

# **Dream GCMs and Superparameterization**

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# GOALS

- ★ Global, nonhydrostatic model of the moist, precipitating atmosphere
- ★ “Exact” treatment of moist thermodynamics
- ★ Inclusion of ice
- ★ Inclusion of precipitation effects on budgets of mass, entropy, momentum
- ★ Clearly defined associated “conservation principles” such as the moist generalizations of the Ertel potential vorticity principle and the Lorenz available potential energy principle

# “GOLDILOCKS” MODEL

*Just right for both climate research and cloud superparameterization research.*

## ★ Mass Types:

- dry air
- water vapor

- dry air
- water vapor
- airborne condensate

- dry air
- water vapor
- airborne condensate
- precipitation

- dry air
- water vapor
- airborne condensate
- rain, snow, graupel, hail, aggregates, ...

## ★ Thermodynamics:

- approximate moist thermodynamics

- exact, moist, equilibrium thermodynamics with precipitation effects on entropy budget

- exact, moist, nonequilibrium thermodynamics

# “GOLDILOCKS” MODEL (continued)

*Just right for both climate research and cloud superparameterization research.*

## ★ Dynamics:

- hydrostatic

- nonhydrostatic
- one vector momentum equation
- precipitation momentum diagnostic

- nonhydrostatic
- many vector momentum equations
- precipitation momentum prognostic

## ★ Microphysics:

- bulk
- no ice

- bulk
- with ice

- spectral
- with curvature and solute effects

# MODELING ASSUMPTIONS

★ Four types of matter:

<b>Type of Matter</b>	<b>Density</b>
dry air	$\rho_a$
water vapor	$\rho_v$
airborne condensate (water or ice)	$\rho_c$
precipitating condensate (water or ice)	$\rho_r$
airborne moisture	$\rho_m = \rho_v + \rho_c$
all	$\rho = \rho_a + \rho_v + \rho_c + \rho_r$

★ Predict  $\rho_m$ , not  $\rho_v$  and  $\rho_c$  separately.

★ Determine  $\rho_v$  and  $\rho_c$  later at the diagnostic stage.

- If the air is not cloudy ( $\rho_c = 0$ ), all of the predicted  $\rho_m$  is  $\rho_v$ .
- If the air is cloudy,  $\rho_v$  is the saturation value and the remainder of  $\rho_m$  is  $\rho_c$ .
- This involves equilibrium thermodynamics.

# MODELING ASSUMPTIONS (continued)

- ★ Dry air and water vapor satisfy the ideal gas law:

$$p_a = \rho_a R_a T \quad \text{and} \quad p_v = \rho_v R_v T$$

$$R_a = 287.05 \text{ J kg}^{-1} \text{ K}^{-1} \quad (\text{gas constant for dry air})$$

$$R_v = 461.51 \text{ J kg}^{-1} \text{ K}^{-1} \quad (\text{gas constant for water vapor})$$

- ★ Dalton's Law of Partial Pressures:

$$p = p_a + p_v$$

- ★ Specific heats are constant:

$$c_{pa} = 1004.675 \text{ J kg}^{-1} \text{ K}^{-1} \quad (\text{dry air at constant pressure})$$

$$c_{pv} = 1850.0 \text{ J kg}^{-1} \text{ K}^{-1} \quad (\text{water vapor at constant pressure})$$

$$c_{va} = c_{pa} - R_a \quad (\text{dry air at constant volume})$$

$$c_{vv} = c_{pv} - R_v \quad (\text{water vapor at constant volume})$$

# THE NEW GOFF FORMULAS

★ For saturation vapor pressure over water:

$$\begin{aligned}\log_{10} E_w(T) = & 10.79574 \left(1 - \frac{T_{tp}}{T}\right) - 5.02800 \log_{10} \left(\frac{T_{tp}}{T}\right) \\ & + 1.50475 \times 10^{-4} \left[1 - 10^{-8.2969(T/T_{tp}-1)}\right] \\ & + 0.42873 \times 10^{-3} \left[10^{-4.76955(1-T_{tp}/T)} - 1\right] + 0.78614\end{aligned}$$

★ For saturation vapor pressure over ice:

$$\begin{aligned}\log_{10} E_i(T) = & -9.09685 \left(\frac{T_{tp}}{T} - 1\right) - 3.56654 \log_{10} \left(\frac{T_{tp}}{T}\right) \\ & + 0.87682 \left(1 - \frac{T}{T_{tp}}\right) + 0.78614\end{aligned}$$

- Note:
- $E_w(T)$  and  $E_i(T)$  are expressed in hectopascals
  - $T$  is expressed in Kelvins
  - $T_{tp} = 273.16$  K is the triple point of water

# SPECIFIED FUNCTIONS OF $T$

★ Given  $E_i(T)$  and  $E_w(T)$  from the new Goff formulas, synthesize  $E(T)$  so that it transitions from  $E_i(T)$  to  $E_w(T)$  as  $T$  passes through  $T_0 = 273.15$  K.

★ From  $E(T)$  compute:

$$L(T) = R_v T^2 \frac{d \ln E(T)}{dT}$$

(Clausius-Clapeyron equation)

$$\rho_v^*(T) = \frac{E(T)}{R_v T}$$

(density of saturated vapor)

$$c_c(T) = c_{pv} - \frac{dL(T)}{dT}$$

(Kirchhoff equation)

★ Entropy of a unit mass of condensate:

