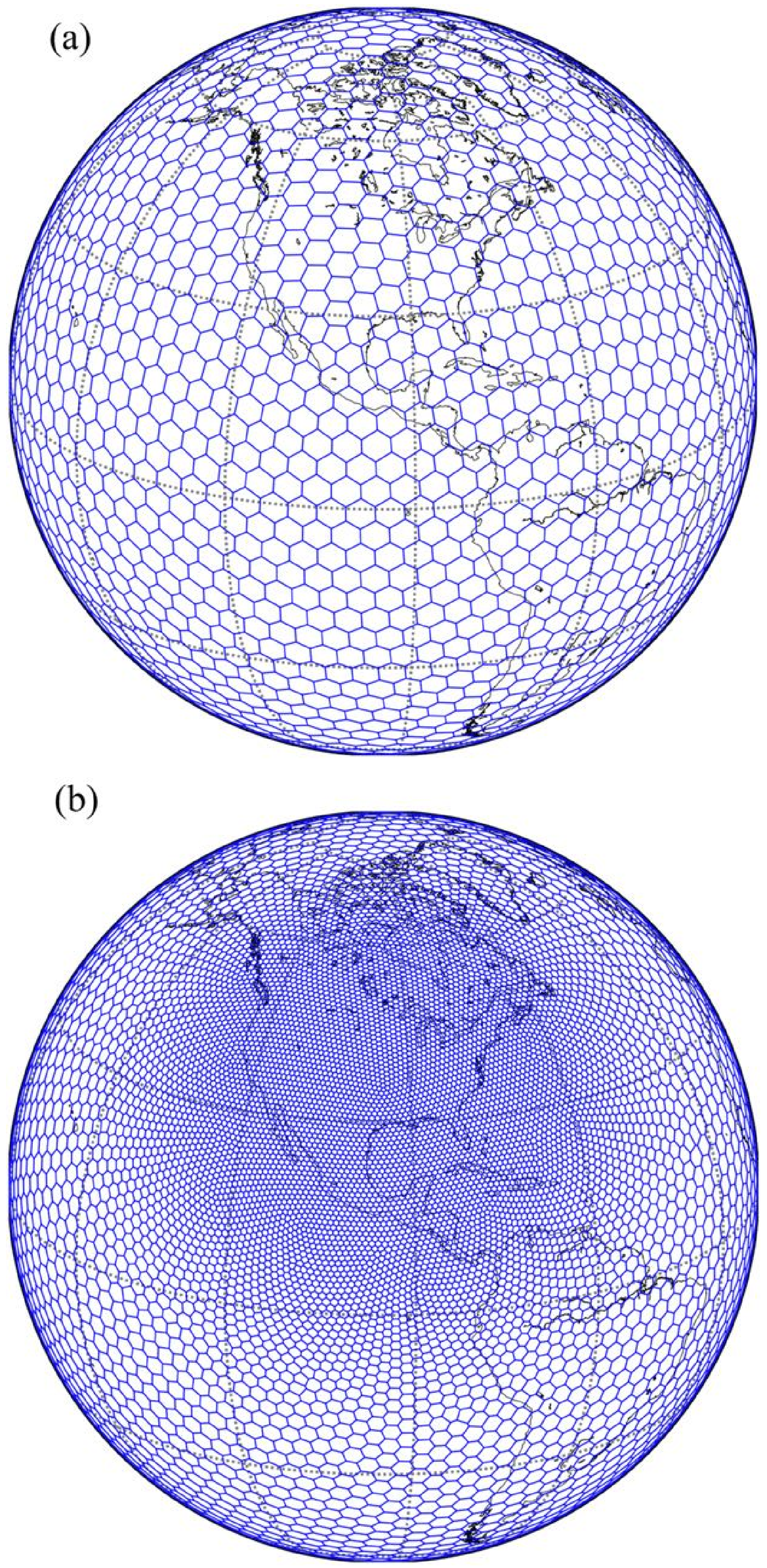
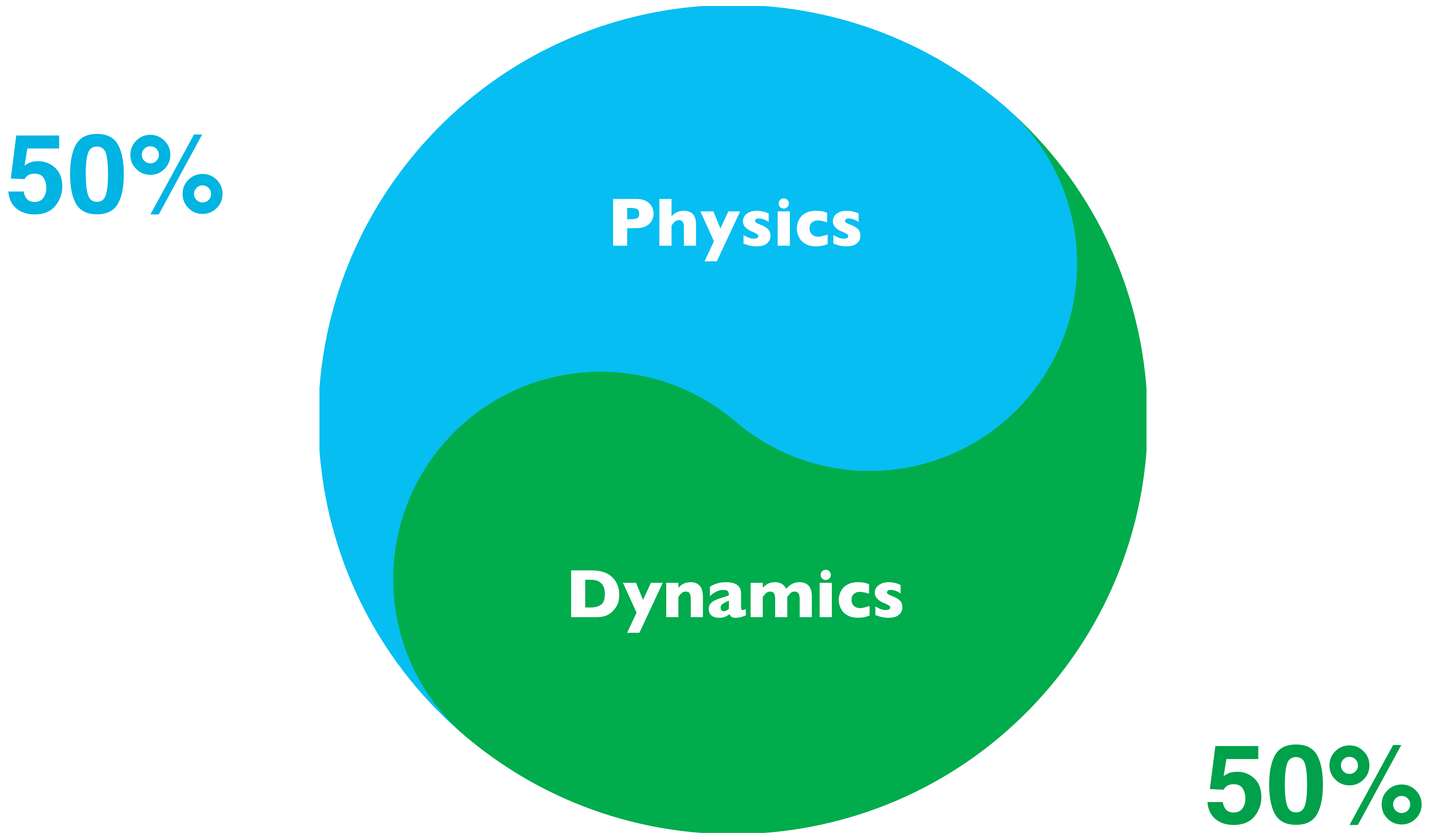
Dynamical Cores



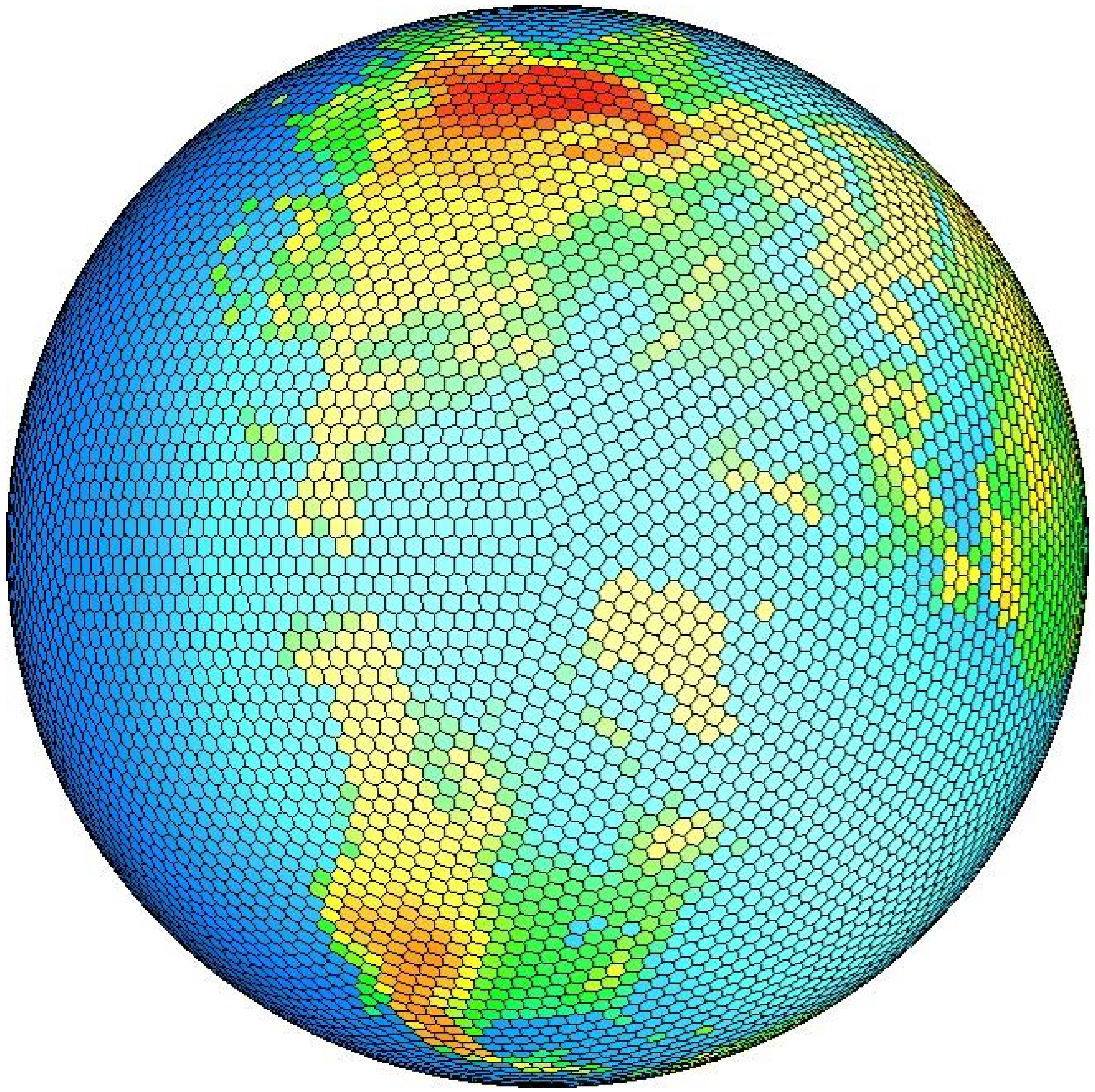
If you want to learn (a lot) more about dynamical cores, take AT604, “Atmospheric Modeling,” which will be offered in Fall semester 2025.

Where does the computer time go?



What is the job of a dynamical core?

