The Inclusion of Entropy Transport by Precipitation and its Effect on Tropical Weather Systems

Scott A. Hausman (CSU)

Katsuyuki V. Ooyama (HRD/AOML/NOAA)

Wayne H. Schubert (CSU)

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MODELING ASSUMPTIONS

★ Four types of matter:

Type of Matter	***Density***
dry air	ρ_a
water vapor	ρ_v
airborne condensate (water or ice)	ρ_c
precipitating condensate (water or ice)	ρ_r
airborne moisture	$\rho_m = \rho_v + \rho_c$
all	$\rho = \rho_a + \rho_v + \rho_c + \rho_r$

★ Predict ρ_m, not ρ_v and ρ_c separately.

★ Determine ρ_v and ρ_c later at the diagnostic stage.

- If the air is not cloudy ($\rho_c = 0$), all of the predicted ρ_m is ρ_v.
- If the air is cloudy, ρ_v is the saturation value and the remainder of ρ_m is ρ_c.
- This involves equilibrium thermodynamics.

MODELING ASSUMPTIONS (continued)

* Dry air and water vapor satisfy the ideal gas law:

$$p_a = \rho_a R_a T \qquad \text{and} \qquad p_v = \rho_v R_v T$$

$$R_a = 287.05 \text{ J kg}^{-1}\text{K}^{-1} \quad \text{(gas constant for dry air)}$$
$$R_v = 461.51 \text{ J kg}^{-1}\text{K}^{-1} \quad \text{(gas constant for water vapor)}$$

* Dalton's Law of Partial Pressures:

$$p = p_a + p_v$$

* Specific heats are constant:

$$c_{pa} = 1004.675 \quad \text{J kg}^{-1}\text{K}^{-1} \quad \text{(dry air at constant pressure)}$$
$$c_{pv} = 1850.0 \quad \text{J kg}^{-1}\text{K}^{-1} \quad \text{(water vapor at constant pressure)}$$
$$c_{va} = c_{pa} - R_a \quad \text{(dry air at constant volume)}$$
$$c_{vv} = c_{pv} - R_v \quad \text{(water vapor at constant volume)}$$

THE NEW GOFF FORMULAS

★ For saturation vapor pressure over water:

$$\log_{10} E_w(T) = 10.79574\left(1-\frac{T_{tp}}{T}\right) - 5.02800\log_{10}\left(\frac{T_{tp}}{T}\right) + 1.50475\times10^{-4}\left[1-10^{-8.2969(T/T_{tp}-1)}\right] + 0.42873\times10^{-3}\left[10^{-4.76955(1-T_{tp}/T)}-1\right] + 0.78614$$

★ For saturation vapor pressure over ice:

$$\log_{10} E_i(T) = -9.09685\left(\frac{T_{tp}}{T}-1\right) - 3.56654\log_{10}\left(\frac{T_{tp}}{T}\right) + 0.87682\left(1-\frac{T}{T_{tp}}\right) + 0.78614$$

Note:

- $E_w(T)$ and $E_i(T)$ are expressed in hectopascals
- T is expressed in Kelvins
- $T_{tp} = 273.16$ K is the triple point of water

SPECIFIED FUNCTIONS OF T

- Given $E_i(T)$ and $E_w(T)$ from the new Goff formulas, synthesize $E(T)$ so that it transitions from $E_i(T)$ to $E_w(T)$ as T passes through $T_0 = 273.15$ K.

- From $E(T)$ compute:

$$L(T) = R_v T^2 \frac{d \ln E(T)}{dT}$$

(Clausius-Clapeyron equation)

$$\rho_v^*(T) = \frac{E(T)}{R_v T}$$

(density of saturated vapor)

$$c_c(T) = c_{pv} - \frac{dL(T)}{dT}$$

(Kirchhoff equation)

- Entropy of a unit mass of condensate:

$$C(T) = \int_{T_0}^{T} \frac{c_c(T')}{T'} dT' \qquad \text{Note}: \quad C(T_0) = 0$$

SPECIFIED FUNCTIONS OF T (continued)

- Using the Kirchhoff equation for $c_c(T)$:

$$C(T) = c_{vv} \ln\left(\frac{T}{T_0}\right) - R_v \ln\left(\frac{\rho_v^*(T)}{\rho_v^*(T_0)}\right) - \frac{L(T)}{T} + \frac{L(T_0)}{T_0}$$