$$C(T) = \int_{T_0}^T \frac{c_c(T')}{T'} dT'$$

Note :  $C(T_0) = 0$

# SPECIFIED FUNCTIONS OF $T$ (continued)

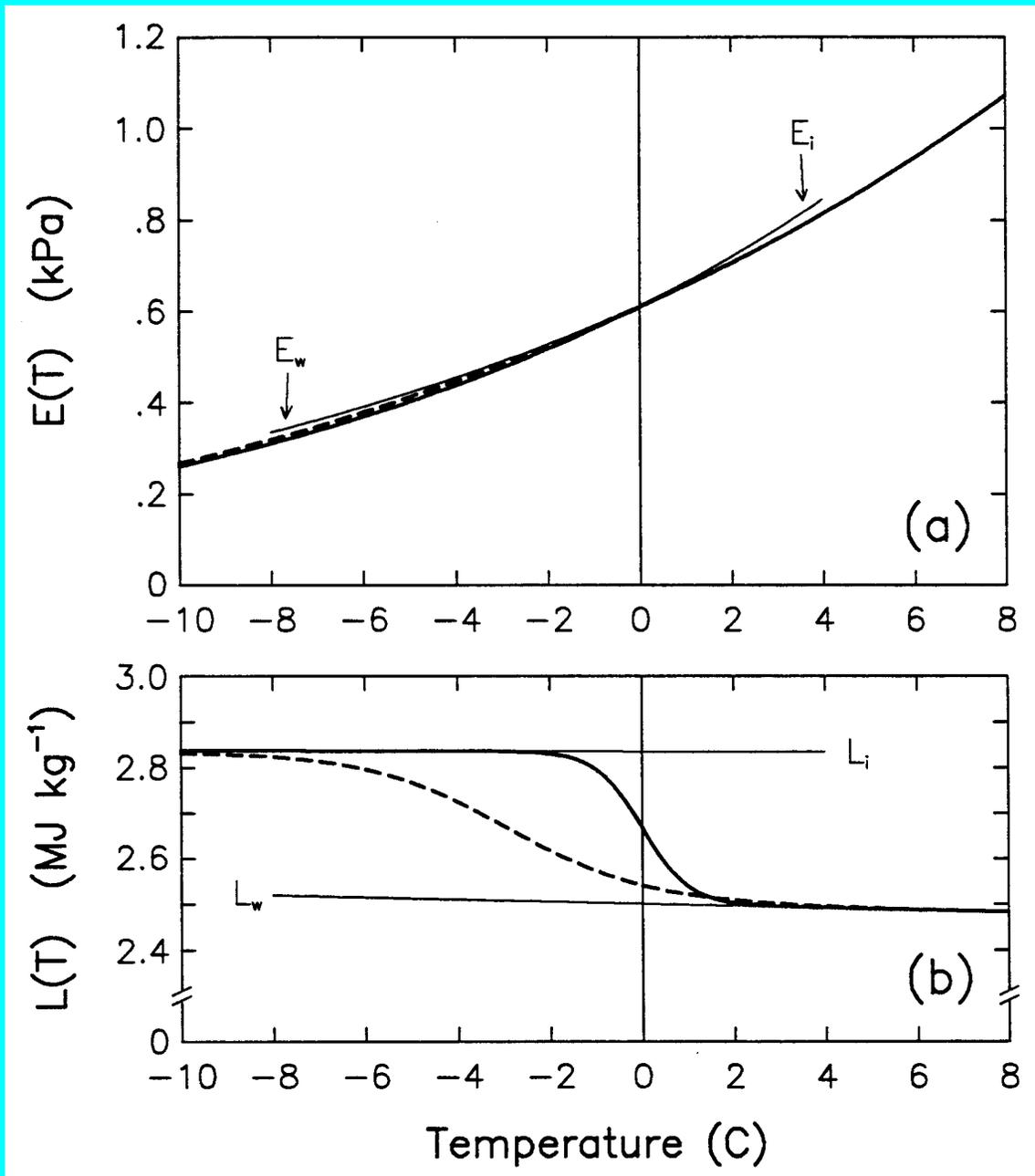
★ Using the Kirchhoff equation for  $c_c(T)$ :

$$C(T) = c_{vv} \ln \left( \frac{T}{T_0} \right) - R_v \ln \left( \frac{\rho_v^*(T)}{\rho_v^*(T_0)} \right) - \frac{L(T)}{T} + \frac{L(T_0)}{T_0}$$

★ Gain of entropy density by evaporating sufficient water to saturate the volume at  $T$ :

$$D(T) = \frac{dE(T)}{dT}$$

# $E(T)$ AND $L(T)$ VERSUS $T$



- Thin solid curves: based on new Goff formulas
- Thick solid curves: 1 K wide freezing zone centered at  $0^\circ\text{C}$
- Dashed curves: 3 K wide freezing zone centered at  $-3^\circ\text{C}$
- From Ooyama (1990)

# PROGNOSTIC EQUATIONS FOR MASS

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{u}) = 0$$

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u} + \mathbf{F}_m) = -Q_r$$

$$\frac{\partial \rho_r}{\partial t} + \nabla \cdot [\rho_r (\mathbf{u} + \mathbf{U})] = Q_r$$

$\mathbf{F}_m$ : Turbulent flux of water vapor

$Q_r$ : Conversion rate of  $\rho_m$  to  $\rho_r$

$\mathbf{U}$ : Terminal fall velocity of precip relative to air

★ The prognostic equation for the total density ( $\rho = \rho_a + \rho_m + \rho_r$ ):

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = -\nabla \cdot (\mathbf{F}_m + q_r \rho \mathbf{U})$$

★ The equations for  $\rho_m$  and  $\rho_r$  can be rewritten in terms of mass fractions:

$$\frac{Dq_m}{Dt} = \frac{1}{\rho} [-Q_r - \nabla \cdot \mathbf{F}_m + q_m \nabla \cdot (\mathbf{F}_m + q_r \rho \mathbf{U})]$$

$$q_m = \rho_m / \rho$$

$$\frac{Dq_r}{Dt} = \frac{1}{\rho} [Q_r - \nabla \cdot (q_r \rho \mathbf{U}) + q_r \nabla \cdot (\mathbf{F}_m + q_r \rho \mathbf{U})]$$

$$q_r = \rho_r / \rho$$

# EQUATION OF MOTION

- ★ Momentum equations for  $\rho_{am} = \rho_a + \rho_m$  and  $\rho_r$ :

$$\frac{\partial(\rho_{am}\mathbf{u})}{\partial t} + \nabla \cdot (\rho_{am}\mathbf{u}\mathbf{u}) + \rho_{am}2\boldsymbol{\Omega} \times \mathbf{u} + \rho_{am}\nabla\Phi + \nabla p = \rho_{am}\mathbf{F}_{am} - \rho_r\mathbf{F}_r - Q_r\mathbf{u} \quad (1)$$

$$\frac{\partial(\rho_r\mathbf{u}_r)}{\partial t} + \nabla \cdot (\rho_r\mathbf{u}_r\mathbf{u}_r) + \rho_r2\boldsymbol{\Omega} \times \mathbf{u}_r + \rho_r\nabla\Phi = \rho_r\mathbf{F}_r + Q_r\mathbf{u}_r \quad (2)$$

where

$\Phi$ : Potential for sum of Newtonian gravitational force and centrifugal force

$\mathbf{F}_r$ : Vertical drag force acting on  $\rho_r$

- ★ No pressure gradient force in (2) since the fractional volume of precipitation is assumed to be negligibly small.
- ★ **Basic Problem:** Serious contradiction in using both (1) and (2), since prediction of both  $\mathbf{u}$  and  $\mathbf{u}_r$  is equivalent to prediction of  $\mathbf{U}$ , which we are assuming is a diagnostic variable.