- Predict the distribution of mass
- Predict the horizontal wind vector
- Determine the vertical motion
- Predict the adiabatic tendency of the temperature
- Advect moisture and other scalars



Choices in the design of a dynamical core

- Choice of scope, including the smallest scale to be represented and the highest model top allowed
- Choice of equation set
- Choice of prognostic variables
- Choice of vertical coordinate system
- Choice of a method to discretize the sphere
- Choice of a method to discretize the equations

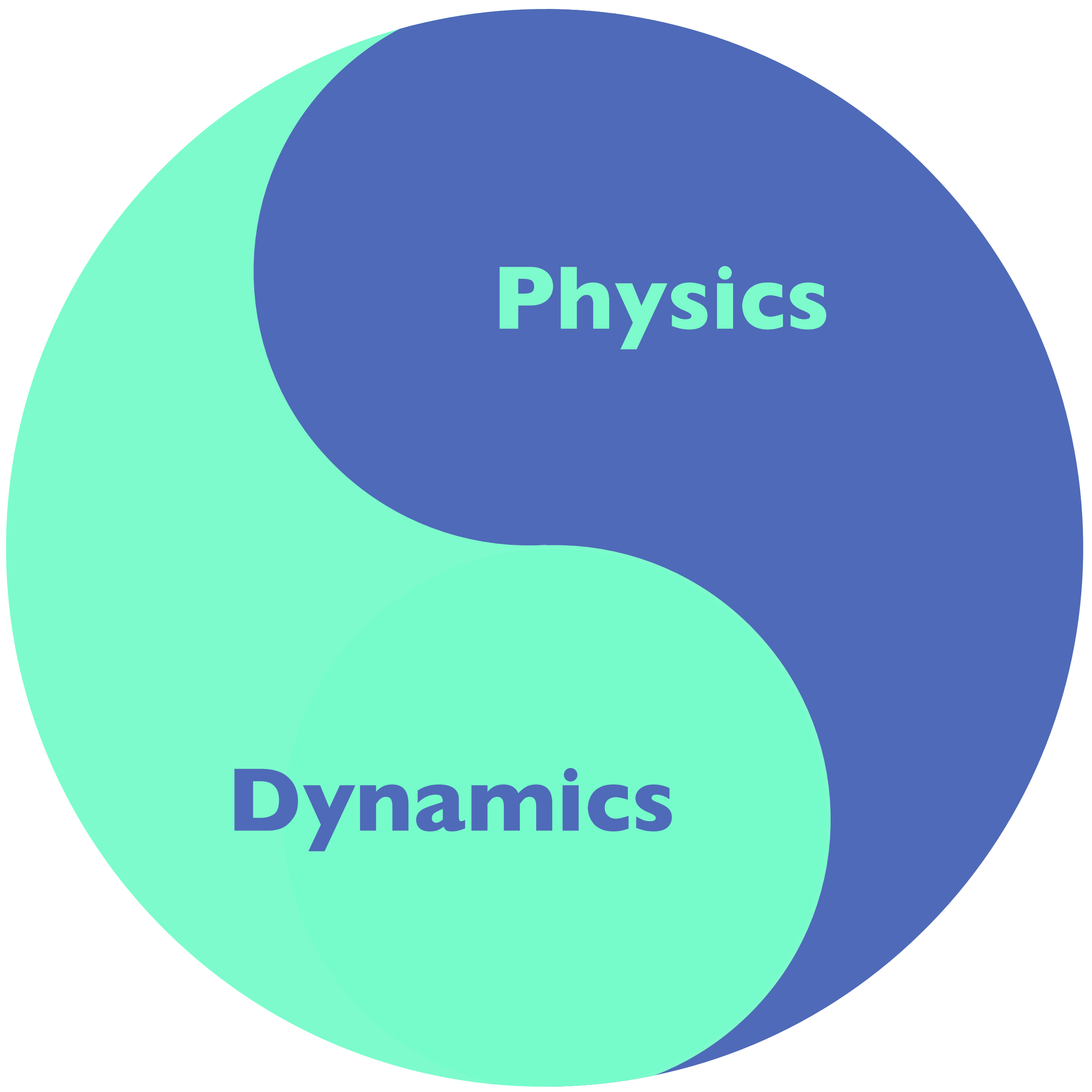
The “exact” equations

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{V} = 0$$

$$\frac{D\mathbf{V}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{V} + \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r} + \nabla\phi_a + \alpha\nabla p = -\alpha\nabla\cdot\mathsf{F}$$

$$c_p\frac{DT}{Dt} - \omega\alpha = -\alpha\nabla\cdot\left(\mathbf{R} + \mathbf{F}_s\right) + LC + \delta$$

$$\frac{Dq_v}{Dt} = -\alpha\nabla\cdot\mathbf{F}_{q_v} - C$$



Choice of equation set:
What approximations should be used?

- Quasi-static approximation
- Various anelastic approximations
- Thin atmosphere approximation
- Neglect of some Coriolis terms
- Neglect of the centrifugal acceleration
- Spherical Earth approximation
- Uniform gravity approximation

We are gradually eliminating these approximations.

The quasi-static approximation has been the most useful, and will be the first to go.

The quasi-static approximation is not the same as the hydrostatic approximation.

1. Approximate the equation of vertical motion by

$$\frac{\partial p}{\partial z} = -\rho g$$

Neglect of the vertical acceleration in the equation of vertical motion is *almost always justified.*

2. Determine the pressure by vertical integration of the hydrostatic equation, and *use that pressure to determine the pressure-gradient term of the horizontal equation of motion.*

This is justified when the vertical velocity is weak and the weather system is much wider than it is tall.

It is *not justified* for vigorous and deep small-scale weather systems.

What happens when the quasi-static approximation is used?

1. The equation of vertical motion can no longer be used to determine the vertical velocity. It has to be determined somehow, though.

2. The number of *prognostic* thermodynamic variables decreases from two to one.

3. Vertically propagating sound waves are filtered.

$$\frac{\partial p}{\partial z} = -\rho g$$

The trend to non-hydrostatic models

Faster, massively parallel computers are allowing us to use finer grids.

Finer grids can resolve weather systems, e.g., thunderstorms, that are not quasi-static.

For this reason, we are now building GCMs that do not use the quasi-static approximation.

These new high-resolution GCMs can borrow ideas from the well established mesoscale modeling community.

Allow a thick atmosphere?

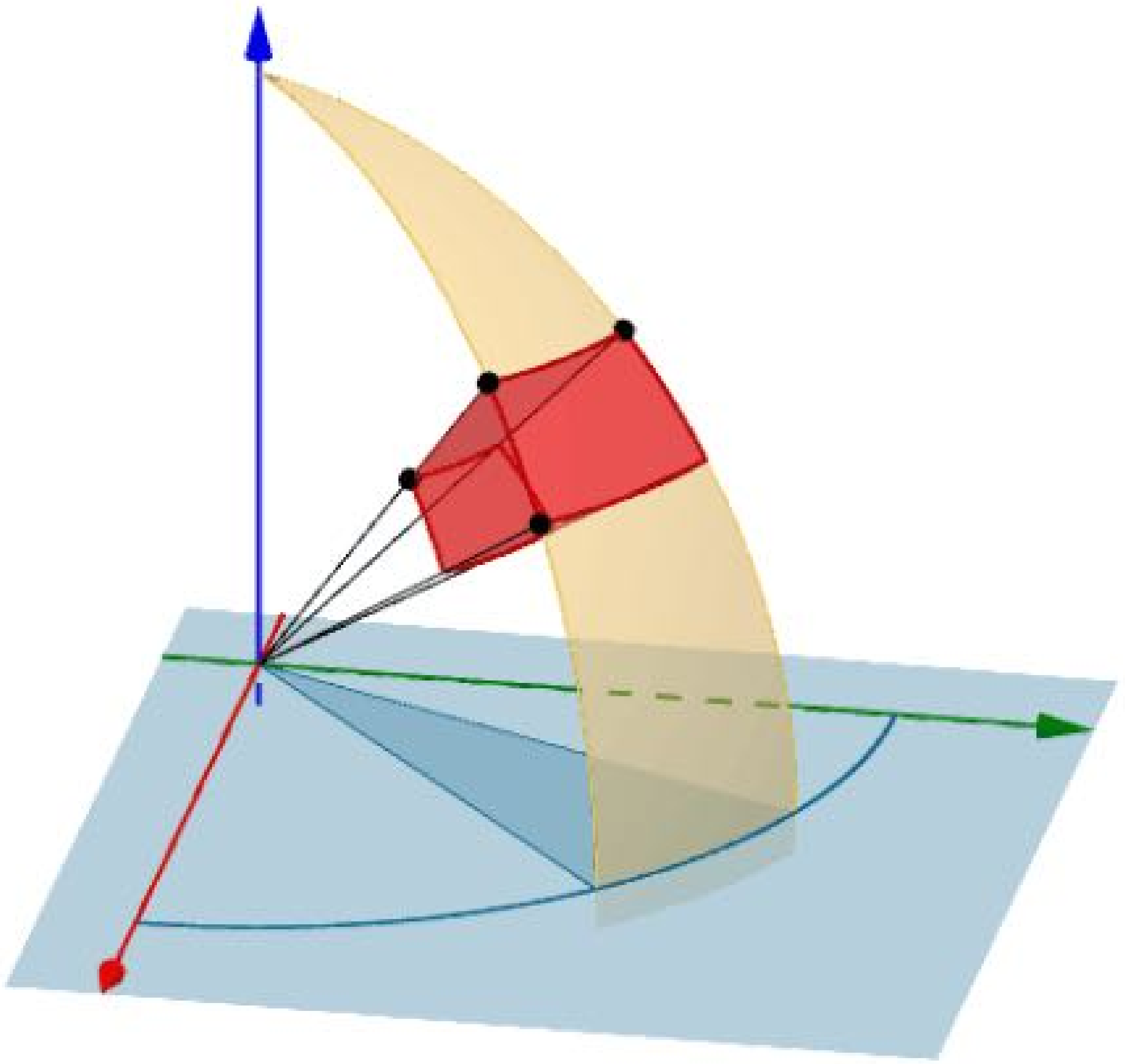
The combined thickness of the troposphere and stratosphere is less than 1% of the radius of the Earth. In that sense, the Earth's atmosphere is "thin."

$$\nabla \cdot \mathbf{H} = \frac{1}{r\cos\varphi}\frac{\partial H_\lambda}{\partial \lambda} + \frac{1}{r\cos\varphi}\frac{\partial}{\partial \varphi}\left(H_\varphi \cos\varphi\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(H_r r^2\right)$$

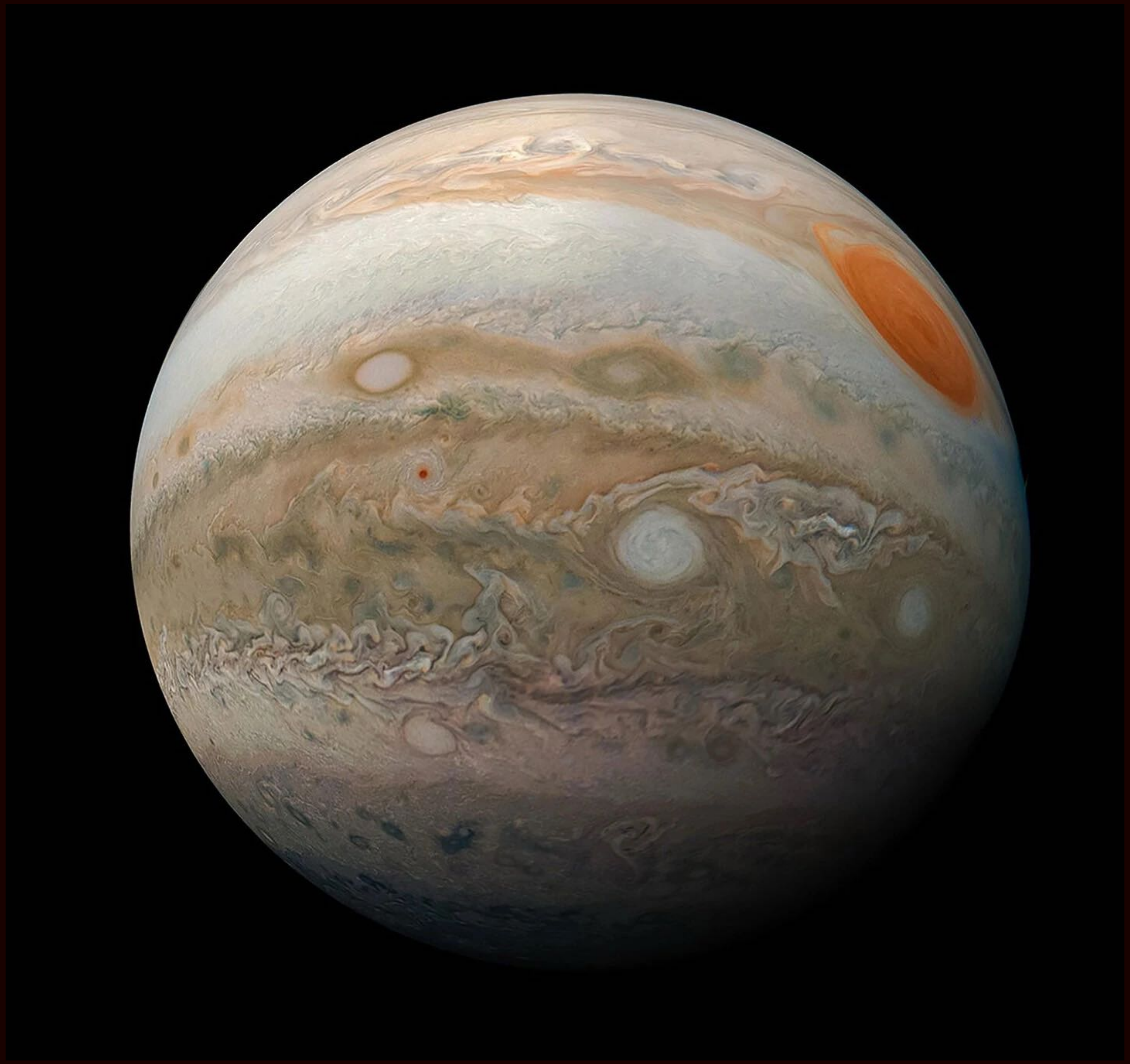
$$\nabla \cdot \mathbf{H} = \frac{1}{r\cos\varphi}\frac{\partial H_\lambda}{\partial \lambda} + \frac{1}{r\cos\varphi}\frac{\partial}{\partial \varphi}\left(H_\varphi \cos\varphi\right) + \frac{\partial H_r}{\partial r} + \cancel{\frac{2H_r}{r}}$$