- Gain of entropy density by evaporating sufficient water to saturate the volume at T:

$$D(T) = \frac{dE(T)}{dT}$$

PROGNOSTIC EQUATIONS FOR MASS

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{u}) = 0$$

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u} + \mathbf{F}_m) = -Q_r$$

$$\frac{\partial \rho_r}{\partial t} + \nabla \cdot [\rho_r(\mathbf{u} + \mathbf{U})] = Q_r$$

Q_r: Conversion rate of ρ_m to ρ_r

$\mathbf{U}$: Terminal fall velocity of precip relative to air

★ The prognostic equation for the total density ($\rho = \rho_a + \rho_m + \rho_r$):

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = -\nabla \cdot (q_r \rho \mathbf{U})$$

★ The equations for ρ_m and ρ_r can be rewritten in terms of mass fractions:

$$\frac{Dq_m}{Dt} = \frac{1}{\rho}[-Q_r + q_m \nabla \cdot (q_r \rho \mathbf{U})]$$

$$\frac{Dq_r}{Dt} = \frac{1}{\rho}[Q_r - (1 - q_r)\nabla \cdot (q_r \rho \mathbf{U})]$$

$$q_m = \rho_m / \rho$$

$$q_r = \rho_r / \rho$$

ENTROPY DENSITY EQUATION

- ⋆ At this stage we don't know how to write the internal energy or enthalpy forms of the first law of thermodynamics for a complicated moist model. However, we can write the budget equation for the total entropy density.

- ⋆ Entropy density is an additive quantity:

$$\sigma = \sigma_a + \sigma_m + \sigma_r$$

σ: Total entropy density
σ_a: Entropy density of dry air
σ_m: Entropy density of airborne moisture
σ_r: Entropy density of precipitation

- ⋆ Since the flux of σ is given by $\sigma_a \mathbf{u} + \sigma_m \mathbf{u} + \sigma_r (\mathbf{u} + \mathbf{U})$ the budget equation for σ is

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \mathbf{u} + \sigma_r \mathbf{U}) = 0$$

where we have neglected radiative effects.

SPECIFIC ENTROPY EQUATION

- Define the specific entropy of moist air as

$$s = \frac{\sigma}{\rho}$$

- The prognostic equation for s is then

$$\frac{Ds}{Dt} = \frac{1}{\rho}\left[s\nabla \cdot (\rho_r \mathbf{U}) - \nabla \cdot (\rho_r s_r \mathbf{U})\right]$$

- Note that in the absence of precipitation ($\rho_r = 0$):

$$\frac{Ds}{Dt} = 0$$

SPECIFIC ENTROPY OF MOIST AIR

★ Specific entropy of moist air:

$$s = \frac{\sigma}{\rho} = \frac{\rho_a s_a + \rho_m s_m + \rho_r s_r}{\rho} = q_a s_a + q_m s_m + q_r s_r$$

★ Specific entropy of dry air:

$$s_a(\rho, q_a, T) = c_{va} \ln\left(\frac{T}{T_0}\right) - R_a \ln\left(\frac{q_a \rho}{\rho_{a0}}\right)$$

★ Specific entropy of precipitation:

$$s_r = C(T_2)$$

SPECIFIC ENTROPY OF MOIST AIR (continued)

- ⋆ Note that the relaxation time for a raindrop to adjust to the wetbulb temperature is 4 to 5 seconds. The rapidity of this adjustment is due to the fact that the amount of heat required to adjust the temperature of the drop is much smaller than that required to evaporate it.

- ⋆ Specific entropy of airborne moisture:

  - State 1 (no airborne condensate):

$$s_m^{(1)}(\rho, q_m, T) = c_{vv} \ln\left(\frac{T}{T_0}\right) - R_v \ln\left(\frac{q_m \rho}{\rho_v^*(T_0)}\right) + \frac{L(T_0)}{T_0}$$

  - State 2 (saturated vapor):

$$s_m^{(2)}(\rho, q_m, T) = C(T) + \frac{D(T)}{q_m \rho}$$

SUMMARY OF PROGNOSTIC EQUATIONS

★ **Prognostic variables:** $\mathbf{u}$, ρ, s, q_m, q_r

- $\mathbf{u}$: velocity
- ρ: total mass density
- s: specific entropy
- q_m: mass fraction of airborne moisture
- q_r: mass fraction of precipitation