# EQUATION OF MOTION (continued)

★ Summing (1) and (2) yields:

$$\frac{D\mathbf{u}}{Dt} + \frac{\rho_r}{\rho} \left( \frac{D^{(r)}\mathbf{U}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{U} \right) + 2\boldsymbol{\Omega} \times \mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = \mathbf{F}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad \frac{D^{(r)}}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_r \cdot \nabla \quad \mathbf{F} = \frac{1}{\rho} [\rho_{am}\mathbf{F}_{am} - \rho_r(\mathbf{U} \cdot \nabla)\mathbf{u}]$$

★ Our approximation is:

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = \mathbf{F}$$

# ENTROPY DENSITY EQUATION

- ★ At this stage we don't know how to write the internal energy or enthalpy forms of the first law of thermodynamics for a complicated moist model. However, we can write the budget equation for the total entropy density.
- ★ Entropy density is an additive quantity:

$$\sigma = \sigma_a + \sigma_m + \sigma_r$$

- $\sigma$ : Total entropy density
- $\sigma_a$ : Entropy density of dry air
- $\sigma_m$ : Entropy density of airborne moisture
- $\sigma_r$ : Entropy density of precipitation

- ★ Since the flux of  $\sigma$  is given by  $\sigma_a \mathbf{u} + \sigma_m \mathbf{u} + \sigma_r (\mathbf{u} + \mathbf{U}) + \mathbf{F}_\sigma$  the budget equation for  $\sigma$  is

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \mathbf{u} + \mathbf{F}_\sigma + \sigma_r \mathbf{U}) = Q_\sigma$$

where

- $\mathbf{F}_\sigma$ : Turbulent flux of entropy
- $Q_\sigma$ : Source term due to radiation

# SPECIFIC ENTROPY EQUATION

- ★ Define the specific entropy of moist air as

$$s = \frac{\sigma}{\rho}$$

- ★ The prognostic equation for  $s$  is then

$$\frac{Ds}{Dt} = \frac{1}{\rho} [Q_\sigma - \nabla \cdot (\mathbf{F}_\sigma + q_r \rho s_r \mathbf{U}) + s \nabla \cdot (\mathbf{F}_m + q_r \rho \mathbf{U})]$$

- ★ Note that in the absence of precipitation ( $q_r = 0$ ), turbulent fluxes of entropy and water vapor ( $\mathbf{F}_\sigma = \mathbf{F}_m = 0$ ), and radiation ( $Q_\sigma = 0$ ):

$$\frac{Ds}{Dt} = 0$$

# SPECIFIC ENTROPY OF MOIST AIR

★ Specific entropy of moist air:

$$s = \frac{\sigma}{\rho} = \frac{\rho_a s_a + \rho_m s_m + \rho_r s_r}{\rho} = q_a s_a + q_m s_m + q_r s_r$$

★ Specific entropy of dry air:

$$s_a(\rho, q_a, T) = c_{va} \ln \left( \frac{T}{T_0} \right) - R_a \ln \left( \frac{q_a \rho}{\rho_{a0}} \right)$$

★ Specific entropy of precipitation:

$$s_r = C(T_2)$$

# SPECIFIC ENTROPY OF MOIST AIR (continued)

- ★ Note that the relaxation time for a raindrop to adjust to the wetbulb temperature is 4 to 5 seconds. The rapidity of this adjustment is due to the fact that the amount of heat required to adjust the temperature of the drop is much smaller than that required to evaporate it.
- ★ Specific entropy of airborne moisture:
  - State 1 (no airborne condensate):

$$s_m^{(1)}(\rho, q_m, T) = c_{vv} \ln \left( \frac{T}{T_0} \right) - R_v \ln \left( \frac{q_m \rho}{\rho_v^*(T_0)} \right) + \frac{L(T_0)}{T_0}$$

- State 2 (saturated vapor):

$$s_m^{(2)}(\rho, q_m, T) = C(T) + \frac{D(T)}{q_m \rho}$$

# SUMMARY OF PROGNOSTIC EQUATIONS

★ **Prognostic variables:**  $\mathbf{u}$ ,  $\rho$ ,  $s$ ,  $q_m$ ,  $q_r$

$\mathbf{u}$ : velocity

$\rho$ : total mass density

$s$ : specific entropy

$q_m$ : mass fraction of airborne moisture

$q_r$ : mass fraction of precipitation

★ **Prognostic equations:**

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = \mathbf{F}$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = -\nabla \cdot (\mathbf{F}_m + q_r\rho\mathbf{U})$$

$$\frac{Ds}{Dt} = \frac{1}{\rho} [Q_\sigma - \nabla \cdot (\mathbf{F}_\sigma + q_r\rho s_r\mathbf{U}) + s\nabla \cdot (\mathbf{F}_m + q_r\rho\mathbf{U})]$$

$$\frac{Dq_m}{Dt} = \frac{1}{\rho} [-Q_r - \nabla \cdot \mathbf{F}_m + q_m\nabla \cdot (\mathbf{F}_m + q_r\rho\mathbf{U})]$$

$$\frac{Dq_r}{Dt} = \frac{1}{\rho} [Q_r - \nabla \cdot (q_r\rho\mathbf{U}) + q_r\nabla \cdot (\mathbf{F}_m + q_r\rho\mathbf{U})]$$

# SUMMARY OF DIAGNOSTIC EQUATIONS

- ★ Associated with the prognostic equations is a set of diagnostic equations that is required to determine

$$p, s_r, \mathbf{U}, Q_r, \mathbf{F}, \mathbf{F}_m, \mathbf{F}_\sigma, Q_\sigma$$

- ★ The diagnostic set of equations can be divided into 4 subsets:
  - Thermodynamic diagnosis ( $p$  and  $s_r$ )
  - Precipitation microphysics diagnosis ( $\mathbf{U}$  and  $Q_r$ )
  - Turbulent boundary layer parameterization ( $\mathbf{F}$ ,  $\mathbf{F}_m$  and  $\mathbf{F}_\sigma$ )
  - Radiative parameterization ( $Q_\sigma$ )
- ★ The thermodynamic diagnosis plays an important part in the derivation of moist available potential energy (the other 3 diagnostic subsets are not involved).

# THERMODYNAMIC DIAGNOSIS

Input  $\{\rho, s, q_m, q_r\} \longrightarrow$  Output  $\{\rho_a, \rho_m, \rho_v, \rho_c, \rho_r, T_1, T_2, T, s_r, p_a, p_v, p\}$

- ★ **Diagnostic sequence:** How to compute the thermodynamic diagnostic variables given the prognostic (or thermodynamic state) variables  $\rho, s, q_m, q_r$ .

$$q_a = 1 - q_m - q_r, \quad \rho_a = q_a \rho, \quad \rho_m = q_m \rho, \quad \rho_r = q_r \rho,$$