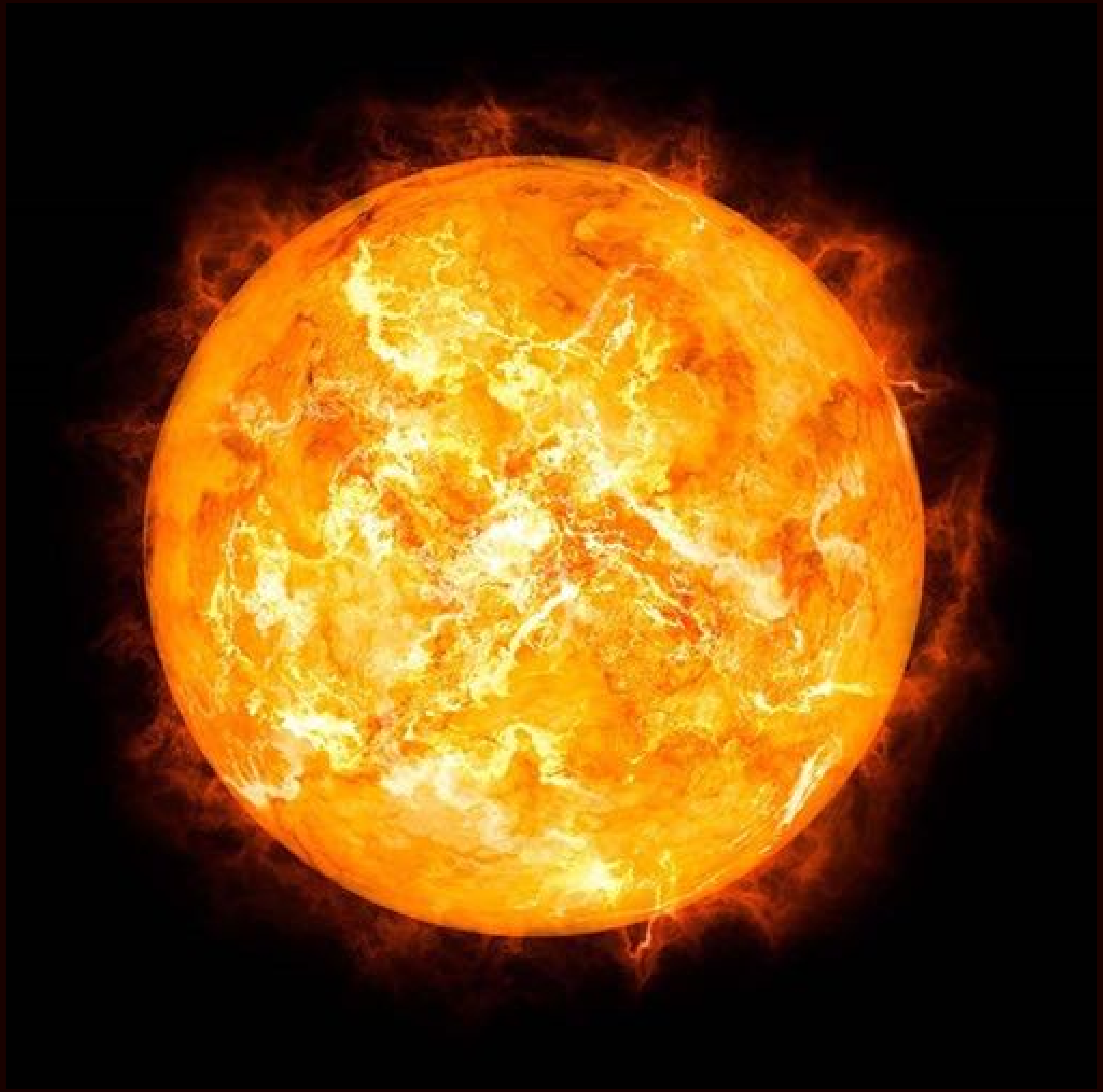
$$r \to a$$

$$\nabla \cdot \mathbf{H} = \frac{1}{a\cos\varphi}\frac{\partial H_\lambda}{\partial \lambda} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left(H_\varphi \cos\varphi\right) + \frac{\partial H_r}{\partial r}$$



Other atmospheres are not so thin.

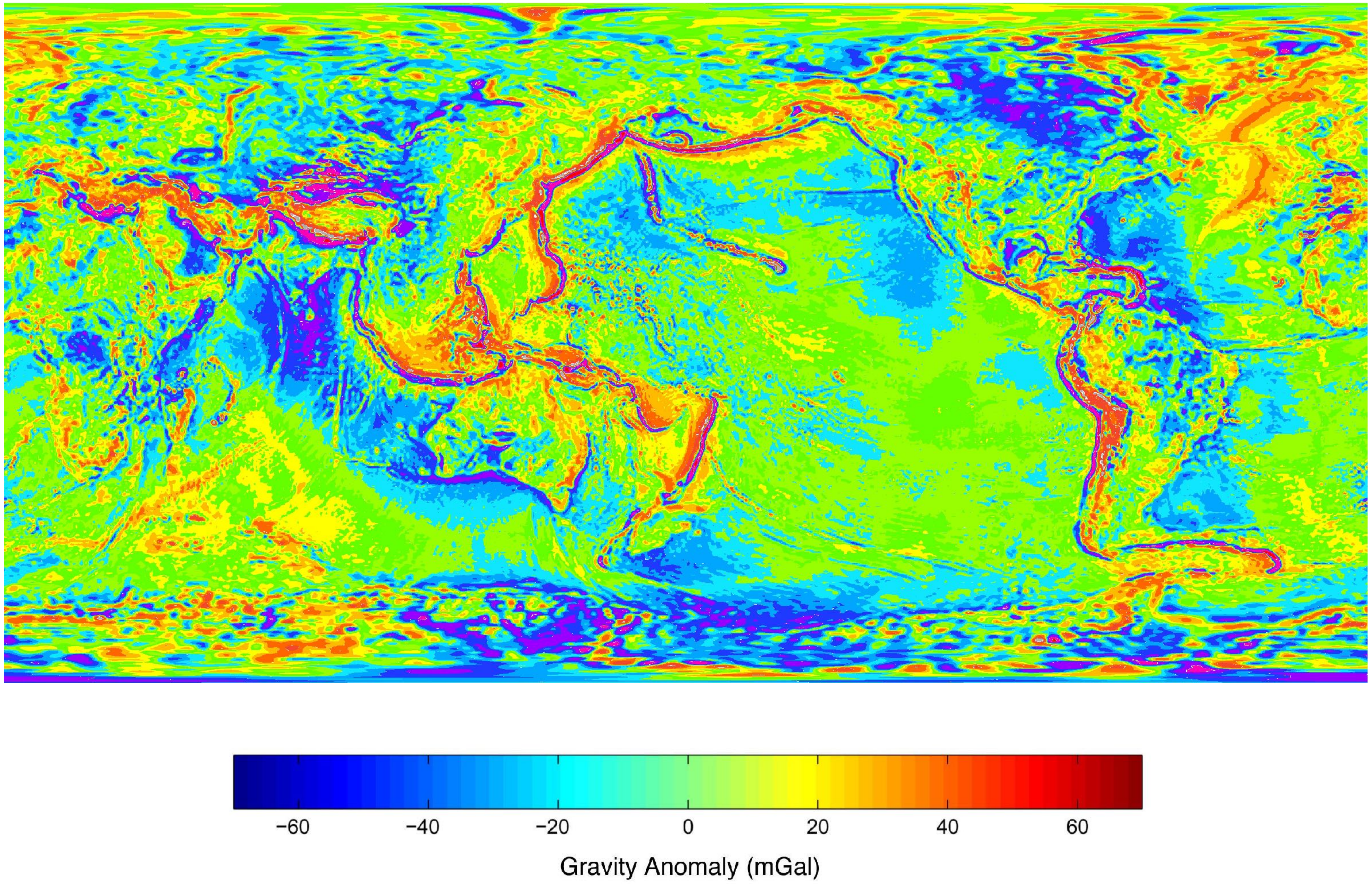




Include oblateness?

The picture on the right is distorted.
In reality, the Equatorial radius is “only” about 20 km larger than the polar radius.

Include nonuniform gravity?



The **gal** (symbol **Gal**), sometimes called **galileo**, is a unit of acceleration used extensively in the science of gravimetry, and defined as 1 centimeter per second squared (1 cm s^{-2}). The mean value of g is about 1000 gals. Gravity also weakens with height, of course.

Grace

Categories of model data

- Prognostic variables (internal)
- Initial conditions (external)
- Diagnostic variables (internal)
- Boundary conditions (external)

How prognostic variables work:

$$\text{Temperature at end of time step} = \text{Temperature at beginning of time step} + \Delta t \times (\text{Warming} - \text{Cooling})$$

Prognostic variables are "remembered" by the model, from one time step to the next.
Diagnostic variables are not remembered. They have to be recomputed from scratch on every time step.

Choice of prognostic variables

(What to time-step?)

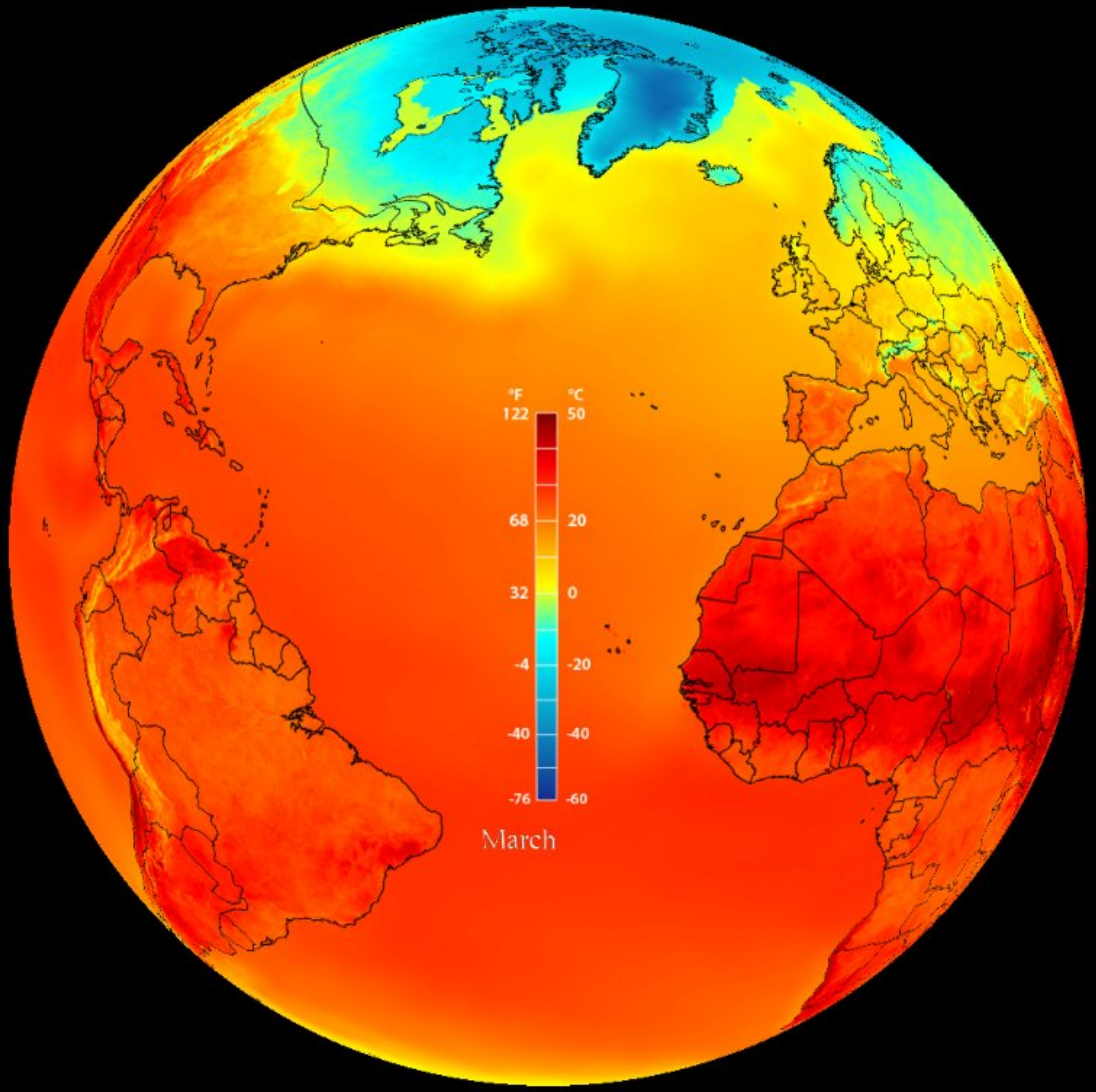
- Winds
  - Zonal, meridional, and vertical components
  - Angular momentum instead of the zonal component
  - Vertical component of the vorticity, and divergence of the horizontal wind vector
  - Horizontal vorticity vector
- Thermodynamic energy
  - Temperature
  - Potential temperature
  - Entropy
  - Moist entropy
- Moisture
  - Water vapor mixing ratio
  - Total water mixing ratio
  - Condensed water species

All choices give the same result in the continuous system, but not in the discrete system.

