l\rho_l,$	$m_l = \frac{\bar\rho_l}{g}\frac{\phi_{l+1/2} - \phi_{l-1/2}}{(\delta\eta)_l} + \frac{1}{g}\left(\frac{\partial\bar\phi}{\partial\eta}\right)_l\rho_l$
$\dot\eta_{l+1/2} = -\frac{\eta_{l+1/2}}{\bar h_1}\frac{\partial z_B}{\partial t} + \frac{w_{l+1/2}}{\bar h_1}$	$\dot\eta_{l+1/2} = -r\left(\frac{\bar z_I - \bar z_{l+1/2}}{\bar h_2}\right)^{r-1}\left(\frac{\bar\theta_{l+1/2} - \theta_I}{\tau_3} + 1\right)\left[\frac{1}{\bar h_2}\left(\frac{\partial z_I}{\partial t} - w_{l+1/2}\right) - \left(\frac{1}{\bar h_2}\right)^2(\bar z_I - \bar z_{l+1/2})\left(\frac{\partial z_I}{\partial t} - \frac{\partial z_B}{\partial t}\right)\right]$	---

Note: Interpolated terms are shaded

2D model results

Model description:

- Finite differencing in vertical and horizontal
- Horizontal staggering -- Arakawa "C" grid
- Time differencing -- Matsuno first order

- Upper boundary -- rigid lid
- Lower boundary -- no topography
- Horizontal boundaries -- periodic

- Vertical coordinate -- can be run using various combinations of vertical-coordinates, i.e. can be configured to use pure theta-coordinates, pure height-based coordinates, hybrid theta-z coordinates, or combinations of these

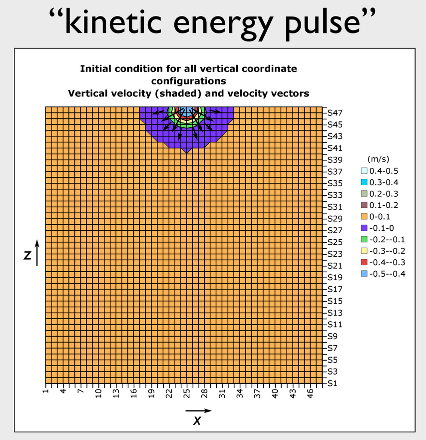
- Vertical domain = 24 km
- Horizontal domain = 24 km

- Vertical grid spacing = 500 m
- Horizontal grid spacing = 500 m
- Time step = 0.5 s

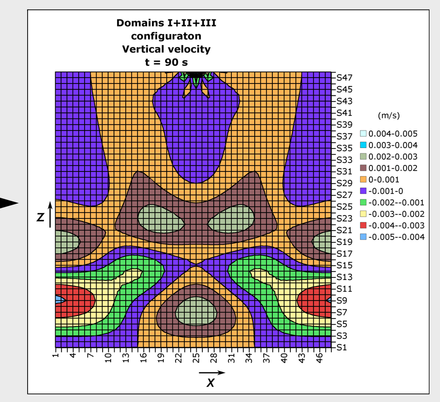
- 48 levels

- Basic state temperature = 300 K
- Basic state surface pressure = 1000 hPa

Initial condition: "kinetic energy pulse"



At t = 90 seconds



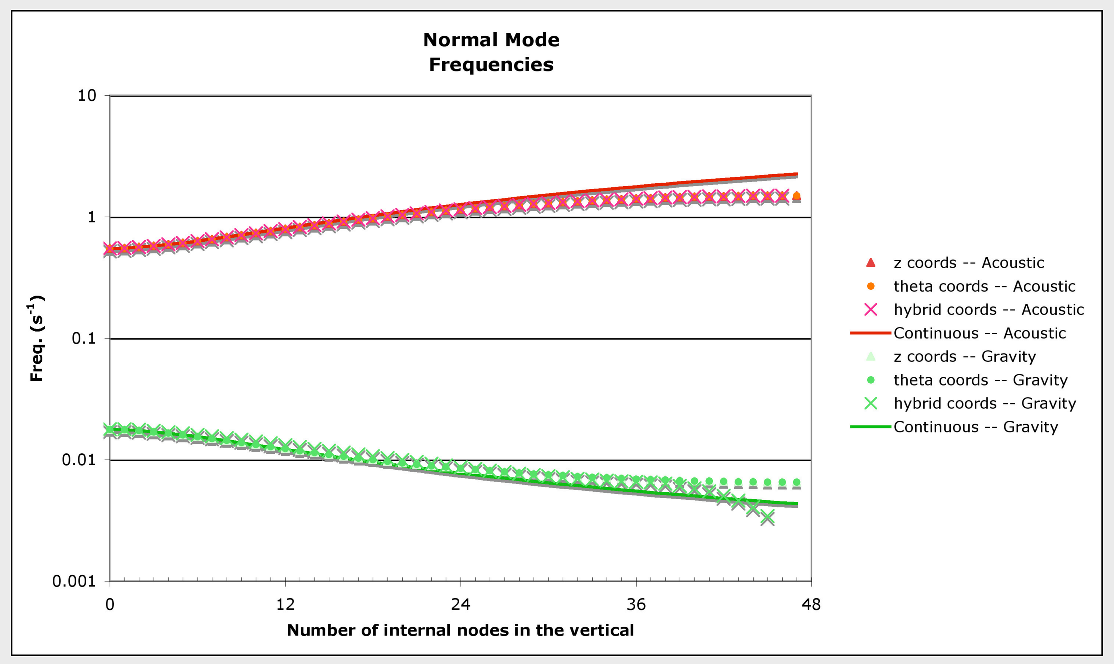
Note: Similar results are obtained for all vertical-coordinate configurations

Normal mode analysis

Normal mode frequencies for the vertically discrete, horizontally and temporally continuous, linear system are shown on the right. Waves of horizontal wave-length = 4 km and variable vertical wave number (shown on the horizontal axis) are represented. Physical and discrete parameters are the same as those used in the 2D model.

The z-coordinate and theta-coordinate frequencies were calculated from analytical dispersion relations. The hybrid coordinate frequencies were calculated from a numerical eigenvalue analysis.

The frequencies of the waves represented in each of the three coordinate systems closely coincide with each other except for gravity waves of high vertical wave number represented by hybrid coordinates.



Conclusion

We have presented the linear characteristics of the equations to be used in an elastic, non-hydrostatic cloud model with multiple vertical coordinates.

The next step is to formulate the discrete nonlinear equations and meet the following goals:

- Maintain mass and energy conservation
- Make optimal use of the hybrid isentropic/height-based coordinate for any profile of static stability
- Use a time differencing scheme which is stable despite the presence of vertically propagating acoustic waves

Results of tests of the nonlinear model, which will include gravity wave breaking, will be forthcoming.

References

Gal-Chen, T., and R.C. J. Somerville, 1975: On the use of a coordinate transformation for the solution of the Navier-Stokes equations. *J. Comp. Phys.*, **17**, 209-228.

Konor, C.S., and A. Arakawa, 1997: Design of an atmospheric model based on a generalized vertical coordinate. *Mon. Wea. Rev.*, **125**, 1649-1673.

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