$$q_a s_a(\rho, q_a, T_2) + q_m s_m^{(2)}(\rho, q_m, T_2) + q_r C(T_2) = s$$

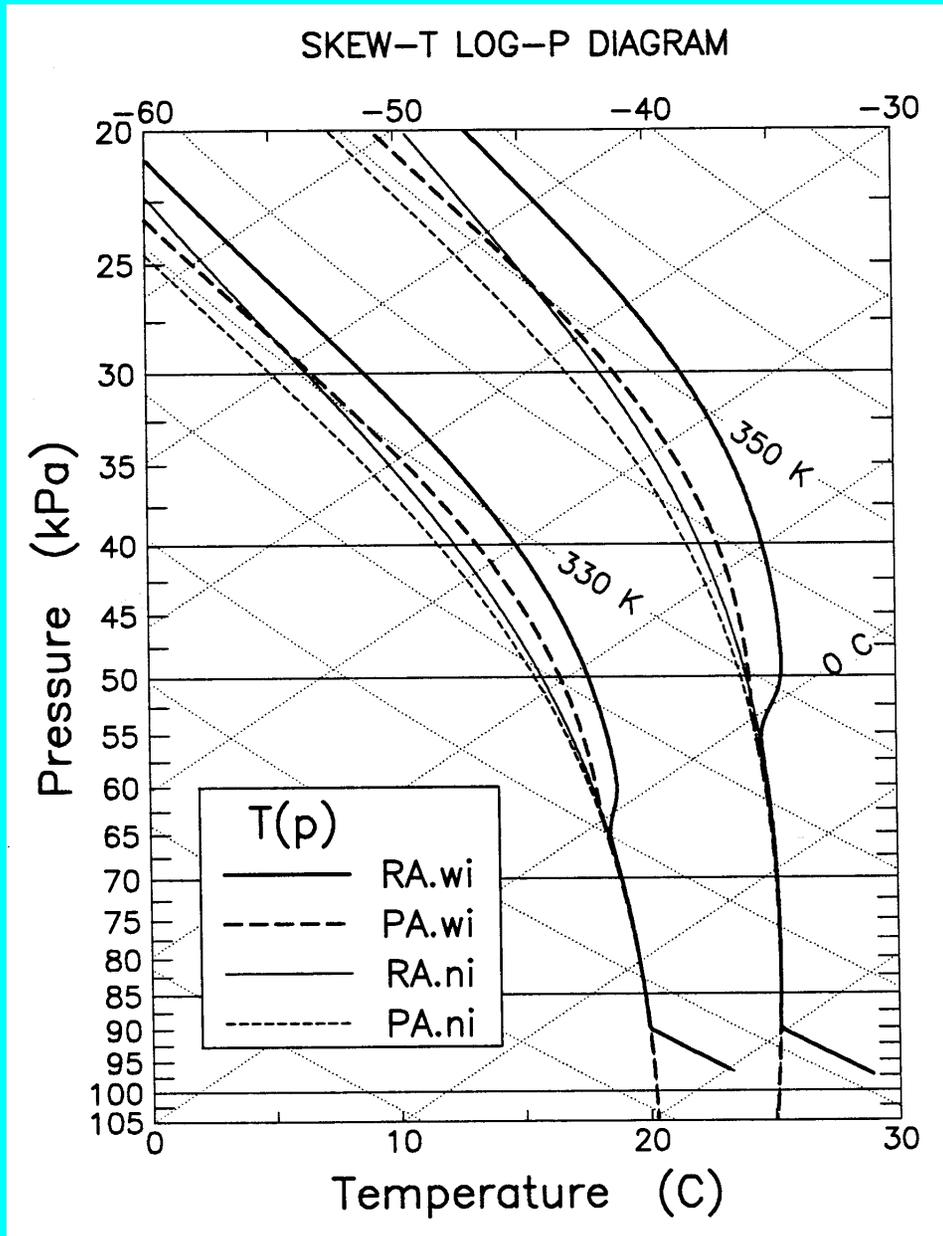
$$q_a s_a(\rho, q_a, T_1) + q_m s_m^{(1)}(\rho, q_m, T_1) + q_r C(T_2) = s$$

$$T = \max(T_1, T_2), \quad p_a = \rho_a R_a T$$

$$\begin{cases} \rho_v = \rho_m, & \rho_c = 0, & p_v = \rho_v R_v T, & \text{if } T = T_1, \\ \rho_v = \rho_v^*(T), & \rho_c = \rho_m - \rho_v, & p_v = E(T), & \text{if } T = T_2, \end{cases}$$

$$p = p_a + p_v, \quad s_r = C(T_2)$$

# SKEW-T LOG-P DIAGRAM SHOWING $T_1$ and $T_2$



★ From Ooyama (1990)

# REDUCTION TO DRY MODEL (ADIABATIC)

- ★ Consider the limiting form of these equations for the special case of a perfectly dry atmosphere:

$$q_m = q_r = 0 \quad \rho = \rho_a \quad p = p_a$$

- ★ The prognostic equations for  $q_m$  and  $q_r$  are dropped and all but two of the diagnostic equations are dropped, so the model equations reduce to:

## Prognostic Equations

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = \mathbf{F}$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0$$

$$\frac{Ds}{Dt} = 0$$

## Diagnostic Equations

$$c_{va} \ln\left(\frac{T}{T_0}\right) - R_a \ln\left(\frac{\rho}{\rho_0}\right) = s$$

$$p = \rho R_a T$$

# MOIST POTENTIAL VORTICITY

- ★ Consider the equation of motion:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) - \mathbf{u} \times \boldsymbol{\zeta} = -\nabla \Phi - \frac{1}{\rho} \nabla p + \mathbf{F}$$

- ★ Taking the curl gives

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\zeta}) = \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nabla \times \mathbf{F}$$

- ★ Scalar multiplication by  $\nabla \psi$  for some scalar field  $\psi$  yields

$$\nabla \psi \cdot \frac{\partial \boldsymbol{\zeta}}{\partial t} + \nabla \cdot [\nabla \psi \times (\mathbf{u} \times \boldsymbol{\zeta})] = \frac{1}{\rho^2} \nabla \psi \cdot (\nabla \rho \times \nabla p) + (\nabla \times \mathbf{F}) \cdot \nabla \psi$$

which can be rewritten as

$$\frac{D}{Dt} (\boldsymbol{\zeta} \cdot \nabla \psi) + (\boldsymbol{\zeta} \cdot \nabla \psi) \nabla \cdot \mathbf{u} = \boldsymbol{\zeta} \cdot \nabla \dot{\psi} + (\nabla \times \mathbf{F}) \cdot \nabla \psi + \frac{1}{\rho^2} \nabla \psi \cdot (\nabla \rho \times \nabla p)$$

# MOIST POTENTIAL VORTICITY (continued)

- ★ Using the continuity equation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} + \nabla \cdot (\mathbf{F}_m + \rho_r \mathbf{U}) = 0$$

to eliminate  $\nabla \cdot \mathbf{u}$  gives

$$\rho \frac{D}{Dt} \left( \frac{1}{\rho} \zeta \cdot \nabla \psi \right) = \zeta \cdot \nabla \psi + (\nabla \times \mathbf{F}) \cdot \nabla \psi + \frac{1}{\rho^2} \nabla \psi \cdot (\nabla \rho \times \nabla p) + (\zeta \cdot \nabla \psi) \nabla \cdot (\rho_r \mathbf{U})$$

- ★ By choosing  $\psi$  properly, this equation for the moist PV can be simplified further. In particular, the solenoidal term

$$\frac{1}{\rho^2} \nabla \psi \cdot (\nabla \rho \times \nabla p)$$

can be eliminated.