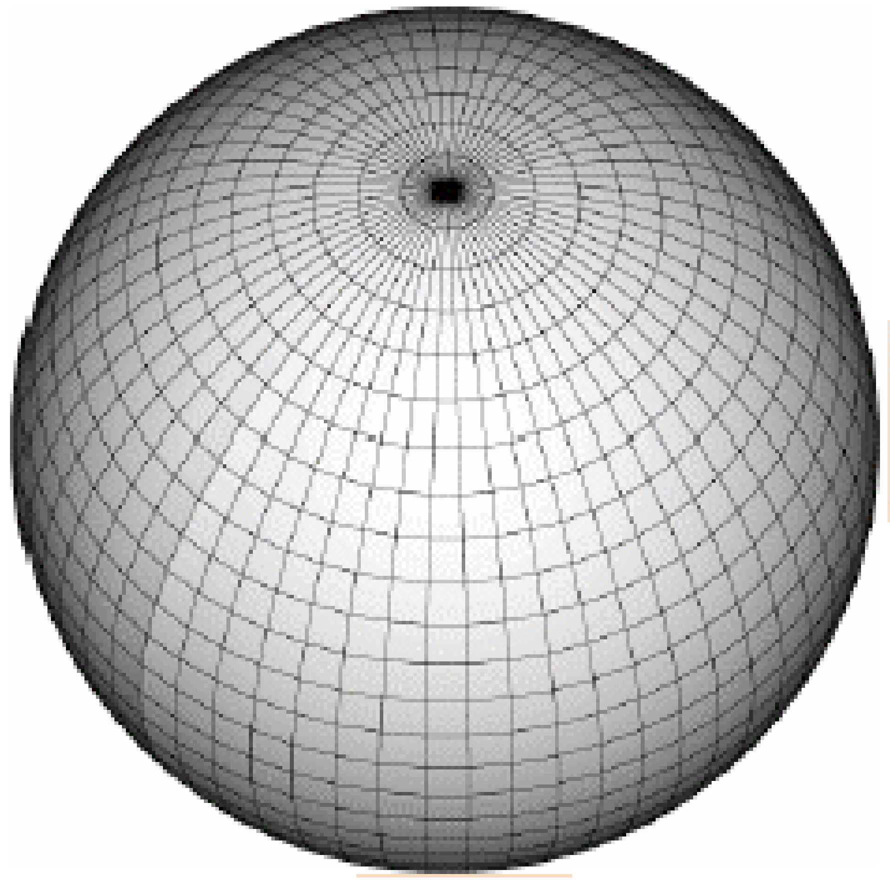
Discretizing the sphere

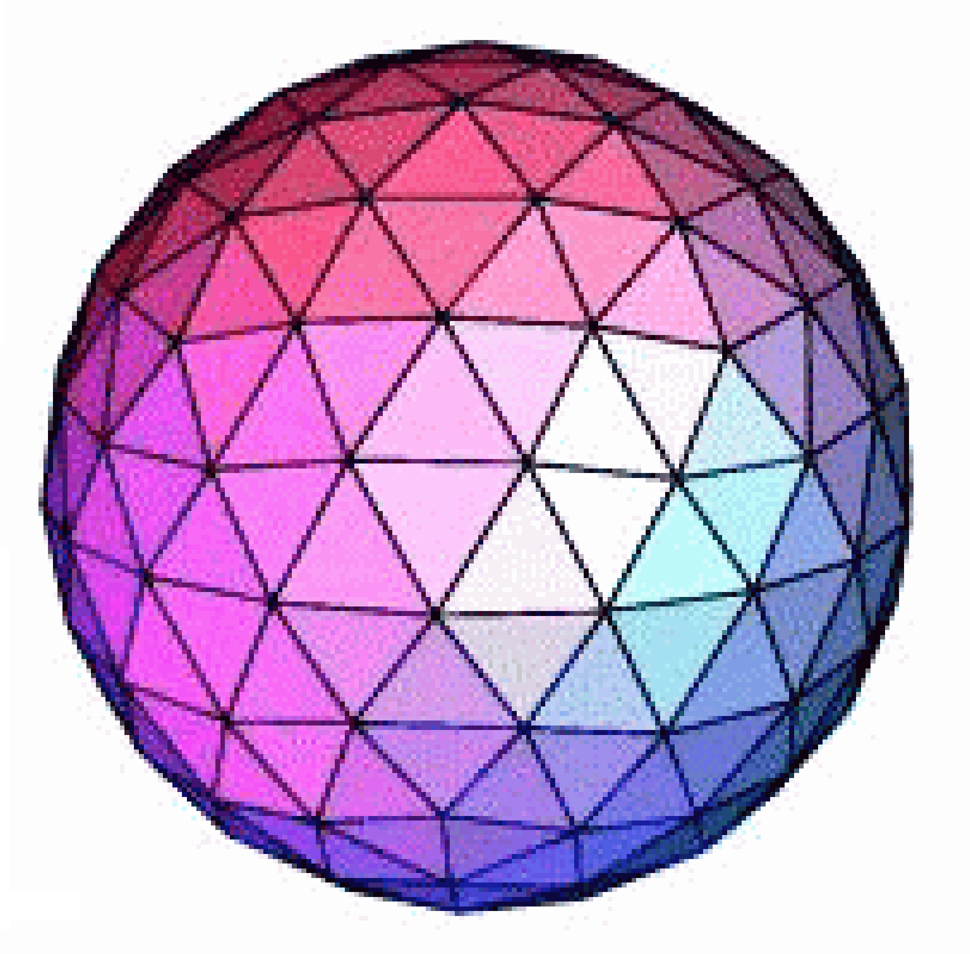
A field (e.g., temperature) that is defined on a continuous spherical surface has infinitely many degrees of freedom.

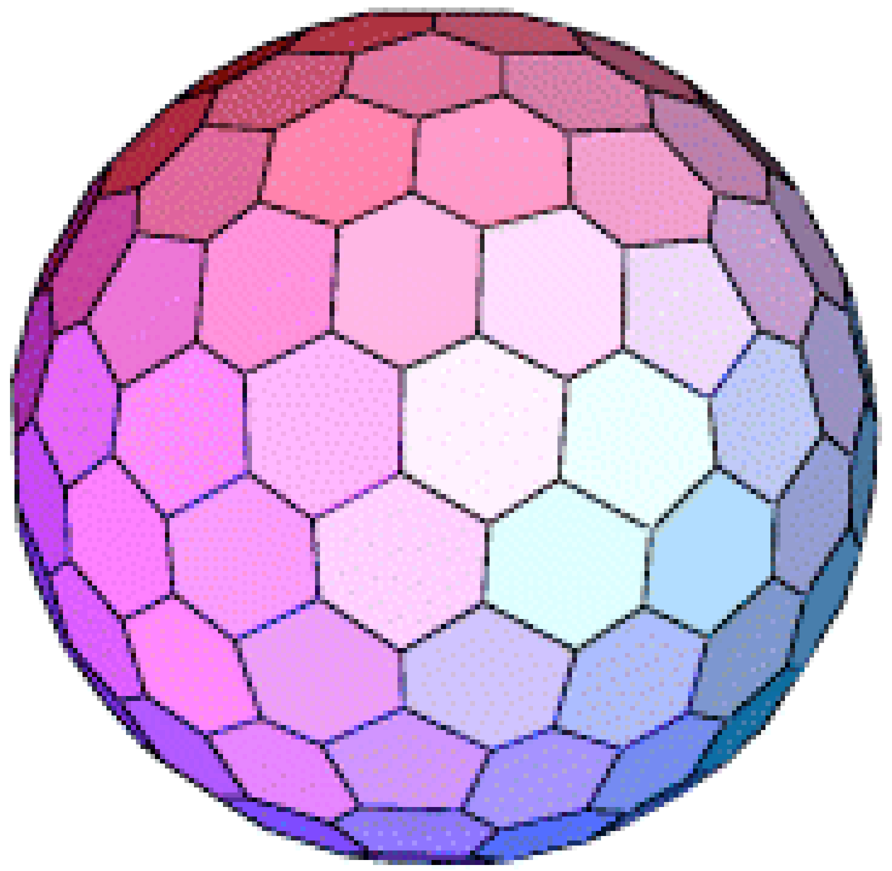
In a model, only a finite number of “discrete” degrees of freedom can be kept.