★ **Prognostic equations:**

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = \mathbf{F}$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = -\nabla \cdot (q_r \rho \mathbf{U})$$

$$\frac{Ds}{Dt} = \frac{1}{\rho}\left[s\nabla \cdot (q_r \rho \mathbf{U}) - \nabla \cdot (q_r \rho s_r \mathbf{U})\right]$$

$$\frac{Dq_m}{Dt} = \frac{1}{\rho}\left[-Q_r + q_m \nabla \cdot (q_r \rho \mathbf{U})\right]$$

$$\frac{Dq_r}{Dt} = \frac{1}{\rho}\left[Q_r - (1 - q_r)\nabla \cdot (q_r \rho \mathbf{U})\right]$$

SUMMARY OF DIAGNOSTIC EQUATIONS

★ Associated with the prognostic equations is a set of diagnostic equations that is required to determine

$$p,\ s_r,\ \mathbf{U},\ Q_r,\ \mathbf{F}$$

★ The diagnostic set of equations can be divided into 3 subsets:

- Thermodynamic diagnosis (p and s_r)
- Precipitation microphysics diagnosis ($\mathbf{U}$ and Q_r)
- Turbulent boundary layer parameterization ($\mathbf{F}$)

★ The thermodynamic diagnosis plays an important part in the derivation of moist available potential energy (the other diagnostic subsets are not involved).

THERMODYNAMIC DIAGNOSIS

Input$\{\rho, s, q_m, q_r\} \longrightarrow$ Output$\{\rho_a, \rho_m, \rho_v, \rho_c, \rho_r, T_1, T_2, T, s_r, p_a, p_v, p\}$

⋆ **Diagnostic sequence:** How to compute the thermodynamic diagnostic variables given the prognostic (or thermodynamic state) variables ρ, s, q_m, q_r.

$$q_a = 1 - q_m - q_r, \quad \rho_a = q_a\rho, \quad \rho_m = q_m\rho, \quad \rho_r = q_r\rho,$$

$$q_a s_a(\rho, q_a, T_2) + q_m s_m^{(2)}(\rho, q_m, T_2) + q_r C(T_2) = s$$

$$q_a s_a(\rho, q_a, T_1) + q_m s_m^{(1)}(\rho, q_m, T_1) + q_r C(T_2) = s$$

$$T = \max(T_1, T_2), \qquad p_a = \rho_a R_a T$$

$$\begin{cases} \rho_v = \rho_m, & \rho_c = 0, & p_v = \rho_v R_v T, & \text{if } T = T_1, \\ \rho_v = \rho_v^*(T), & \rho_c = \rho_m - \rho_v, & p_v = E(T), & \text{if } T = T_2, \end{cases}$$

$$p = p_a + p_v, \qquad s_r = C(T_2)$$

REDUCTION TO DRY MODEL (ADIABATIC)

- ⋆ Consider the limiting form of these equations for the special case of a perfectly dry atmosphere:

$$q_m = q_r = 0 \qquad \rho = \rho_a \qquad p = p_a$$

- ⋆ The prognostic equations for q_m and q_r are dropped and all but two of the diagnostic equations are dropped, so the model equations reduce to:

Prognostic Equations

$$\frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = \mathbf{F}$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0$$

$$\frac{Ds}{Dt} = 0$$

Diagnostic Equations

$$c_{va} \ln\left(\frac{T}{T_0}\right) - R_a \ln\left(\frac{\rho}{\rho_0}\right) = s$$

$$p = \rho R_a T$$

EFFECT OF ENTROPY TRANSPORT BY PRECIPITATION IN AN AXISYMMETRIC TROPICAL CYCLONE MODEL

- ⋆ Control experiment (CNTL) uses:

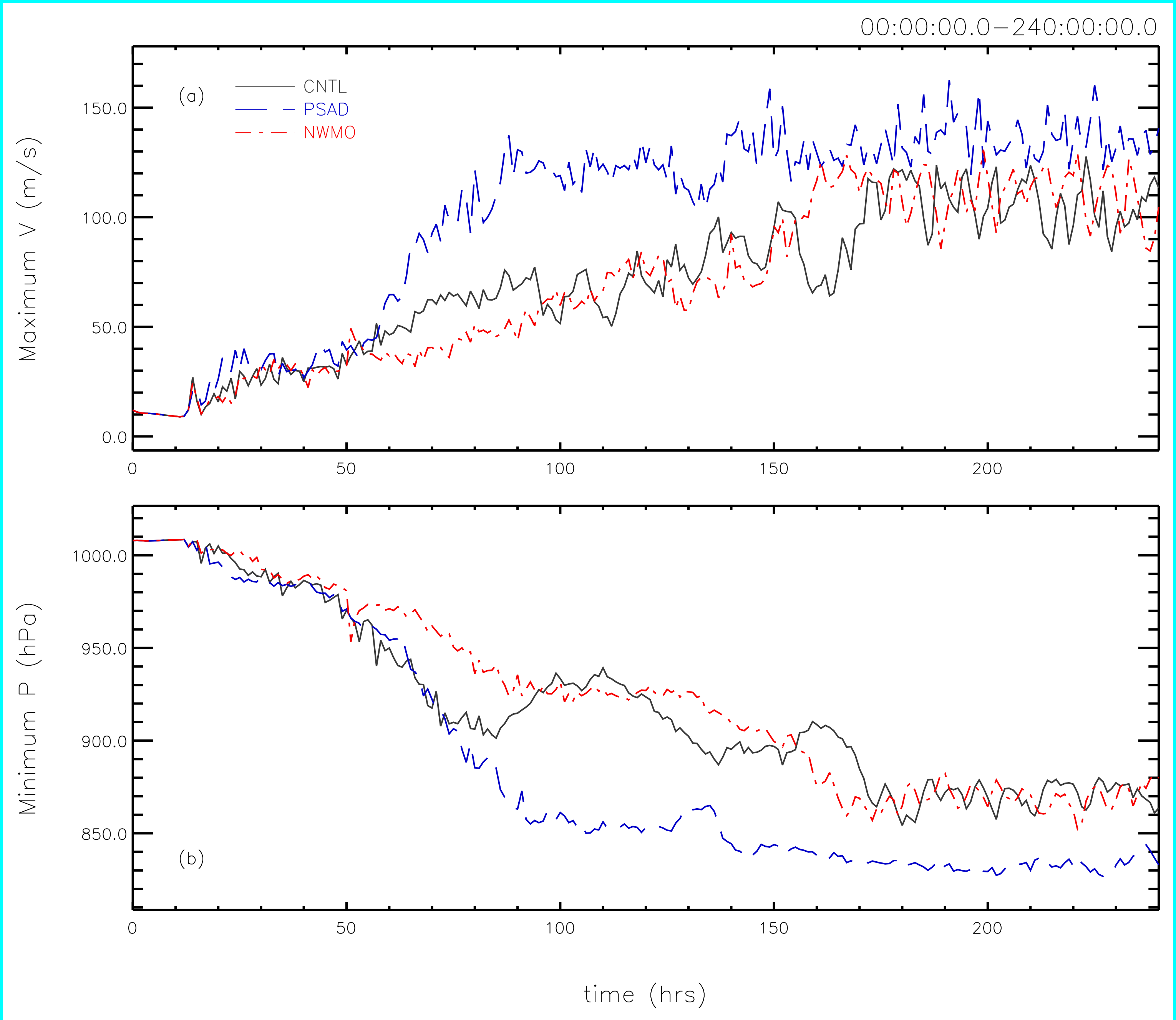
$$\frac{Ds}{Dt} = \frac{1}{\rho}\left[s\nabla \cdot (q_r \rho \mathbf{U}) - \nabla \cdot (q_r \rho s_r \mathbf{U})\right]$$

- ⋆ Pseudo-adiabatic experiment (PSAD) uses:

$$\frac{Ds}{Dt} = 0$$

- ⋆ Reference:

Hausman, S. A., K. V. Ooyama, and W. H. Schubert, 2005: Potential vorticity structure of simulated hurricanes. *J. Atmos. Sci.*, in press.



★ From Hausman et al. (2005)

SUMMARY

- ⋆ The effect of vertical transport of entropy by precipitation is easily included in the thermodynamic equation.
- ⋆ An axisymmetric tropical cyclone model has been run with and without the vertical transport of entropy by precipitation.
- ⋆ With this vertical transport, convective downdrafts are more intense and the boundary layer θ_e is lower.
- ⋆ Without this vertical transport, model storms are stronger and have approximately 10 K higher values of boundary layer θ_e.
- ⋆ The inclusion of vertical transport of entropy by precipitation may be important in future MMAP models.