# ANNIHILATION OF THE SOLENOIDAL TERM

- ★ **Virtual temperature**  $T_\rho$ : Temperature that dry air would have if its pressure and density were equal to those of the given sample of moist air

$$T_\rho = \frac{p}{\rho R_a} = \frac{p_a + p_v}{(\rho_a + \rho_m + \rho_r) R_a}$$

- ★ **Virtual potential temperature**  $\theta_\rho$ :

$$\theta_\rho = T_\rho \left( \frac{p_0}{p} \right)^\kappa = \frac{p}{\rho R_a} \left( \frac{p_0}{p} \right)^\kappa \quad \left( \kappa = \frac{R_a}{c_{pa}} \right)$$

- ★ Now choose  $\psi = \theta_\rho$ .

Since  $\theta_\rho$  can be written as a function of  $p$  and  $\rho$  only,

$$\nabla \theta_\rho \cdot (\nabla p \times \nabla \rho) = \left[ \left( \frac{\partial \theta_\rho}{\partial p} \right)_\rho \nabla p + \left( \frac{\partial \theta_\rho}{\partial \rho} \right)_p \nabla \rho \right] \cdot (\nabla p \times \nabla \rho) = 0$$

## THE SOLENOIDAL TERM (continued)

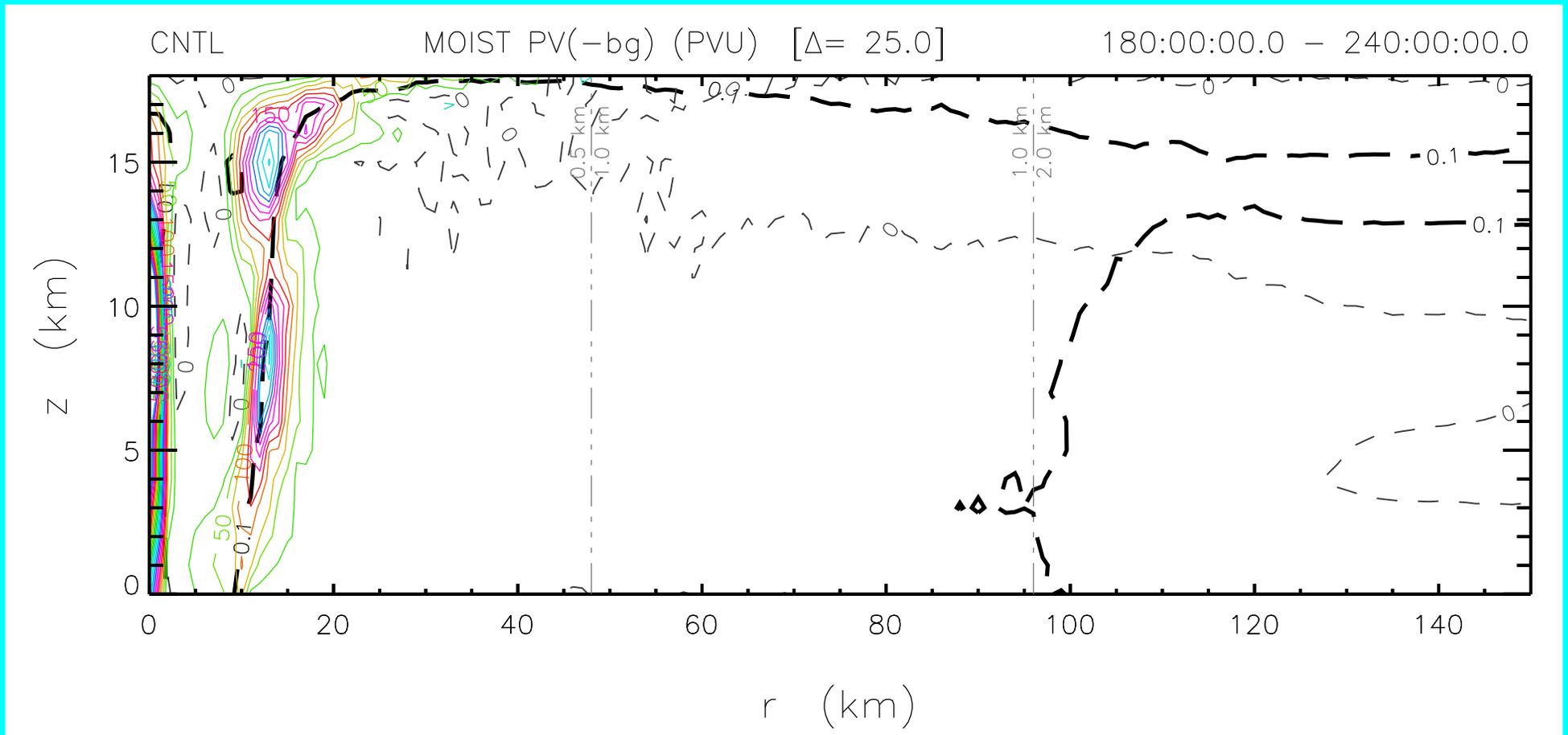
★ With the choice  $\psi = \theta_\rho$ , the moist potential vorticity equation simplifies to

$$\frac{DP}{Dt} = \frac{1}{\rho} \zeta \cdot \nabla \dot{\theta}_\rho + \frac{1}{\rho} (\nabla \times \mathbf{F}) \cdot \nabla \theta_\rho + P \nabla \cdot (\rho_r \mathbf{U})$$

where  $P$  is the **moist potential vorticity** defined as

$$P = \frac{1}{\rho} \zeta \cdot \nabla \theta_\rho$$

# MOIST PV IN AN AXISYMMETRIC HURRICANE



★ From Hausman et al. (2003)

# INVERTIBILITY PRINCIPLE FOR $P$

- ★ **Does there exist an invertibility principle for  $P$ ? Yes!**

The importance of the existence of an invertibility principle is hard to overemphasize. It is the existence of such a principle that makes  $P$  such a dynamically interesting quantity.

- ★ With an invertibility principle, the balanced wind and mass fields can be determined diagnostically from the potential vorticity field  $P$ .

- ★ The choice of vertical coordinate has a significant effect on the form and complexity of the invertibility principle.

# INVERTIBILITY PRINCIPLE IN THE $Z$ -COORDINATE

- ★ Consider an  $f$ -plane case in which a large-scale axisymmetric flow has a slowly evolving  $P$  field. Since the evolution is slow, we can consider the tangential wind and mass fields as continuously changing from one hydrostatic and gradient balanced state to another.
- ★ The invertibility problem is to diagnostically determine, at each time, the tangential wind  $v(r, z, t)$ , the total density  $\rho(r, z, t)$ , the total pressure  $p(r, z, t)$ , and the virtual potential temperature  $\theta_\rho(r, z, t)$  from

$$\rho \left( f + \frac{v}{r} \right) v = \frac{\partial p}{\partial r}$$

$$-g\rho = \frac{\partial p}{\partial z}$$

$$\theta_\rho = \frac{p}{\rho R_a} \left( \frac{p_0}{p} \right)^\kappa$$

$$\frac{1}{\rho} \left[ -\frac{\partial v}{\partial z} \frac{\partial \theta_\rho}{\partial r} + \left( f + \frac{\partial(rv)}{r\partial r} \right) \frac{\partial \theta_\rho}{\partial z} \right] = P$$

- ★ For a given  $P(r, z, t)$ , this constitutes a system of four equations for the four unknowns  $v(r, z, t)$ ,  $\rho(r, z, t)$ ,  $p(r, z, t)$ , and  $\theta_\rho(r, z, t)$ .