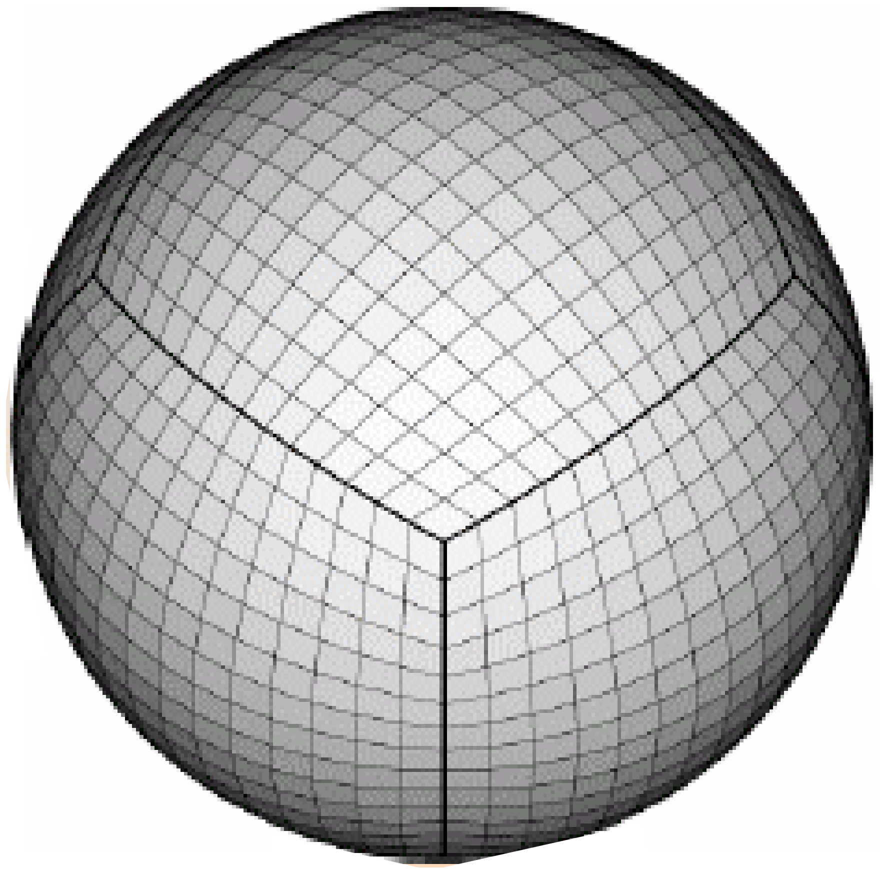


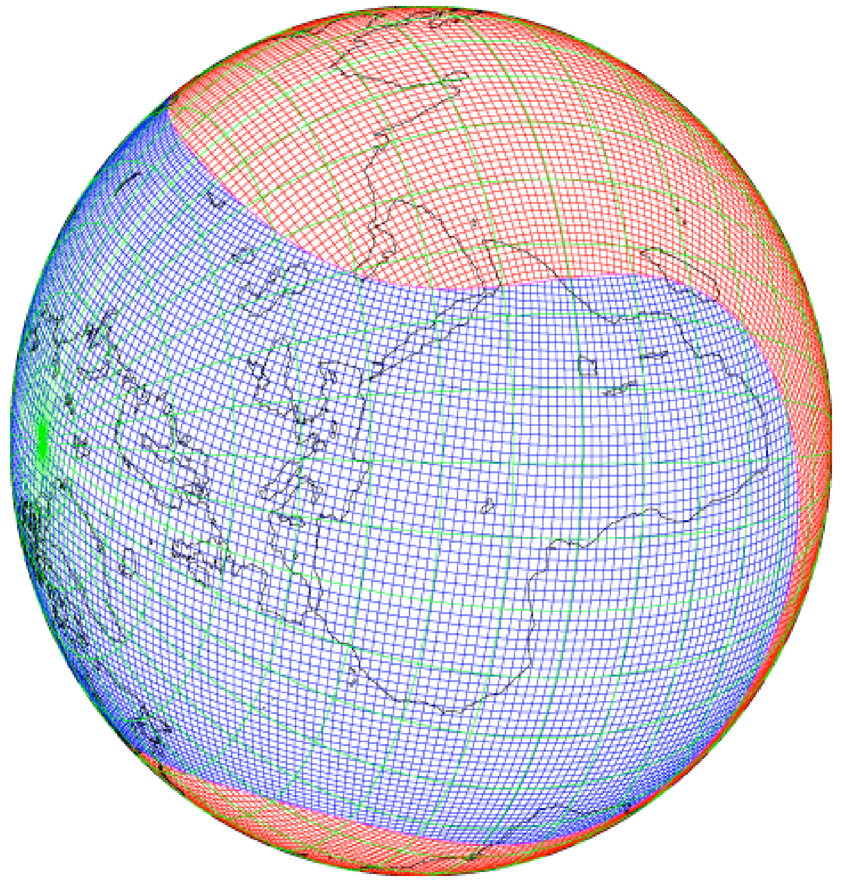
Discretizing the sphere with grids

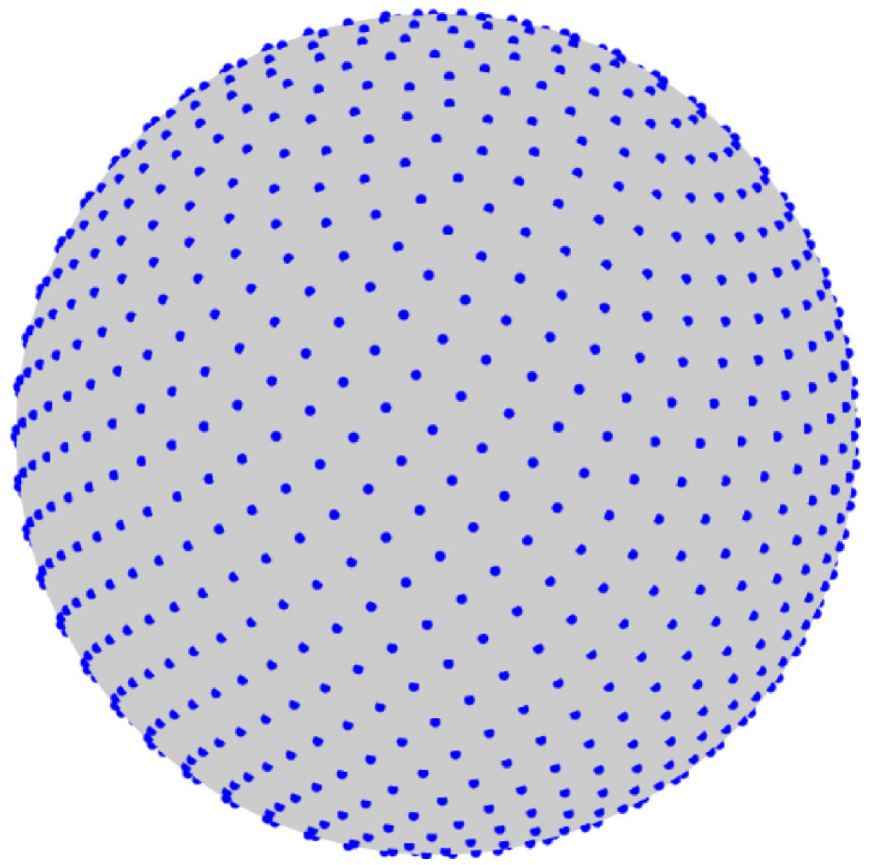

Lat-Lon

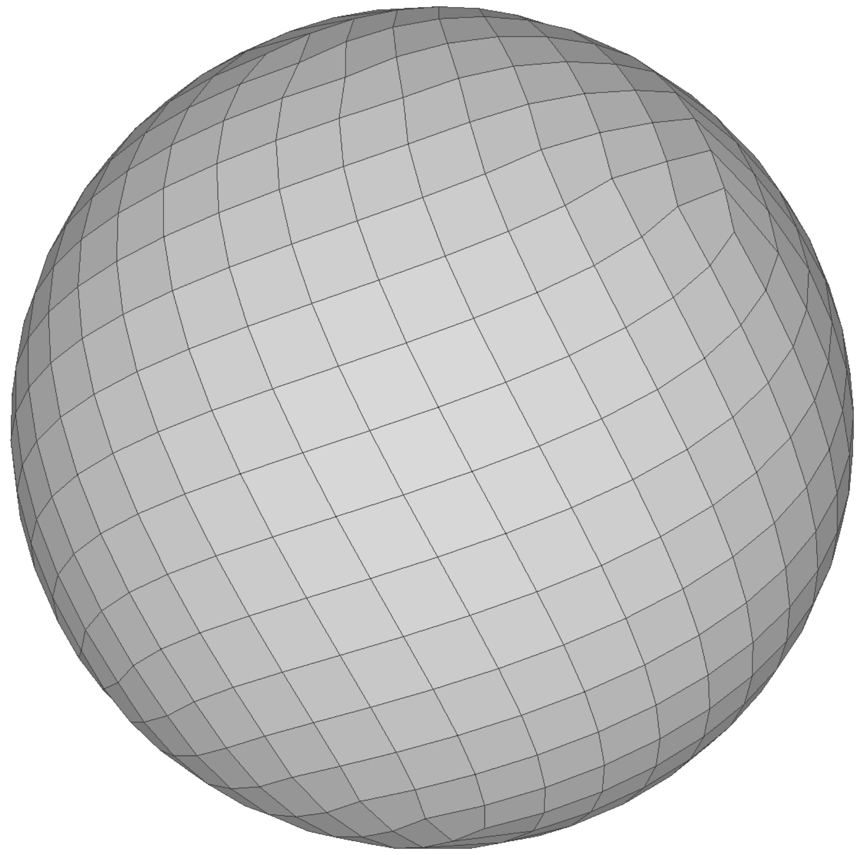

Icosahedral-triangles


Icosahedral-hexagons

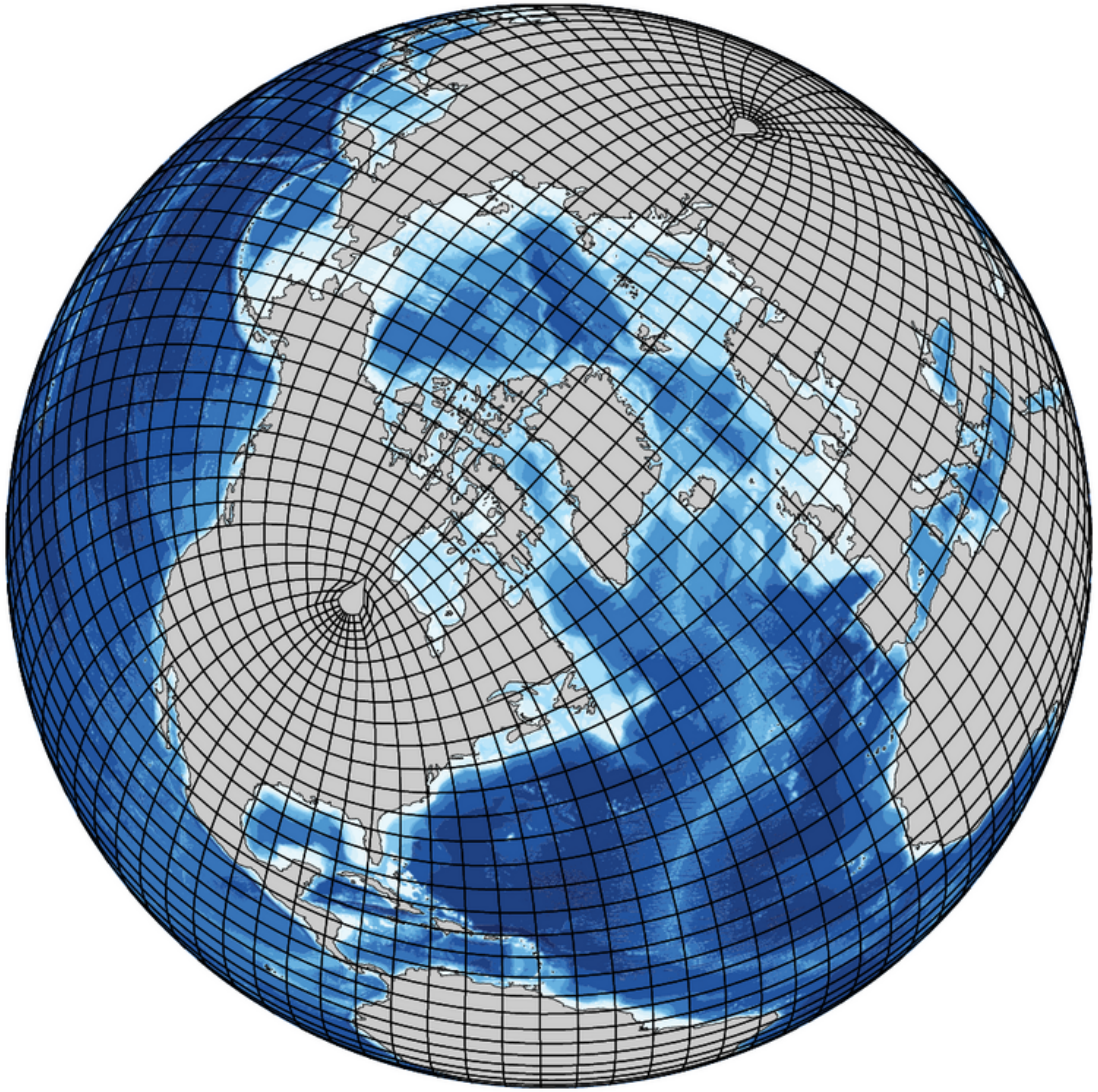

Cubed Sphere


Yin-Yang

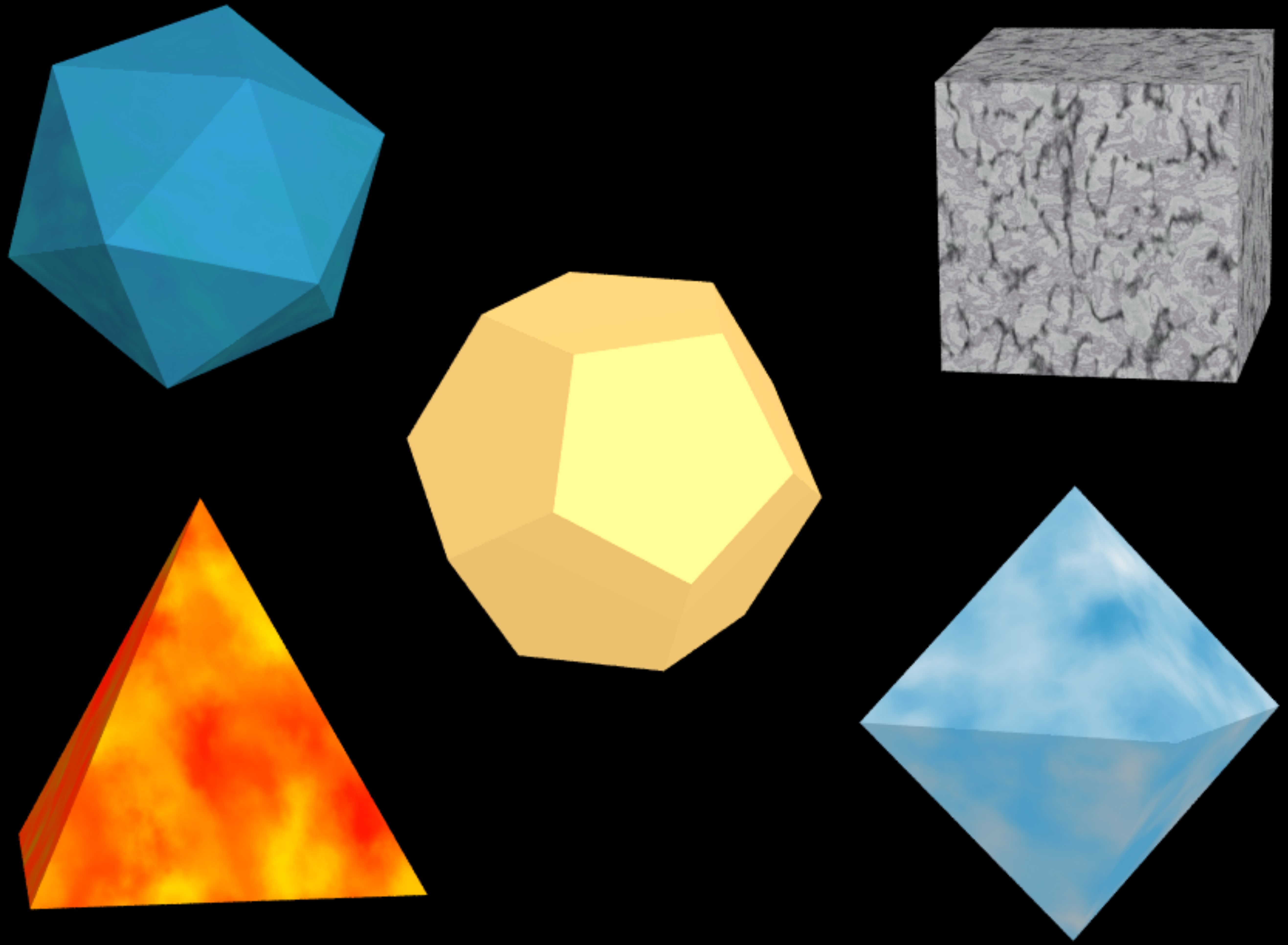

Fibonacci


HEALpix

Tri-polar ocean grid

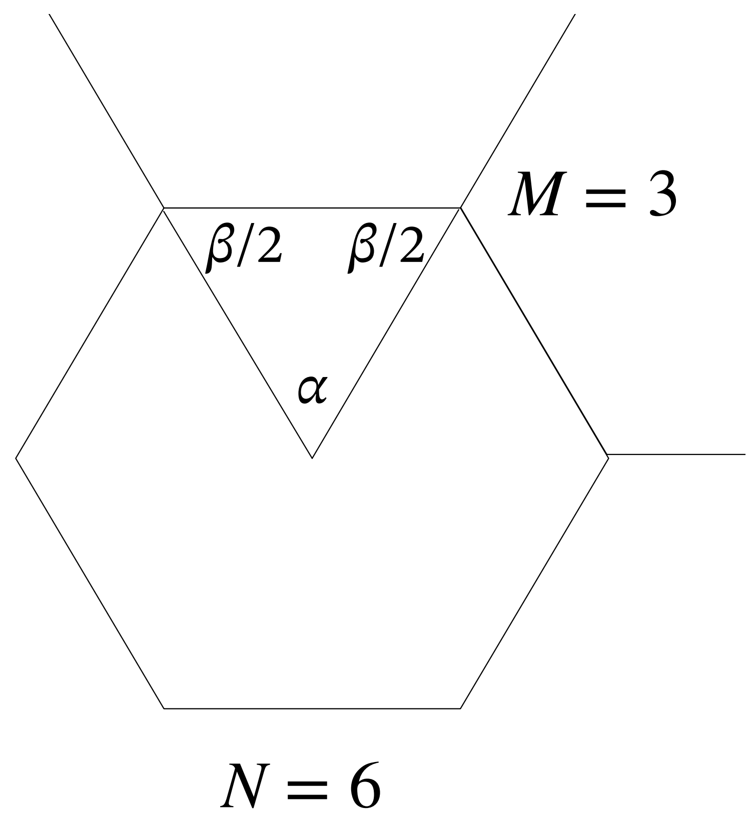


The Platonic Solids



These are the only regular polyhedrons.

Tiling the plane



Suppose that a regular convex polygon has N sides. The interior angles, α, must sum to 2π, so

$$N\alpha = 2\pi.$$

Next, suppose that M polygons come together at each vertex. The exterior angles, β, must sum to 2π, so

$$M\beta = 2\pi.$$

Since the interior angles of a triangle must sum to π, we can write

$$\alpha + 2\left(\frac{\beta}{2}\right) = \pi,$$

or

$$\alpha + \beta = \pi.$$

Combining these three equations, we find that

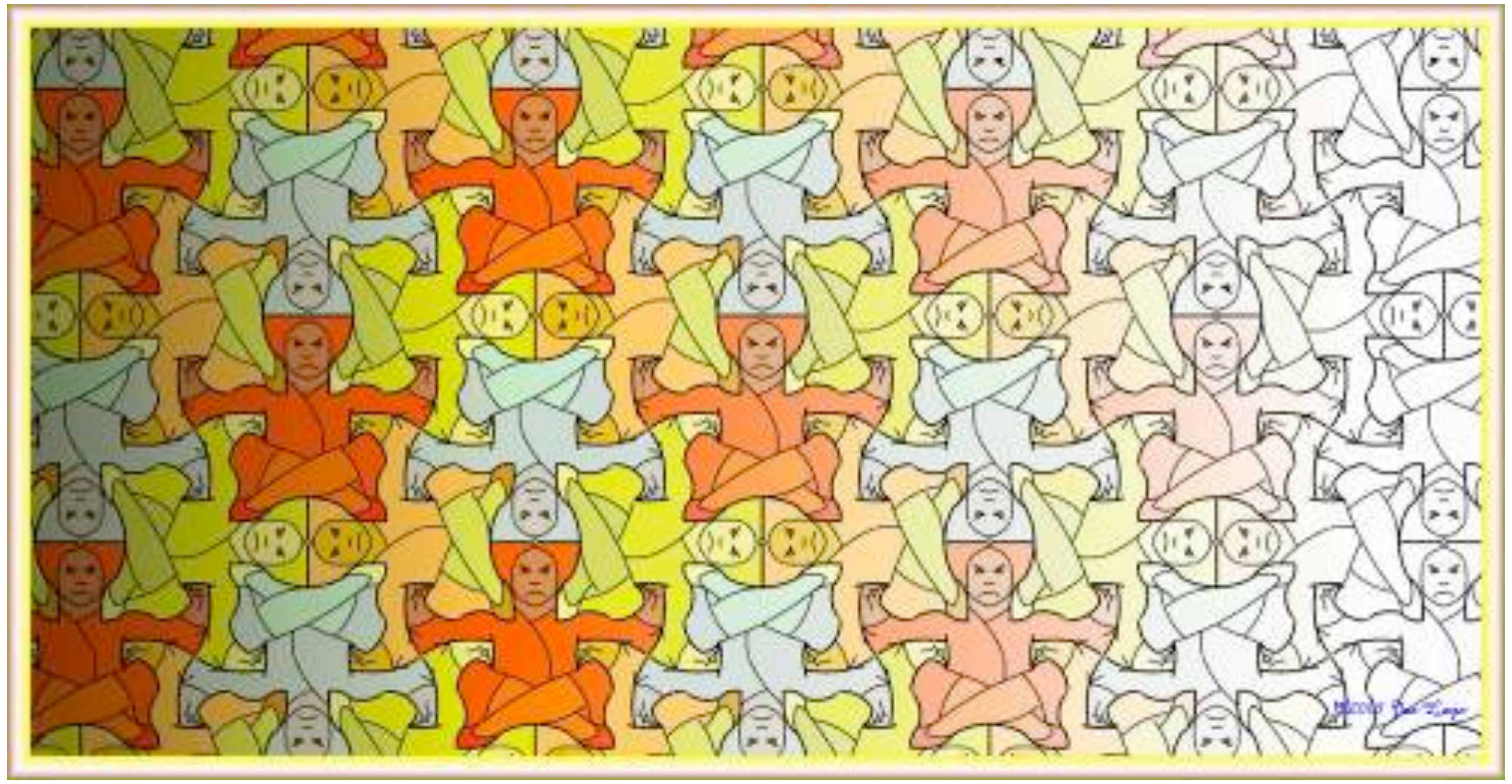
$$M = \frac{2N}{N-2}.$$

Trying different values of N, we find that M is an integer only for $N = 3$, 4, or 6.

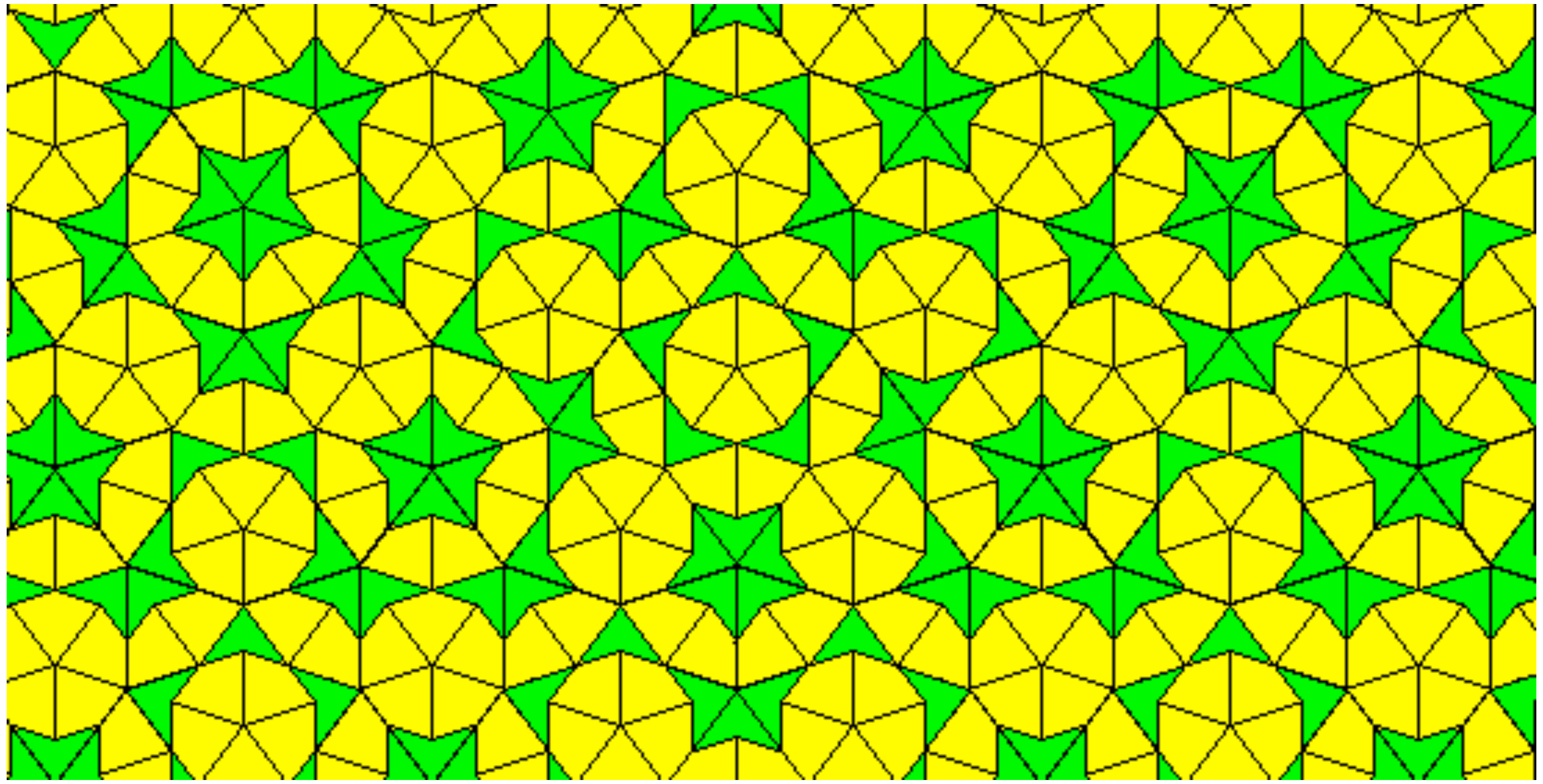
N	M
3	6
4	4
5	10/3
6	3
7	14/5
∞	2

Conclusion: *Triangles, squares, and hexagons are the only convex regular polygons that tile the plane.*