# “SPICE”

- ★ The solution of the invertibility problem provides the total pressure  $p(r, z, t)$ , the total density  $\rho(r, z, t)$ , and the virtual potential temperature  $\theta_\rho(r, z, t)$ .
- ★ The partitioning of  $p$  between  $p_a$  and  $p_v$  and of  $\rho$  between  $\rho_a$ ,  $\rho_m$  and  $\rho_r$ , however, cannot be determined from knowledge of  $P$  only.

***The solution of the invertibility problem only gives the parts of the mass field which are of direct dynamical significance.***

- ★ If the PV anomaly is localized, the resulting anomalies in the  $v, p, \rho, T_\rho, \theta_\rho$  fields decay smoothly over a Rossby length in the horizontal and over a Rossby depth in the vertical.
- ★ However,  $T$  can have smaller scale fluctuations which are compensated for by moisture fluctuations so that  $T_\rho$  remains smooth.

***A dynamically adjusted moist atmosphere can have a certain amount of “spice” in its temperature and moisture fields.***

# INVERTIBILITY PRINCIPLE IN THE $\hat{z}$ -COORDINATE

- ★ Pseudo-height coordinate:  
(based on total pressure)

$$\hat{z} = \frac{c_{pa} T_0}{g} \left[ 1 - \left( \frac{p}{p_0} \right)^\kappa \right]$$

- ★ Transformation of the invertibility principle:

$$\frac{T_0}{g \hat{\rho}} \left\{ \left[ f^2 + \frac{\partial}{r^3 \partial r} \left( r^3 \frac{\partial \phi}{\partial r} \right) \right] \frac{\partial^2 \phi}{\partial \hat{z}^2} - \left( \frac{\partial^2 \phi}{\partial r \partial \hat{z}} \right)^2 \right\} \left( f^2 + \frac{4}{r} \frac{\partial \phi}{\partial r} \right)^{-\frac{1}{2}} = P$$

$$\phi \rightarrow \tilde{\phi}(\hat{z}) \quad \text{as } r \rightarrow \infty$$

$$\frac{\partial \phi}{\partial \hat{z}} = \frac{g}{T_0} \theta_\rho \quad \text{at lower and upper boundaries}$$

where

$$\hat{\rho}(\hat{z}) = \rho_0 \left( 1 - \frac{g \hat{z}}{c_{pa} T_0} \right)^{(1-\kappa)/\kappa}$$

# INVERTIBILITY PRINCIPLE IN THE $\hat{z}$ -COORDINATE (continued)

- ★ This invertibility principle is a single nonlinear, second order PDE with the indicated boundary conditions.
- ★ Given a potential vorticity field  $P(r, \hat{z}, t)$ , the invertibility principle yields  $\phi(r, \hat{z}, t)$ , from which  $v(r, \hat{z}, t)$  and  $\theta_\rho(r, \hat{z}, t)$  can then be calculated using

$$\left(f + \frac{v}{r}\right) v = \frac{\partial \phi}{\partial r}$$

$$\frac{g}{T_0} \theta_\rho = \frac{\partial \phi}{\partial \hat{z}}$$

# INVERTIBILITY PRINCIPLE IN THE $\theta_\rho$ -COORDINATE

- ★ Montgomery potential based on virtual temperature:

$$M = c_{pa}T_\rho + gz$$

- ★ Exner function and its derivative, based on total pressure:

$$\Pi = c_{pa} \left( \frac{p}{p_0} \right)^\kappa \quad \Gamma = \frac{d\Pi}{dp} = \frac{\kappa\Pi}{p}$$

- ★ Transformation of the invertibility principle:

$$-g\Gamma \left[ f^2 + \frac{\partial}{r^3 \partial r} \left( r^3 \frac{\partial M}{\partial r} \right) \right] \left( f^2 + \frac{4}{r} \frac{\partial M}{\partial r} \right)^{-\frac{1}{2}} \left( \frac{\partial^2 M}{\partial \theta_\rho^2} \right)^{-1} = P$$

$$\begin{aligned} M &\rightarrow \tilde{M}(\theta_\rho) && \text{as } r \rightarrow \infty \\ \frac{\partial M}{\partial \theta_\rho} &= \Pi && \text{at lower and upper boundaries} \end{aligned}$$

# INVERTIBILITY PRINCIPLE IN THE $\theta_\rho$ -COORDINATE (continued)

- ★ This invertibility principle is a single nonlinear, second order PDE with the indicated boundary conditions.
- ★ Given a potential vorticity field  $P(r, \hat{z}, t)$ , the invertibility principle yields  $M(r, \hat{z}, t)$ , from which  $\Pi(r, \hat{z}, t)$  and  $\theta_\rho(r, \hat{z}, t)$  can then be calculated using

$$\left(f + \frac{v}{r}\right) v = \frac{\partial M}{\partial r}$$

$$\Pi = \frac{\partial M}{\partial \theta_\rho}$$

# INTERNAL ENERGY EQUATION

- ★ To derive the moist available potential energy, the entropy equation must be converted into the internal energy or enthalpy form of the first law:

$$\frac{D\mathcal{E}}{Dt} = \mathcal{E}_\rho \frac{D\rho}{Dt} + \mathcal{E}_s \frac{Ds}{Dt} + \mathcal{E}_{q_m} \frac{Dq_m}{Dt} + \mathcal{E}_{q_r} \frac{Dq_r}{Dt}$$

where

$$\mathcal{E}(\rho, s, q_m, q_r) = (q_a c_{va} + q_m c_{vv})T + q_r c_{vv} T_2 - (q_c + q_r)[L(T_2) - R_v T_2]$$

- ★ This formula involves  $q_a, q_m, q_c, q_r, T_2, T$ , all of which are determined by the thermodynamic diagnosis.
- ★ The internal energy  $\mathcal{E}(\rho, s, q_m, q_r)$  is considered to be a function of the four prognostic variables  $\rho, s, q_m, q_r$ , since these are the inputs to the thermodynamic diagnosis.
- ★ Special cases:

$$\mathcal{E}(\rho, s, 0, 0) = c_{va} T \quad (\text{for dry air})$$

$$\mathcal{E}(\rho, s, q_m, 0) = (q_a c_{va} + q_m c_{vv}) T \quad (\text{for dry air plus water vapor})$$

# INTERNAL ENERGY EQUATION (continued)

★  $\mathcal{E}_\rho$ ,  $\mathcal{E}_s$ ,  $\mathcal{E}_{q_m}$ , and  $\mathcal{E}_{q_r}$  denote the four partial derivatives of  $\mathcal{E}(\rho, s, q_m, q_r)$ :

$$\mathcal{E}_\rho = \frac{p}{\rho^2} - q_r(T - T_2) \left( \frac{\partial s_r}{\partial \rho} \right)_{s, q_m, q_r}$$

$$\mathcal{E}_s = T - q_r(T - T_2) \left( \frac{\partial s_r}{\partial s} \right)_{\rho, q_m, q_r}$$

$$\mathcal{E}_{q_m} = T(s_a - c_{pa}) - T(s_m - c_{pv}) - \frac{q_c L(T_2)}{q_m} - q_r(T - T_2) \left( \frac{\partial s_r}{\partial q_m} \right)_{s, \rho, q_r}$$

$$\mathcal{E}_{q_r} = T(s_a - c_{pa}) - T_2(s_r - c_{pv}) - L(T_2) - (T - T_2) \left( \frac{\partial(q_r s_r)}{\partial q_r} \right)_{s, \rho, q_m}$$

★ Note that the last term in each of the above is nonzero only when precipitation falls through a cloudless atmosphere (i.e.,  $q_r \neq 0$  and  $T = T_1$ ).

# MOIST AVAILABLE POTENTIAL ENERGY

- ★ Taking the dot product of  $\mathbf{u}$  and the momentum equation yields:

$$\frac{D}{Dt} \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi \right) + \frac{1}{\rho} \nabla \cdot (p\mathbf{u}) = \mathbf{u} \cdot \mathbf{F} + \frac{p}{\rho} \nabla \cdot \mathbf{u}$$

Note that  $\mathbf{u} \cdot \nabla \Phi = D\Phi/Dt$  since  $\partial\Phi/\partial t = 0$ .

- ★ Adding this result to the internal energy equation gives

$$\begin{aligned} \frac{D}{Dt} \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + \mathcal{E} \right) + \frac{1}{\rho} \nabla \cdot (p\mathbf{u}) = \mathbf{u} \cdot \mathbf{F} - \frac{p}{\rho^2} \nabla \cdot (\mathbf{F}_m + q_r \rho \mathbf{U}) \\ + \left( \mathcal{E}_\rho - \frac{p}{\rho^2} \right) \frac{D\rho}{Dt} + \mathcal{E}_s \frac{Ds}{Dt} + \mathcal{E}_{q_m} \frac{Dq_m}{Dt} + \mathcal{E}_{q_r} \frac{Dq_r}{Dt} \end{aligned}$$

where the continuity equation was used to replace the  $\nabla \cdot \mathbf{u}$  term.

# REFERENCE STATE FOR MOIST APE

- ★ Useful definitions of “virtual” quantities:

$$T_\rho = \frac{p}{\rho R_a} \quad \theta_\rho = T_\rho \left( \frac{p_{00}}{p} \right)^\kappa \quad s_\rho = c_{pa} \ln \left( \frac{\theta_\rho}{T_{00}} \right)$$

- ★ The moist available potential energy is defined with respect to a specified reference state, denoted by a “0” subscript. For example, the reference state density, pressure, entropy, airborne moisture mass fraction, temperature, and virtual entropy are denoted by

$$\rho_0(z), p_0(z), s_0(z), q_{m0}(z), T_0(z), \text{ and } s_{\rho 0}(z)$$

respectively, with  $p_0(z)$  and  $\rho_0(z)$  related by

$$\nabla p_0 = -\rho_0 \nabla \Phi$$

- ★ The reference state is assumed to be a horizontally uniform, resting atmosphere in hydrostatic equilibrium and to be free of precipitation, although it may be cloudy.

# REFERENCE STATE FOR MOIST APE (continued)

- ★ For a reference state designed to resemble the mean tropical atmosphere,  $s_0(z)$  will not be monotonic in  $z$ , but will typically have a minimum near  $p_0(z) \approx 70$  kPa. The reference state virtual entropy  $s_{\rho 0}(z)$ , however, is assumed to increase monotonically with  $z$ , so that the reference state variables can be expressed as functions of  $s_\rho$ .
- ★ Expressed as functions of  $s_\rho$ , the reference state density, pressure, entropy, potential, airborne moisture mass fraction, temperature, and virtual temperature are

$$\tilde{\rho}(s_\rho), \tilde{p}(s_\rho), \tilde{s}(s_\rho), \tilde{\Phi}(s_\rho), \tilde{q}_m(s_\rho), \tilde{T}(s_\rho), \text{ and } \tilde{T}_\rho(s_\rho)$$

respectively, with  $\tilde{p}(s_\rho)$  and  $\tilde{\rho}(s_\rho)$  related by

$$\frac{d\tilde{p}}{ds_\rho} = -\tilde{\rho} \frac{d\tilde{\Phi}}{ds_\rho}$$

# MORE ON MOIST AVAILABLE POTENTIAL ENERGY

- ★ Let  $\mathcal{C}(s, q_m, q_r)$  denote a yet-to-be-determined Casimir function

$$\frac{D\mathcal{C}}{Dt} = C_s \frac{Ds}{Dt} + C_{q_m} \frac{Dq_m}{Dt} + C_{q_r} \frac{Dq_r}{Dt}$$

- ★ From the continuity equation it follows that

$$\frac{D}{Dt} \left( \frac{p_0}{\rho} \right) - \frac{1}{\rho} \nabla \cdot (p_0 \mathbf{u}) = \frac{p_0}{\rho^2} \nabla \cdot (\mathbf{F}_m + q_r \rho \mathbf{U})$$

- ★ Adding the above two equations and the previous energy equation and then converting to flux form gives:

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \rho \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + \mathcal{E} + \mathcal{C} + p_0/\rho \right] \right\} \\ & + \nabla \cdot \left\{ \rho \mathbf{u} \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + \mathcal{E} + \mathcal{C} + p_0/\rho \right] + \mathbf{u} (p - p_0) \right\} = \rho S \end{aligned}$$

## MORE ON MOIST APE (continued)

★ Note that

$$S = \mathbf{u} \cdot \mathbf{F} + \left( \varepsilon_\rho - \frac{p}{\rho^2} \right) \frac{D\rho}{Dt} + (\varepsilon_s + c_s) \frac{Ds}{Dt} + (\varepsilon_{q_m} + c_{q_m}) \frac{Dq_m}{Dt} \\ + (\varepsilon_{q_r} + c_{q_r}) \frac{Dq_r}{Dt} - \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + \varepsilon + c + \frac{p}{\rho} \right) \nabla \cdot (\mathbf{F}_m + q_r \rho \mathbf{U})$$

- ★ This constitutes a family of conservation relations, with each member of the family resulting from a particular choice of the Casimir function  $C(s, q_m, q_r)$ .
- ★ Our conservation equation involving moist available potential energy will result from one such choice.