Other ways to tile the plane

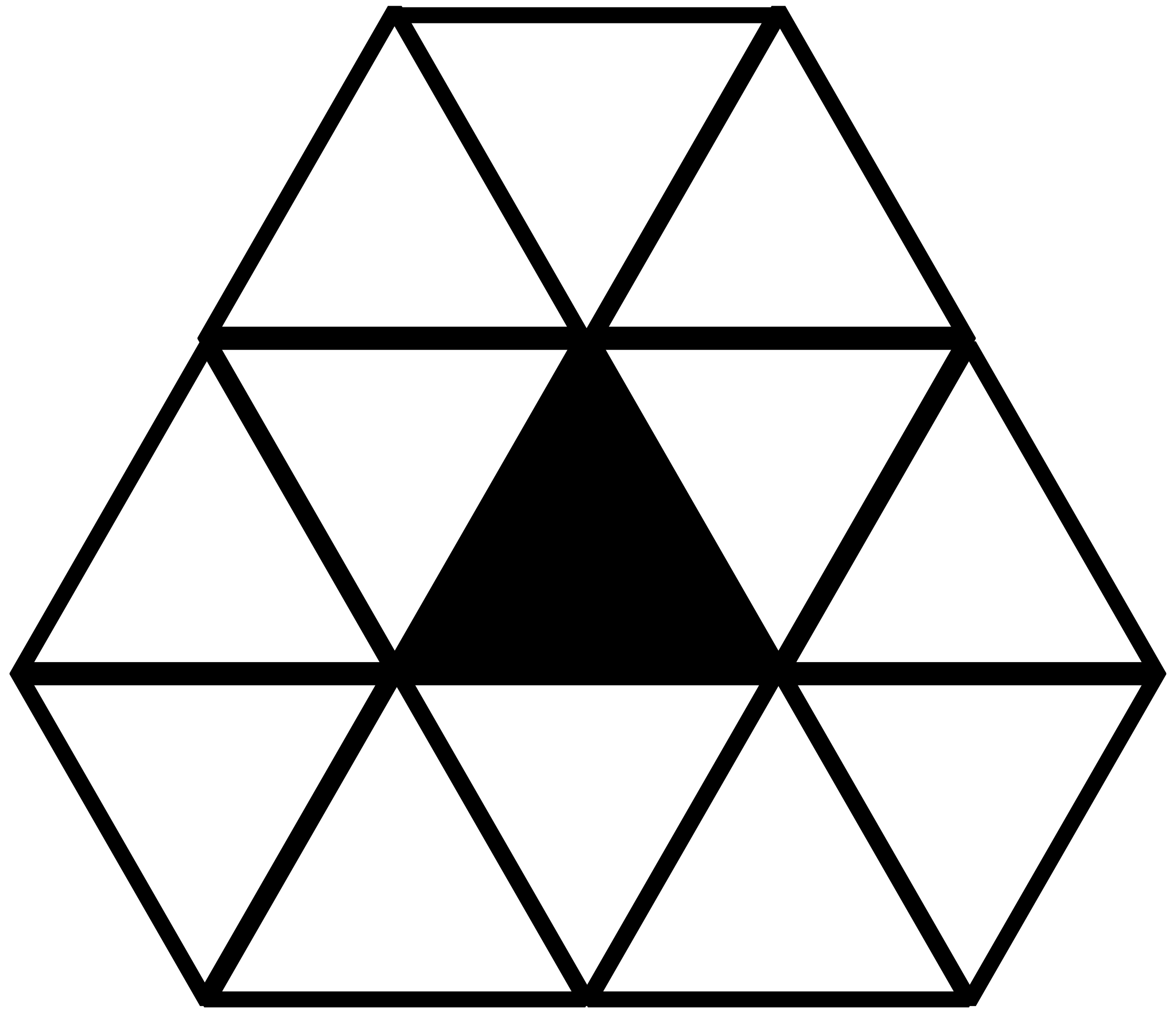


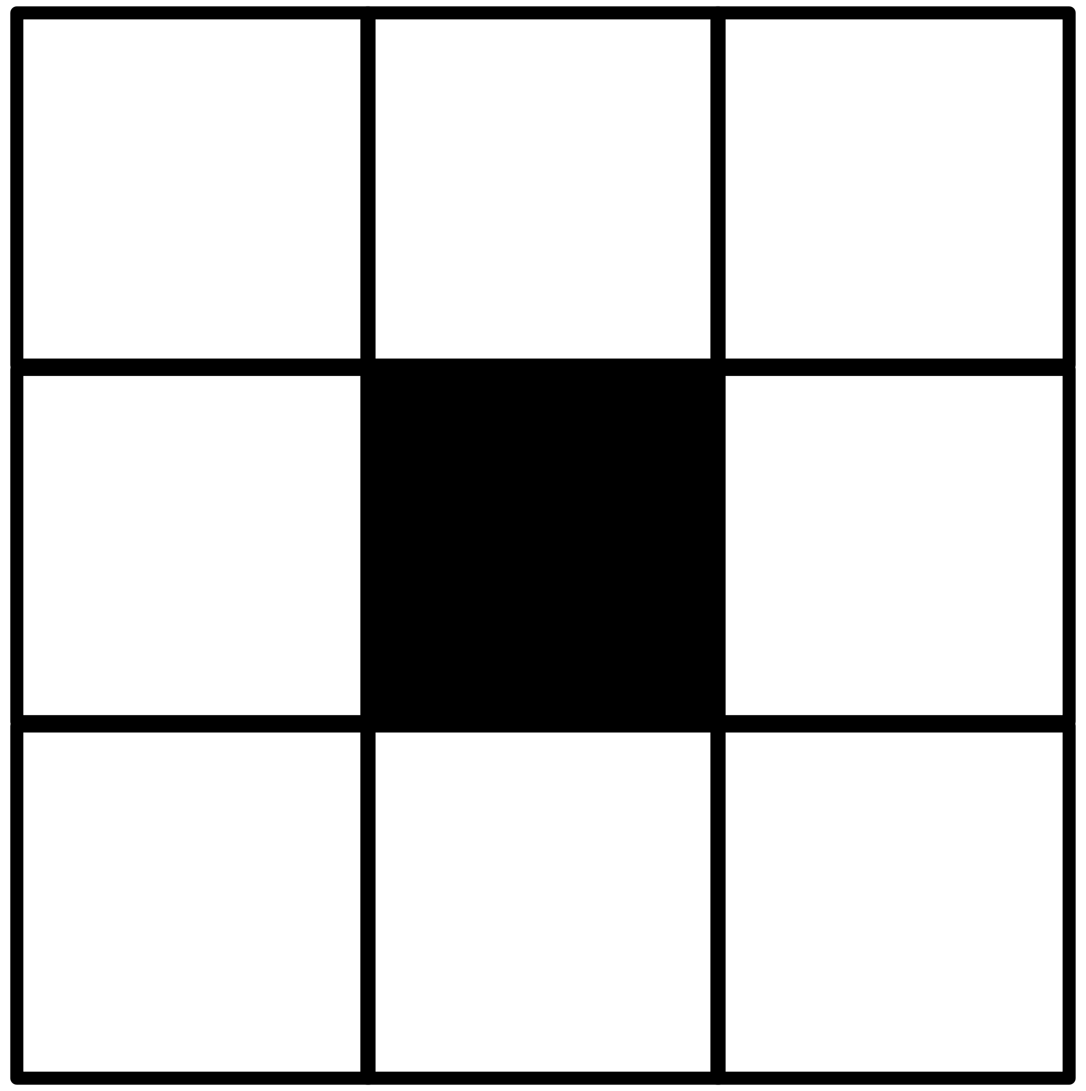
Escher

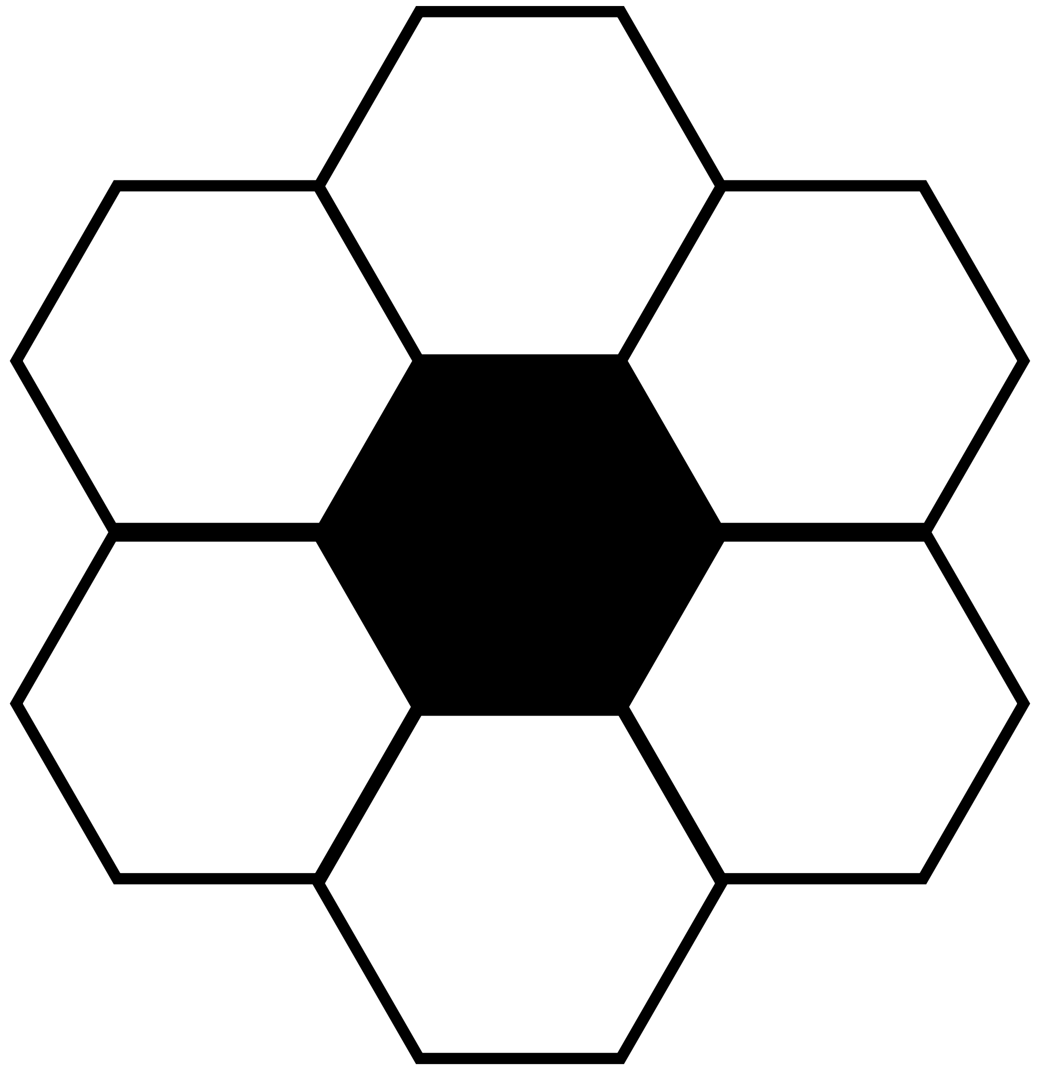


Penrose

Tiling the Plane







12 neighbors
3 wall neighbors

8 neighbors
4 wall neighbors

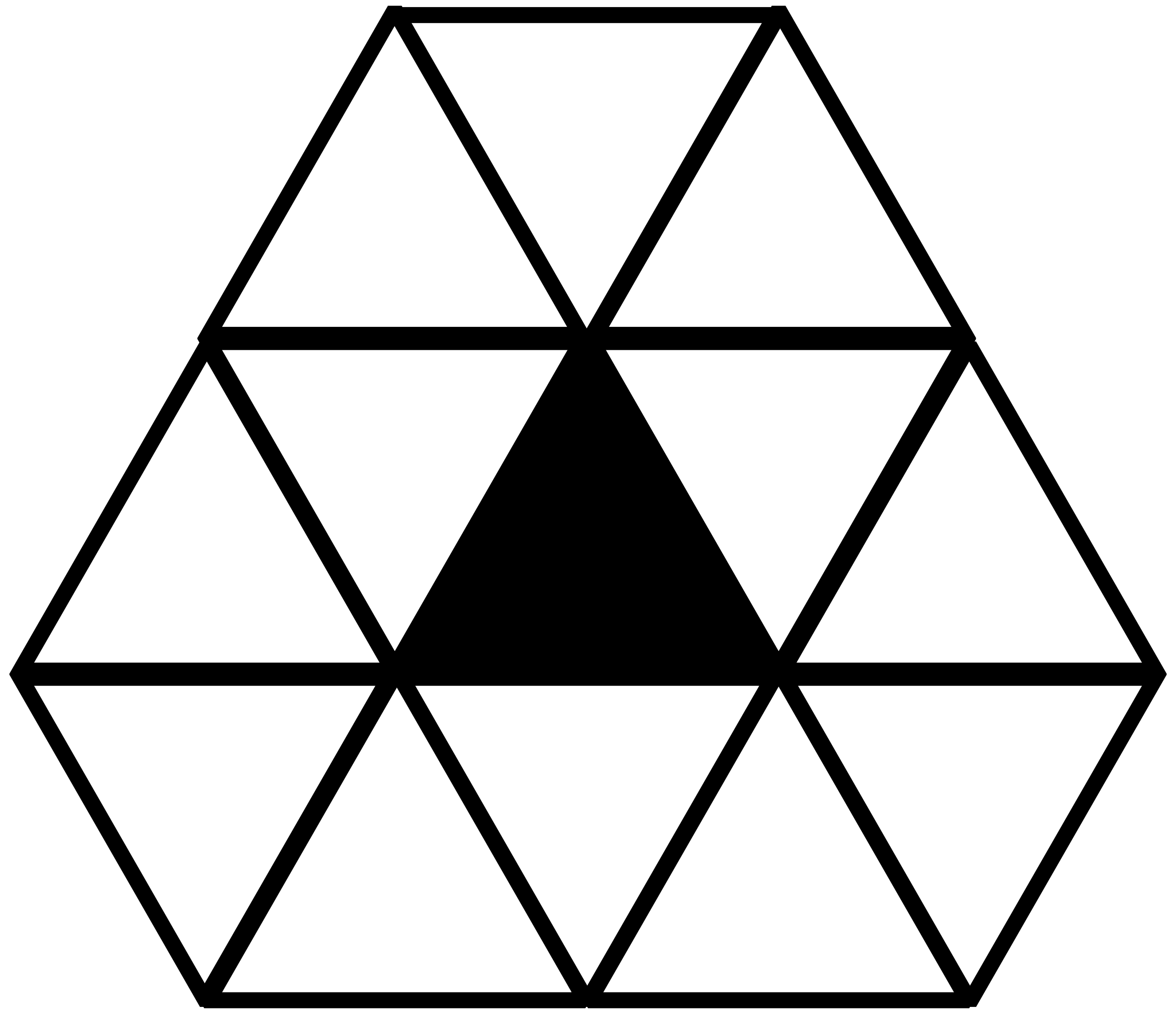
6 neighbors
6 wall neighbors

Triangles
nest.