# CHOICE OF CASIMIR FUNCTION

- ★ Choose  $\mathcal{C}(s, q_m, q_r)$  so that the disturbance (or eddy) part of

$$\rho \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + \mathcal{E}(\rho, s, q_m, q_r) + \mathcal{C}(s, q_m, q_r) + p_0/\rho \right]$$

can be written in a form which is explicitly second order in disturbance amplitude:

$$\begin{aligned} & \left\{ \rho \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + \mathcal{E}(\rho, s, q_m, q_r) + \mathcal{C}(s, q_m, q_r) + p_0/\rho \right] \right\}_e \\ &= \rho \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + \mathcal{E}(\rho, s, q_m, q_r) + \mathcal{C}(s, q_m, q_r) \right] \\ & \quad - \rho_0 \left[ \Phi + \mathcal{E}(\rho_0, s_0, q_{m0}, 0) + \mathcal{C}(s_0, q_{m0}, 0) \right] \end{aligned}$$

- ★ Rearrange the right hand side by adding and subtracting terms to obtain ...

# CHOICE OF CASIMIR FUNCTION (continued)

$$\begin{aligned}
 & \left\{ \rho \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + \mathcal{E}(\rho, s, q_m, q_r) + \mathcal{C}(s, q_m, q_r) + p_0/\rho \right] \right\}_e \\
 &= \rho \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \mathcal{E}(\rho, s, q_m, q_r) - \mathcal{E}(\rho_0, s_0, q_{m0}, 0) \right. \\
 &\quad - \mathcal{E}_\rho(\rho_0, s_0, q_{m0}, 0)(\rho - \rho_0) - \mathcal{E}_s(\rho_0, s_0, q_{m0}, 0)(s - s_0) \\
 &\quad - \mathcal{E}_{q_m}(\rho_0, s_0, q_{m0}, 0)(q_m - q_{m0}) - \mathcal{E}_{q_r}(\rho_0, s_0, q_{m0}, 0)q_r \\
 &\quad + \mathcal{C}(s, q_m, q_r) - \mathcal{C}(s_0, q_{m0}, 0) - \mathcal{C}_s(s_0, q_{m0}, 0)(s - s_0) \\
 &\quad \left. - \mathcal{C}_{q_m}(s_0, q_{m0}, 0)(q_m - q_{m0}) - \mathcal{C}_{q_r}(s_0, q_{m0}, 0)q_r \right] \\
 &+ \mathcal{E}_\rho(\rho_0, s_0, q_{m0}, 0)(\rho - \rho_0)^2 \\
 &+ [\Phi + \mathcal{E}(\rho_0, s_0, q_{m0}, 0) + \rho_0 \mathcal{E}_\rho(\rho_0, s_0, q_{m0}, 0) + \mathcal{C}(s_0, q_{m0}, 0)] (\rho - \rho_0) \\
 &+ \rho [\mathcal{E}_s(\rho_0, s_0, q_{m0}, 0) + \mathcal{C}_s(s_0, q_{m0}, 0)] (s - s_0) \\
 &+ \rho [\mathcal{E}_{q_m}(\rho_0, s_0, q_{m0}, 0) + \mathcal{C}_{q_m}(s_0, q_{m0}, 0)] (q_m - q_{m0}) \\
 &+ \rho [\mathcal{E}_{q_r}(\rho_0, s_0, q_{m0}, 0) + \mathcal{C}_{q_r}(s_0, q_{m0}, 0)] q_r
 \end{aligned}$$

# CHOICE OF CASIMIR FUNCTION (continued)

★ Choose the Casimir function so that the first order terms disappear:

$$-\mathcal{C}(s_0, q_{m0}, 0) = \Phi + \mathcal{E}(\rho_0, s_0, q_{m0}, 0) + \frac{p_0}{\rho_0}$$

$$-\mathcal{C}_s(s_0, q_{m0}, 0) = \mathcal{E}_s(\rho_0, s_0, q_{m0}, 0)$$

$$-\mathcal{C}_{q_m}(s_0, q_{m0}, 0) = \mathcal{E}_{q_m}(\rho_0, s_0, q_{m0}, 0)$$

$$-\mathcal{C}_{q_r}(s_0, q_{m0}, 0) = \mathcal{E}_{q_r}(\rho_0, s_0, q_{m0}, 0)$$

★ The following Casimir satisfies these four conditions:

$$-\mathcal{C}(s, q_m, q_r) = \tilde{\Phi}(\tilde{\rho}(s, q_m)) + \mathcal{E}(\tilde{\rho}(s, q_m), s, q_m, 0) + \frac{\tilde{p}(\tilde{\rho}(s, q_m))}{\tilde{\rho}(s, q_m)}$$

# CHOICE OF CASIMIR FUNCTION (continued)

★ With this Casimir, we have

$$\frac{\partial}{\partial t} \left\{ \rho \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + A \right) \right\} + \nabla \cdot \left\{ \rho \mathbf{u} \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + A \right) + \mathbf{u} (p - p_0) \right\} = \rho S$$

where

$$A = \Phi - \tilde{\Phi}(\tilde{\rho}(s, q_m)) + \mathcal{E}(\rho, s, q_m, q_r) - \mathcal{E}(\tilde{\rho}(s, q_m), s, q_m, 0) \\ + \frac{p_0(z)}{\rho} - \frac{\tilde{p}(\tilde{\rho}(s, q_m))}{\tilde{\rho}(s, q_m)}$$

is the moist available potential energy per unit mass.

★ Note that  $A = 0$  when  $\rho = \rho_0$ ,  $s = s_0$ ,  $q_m = q_{m0}$  and  $q_r = 0$ .

# PHYSICAL INTERPRETATION OF MOIST APE

- ★ Consider a moist parcel of unit mass at position  $\mathbf{x}$  and time  $t$ .
- ★ Let  $\rho(\mathbf{x}, t)$ ,  $s(\mathbf{x}, t)$ ,  $q_m(\mathbf{x}, t)$ ,  $T(\mathbf{x}, t)$ ,  $p(\mathbf{x}, t)$ , and  $\mathcal{E}(\mathbf{x}, t)$  denote this parcel's density, entropy, specific humidity, temperature, pressure, and internal energy, respectively.
- ★ Along a reversible adiabat,  $s$  and  $q_m$  are fixed, but  $\rho$  varies.
- ★ How much work  $W$  is required to move a unit mass from the reference state to its actual state along a path  $\Gamma$  that lies on a reversible adiabat?

$$W = \int_{\Gamma} (\nabla\Phi + \rho^{-1}\nabla p_0) \cdot d\mathbf{x} - \int_{\Gamma} (p - p_0) d\rho^{-1} = \int_{\Gamma} d\left(\Phi + \mathcal{E} + \frac{p_0}{\rho}\right) = A$$

Note that the internal energy equation in the form  $d\mathcal{E} = -pd\rho^{-1}$  has been used.

***Therefore, the moist available potential energy  $A$  equals the work done on the unit mass in moving it from the reference state to the actual state.***

# SUMMARY

We have discussed a nonhydrostatic model of the moist precipitating atmosphere. This model . . .

- ★ has an “exact” treatment of moist thermodynamics
- ★ includes the effects of ice
- ★ includes precipitation effects on budgets of mass, entropy, and momentum
- ★ has associated moist generalizations of the Ertel potential vorticity principle and the Lorenz available potential energy principle
- ★ can be run as a GCM on a global domain and as a cloud ensemble model in a limited-area domain
- ★ is ideal for superparameterization studies