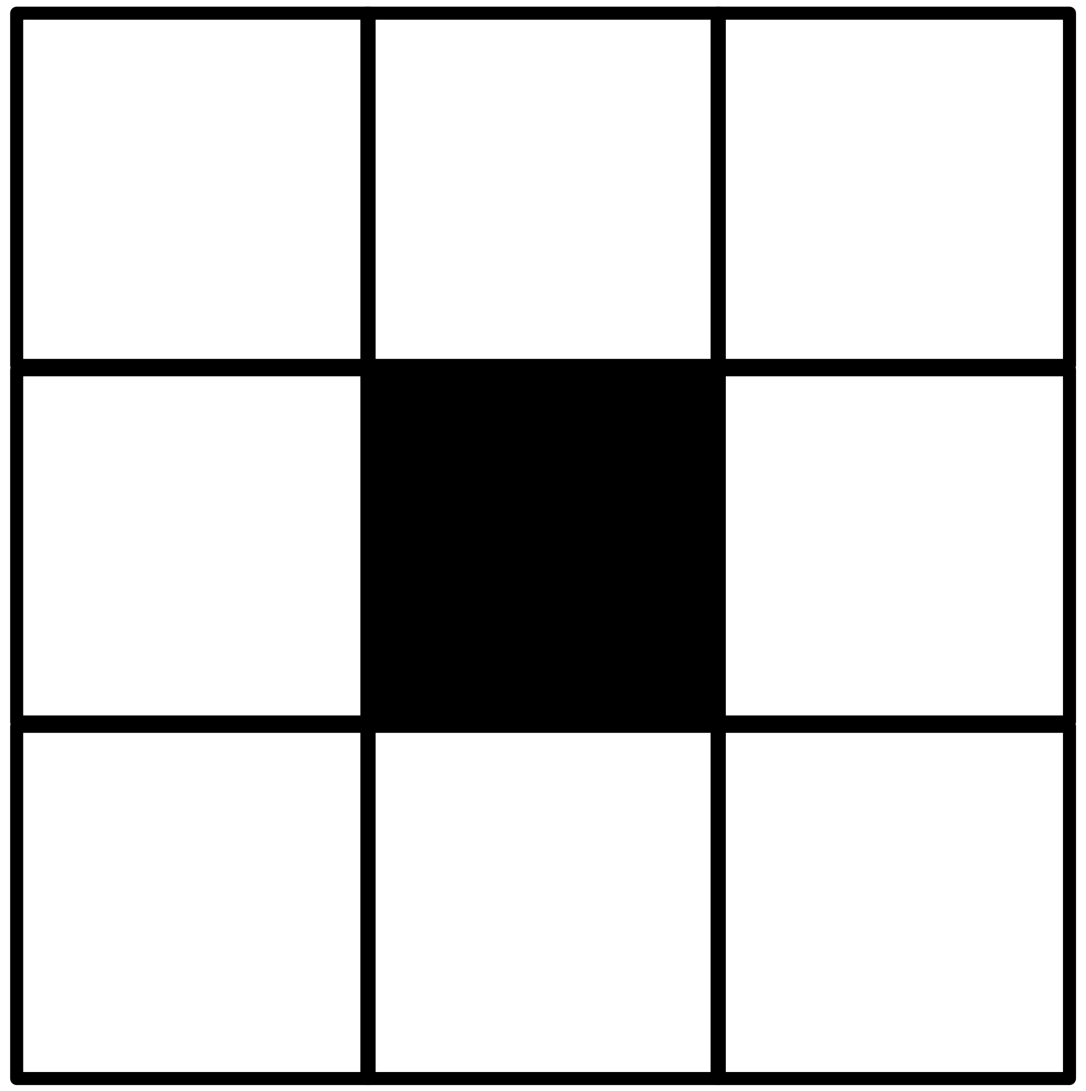
Squares
nest.

Hexagons
don’t nest.

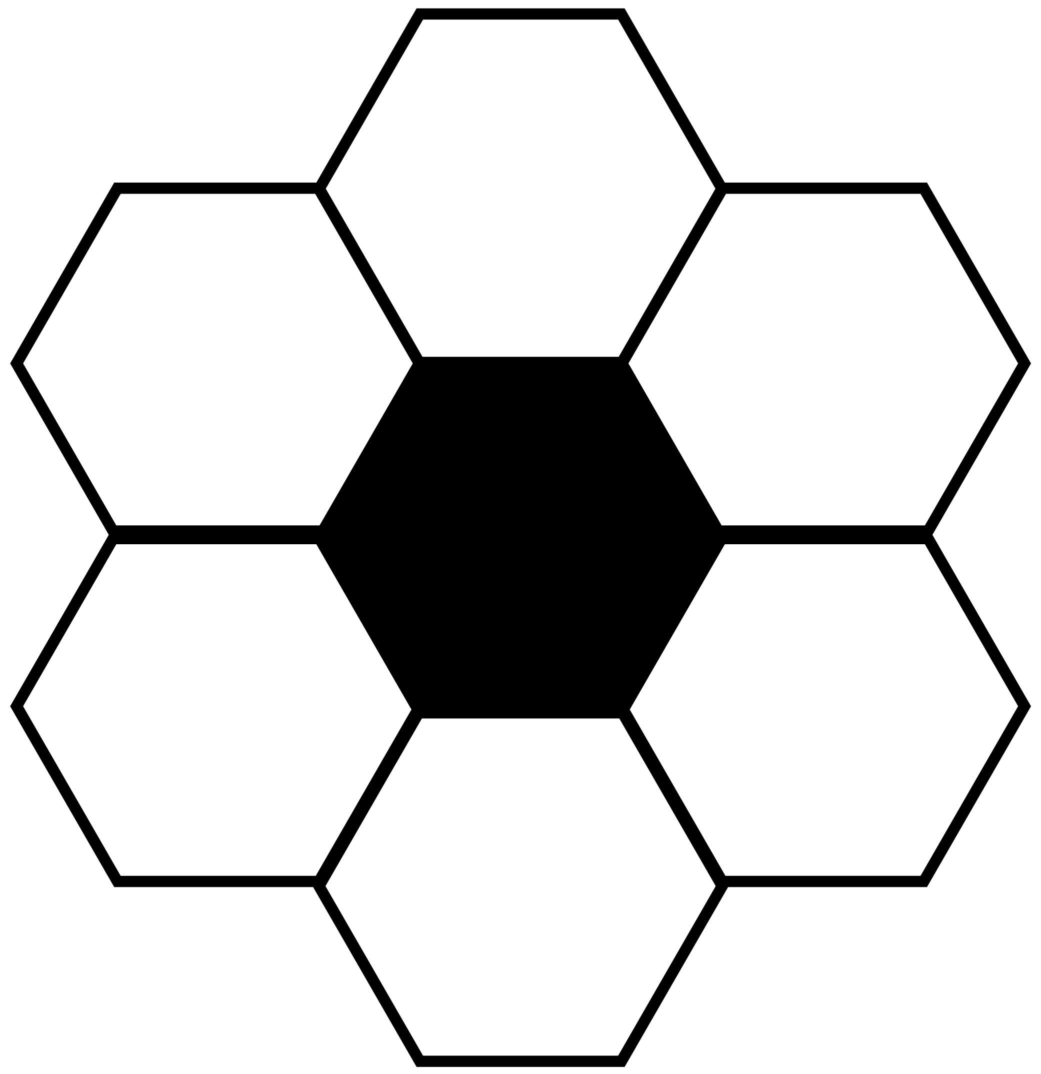
All places the same.
All directions the same.
All neighbors the same.



12 neighbors
3 wall neighbors

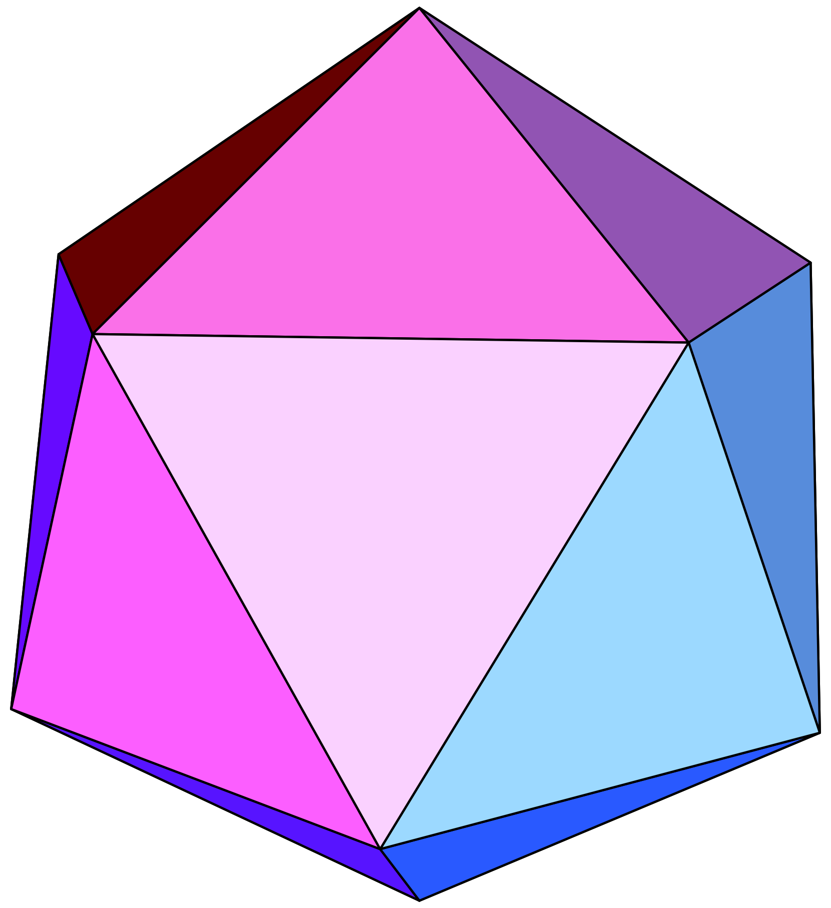


8 neighbors
4 wall neighbors

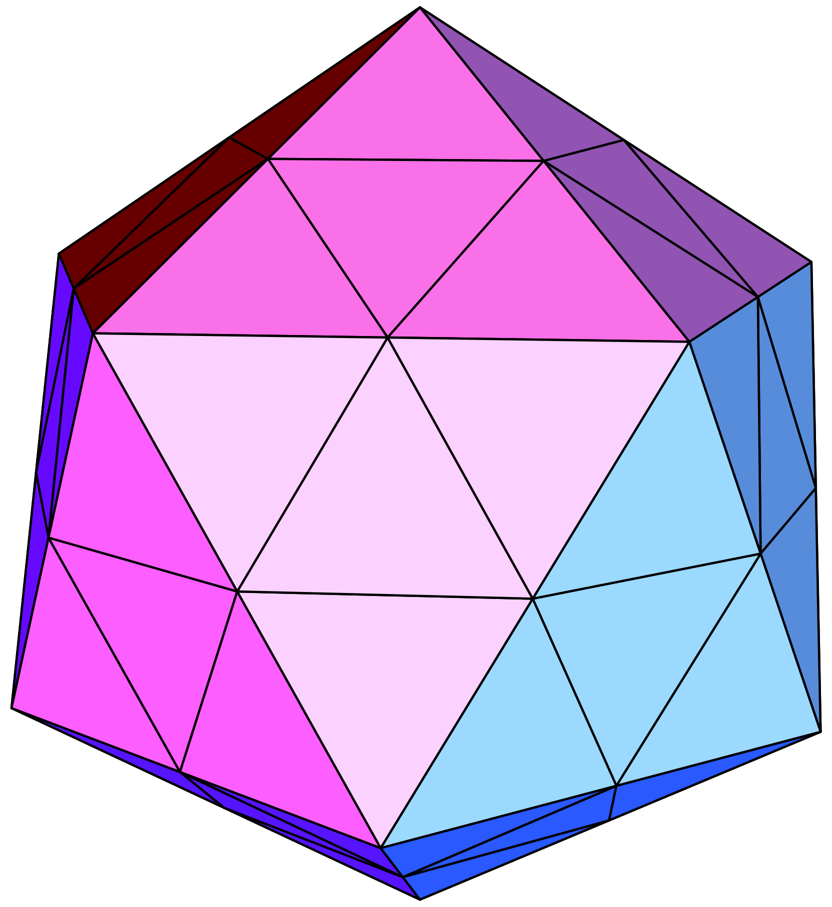


6 neighbors
6 wall neighbors

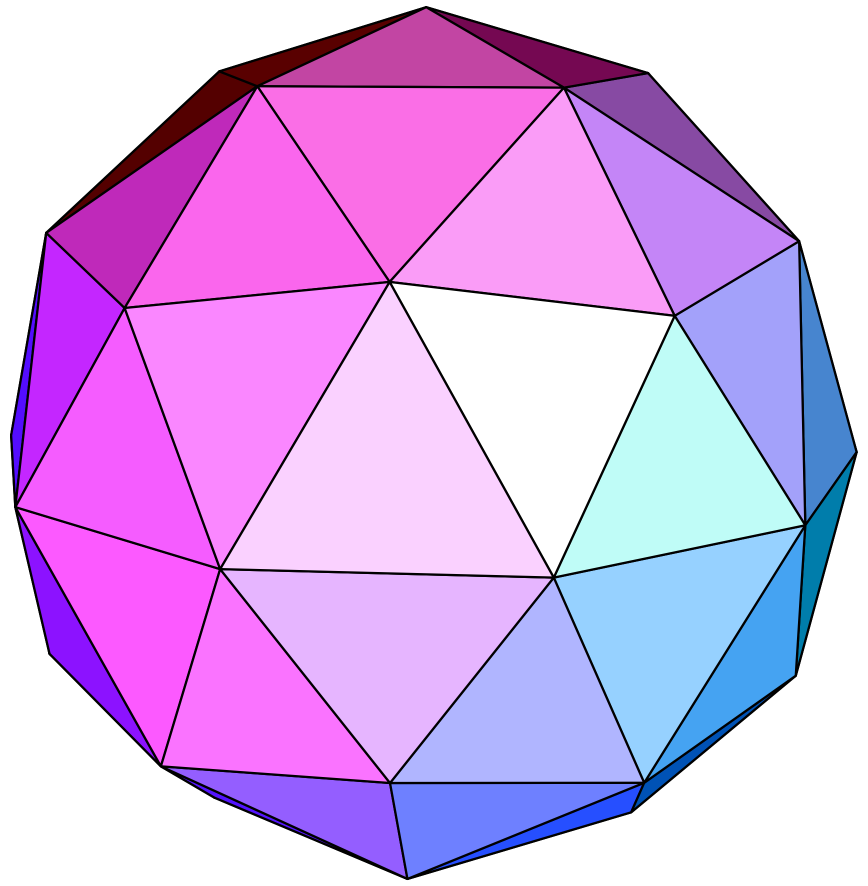
Geodesic Grids



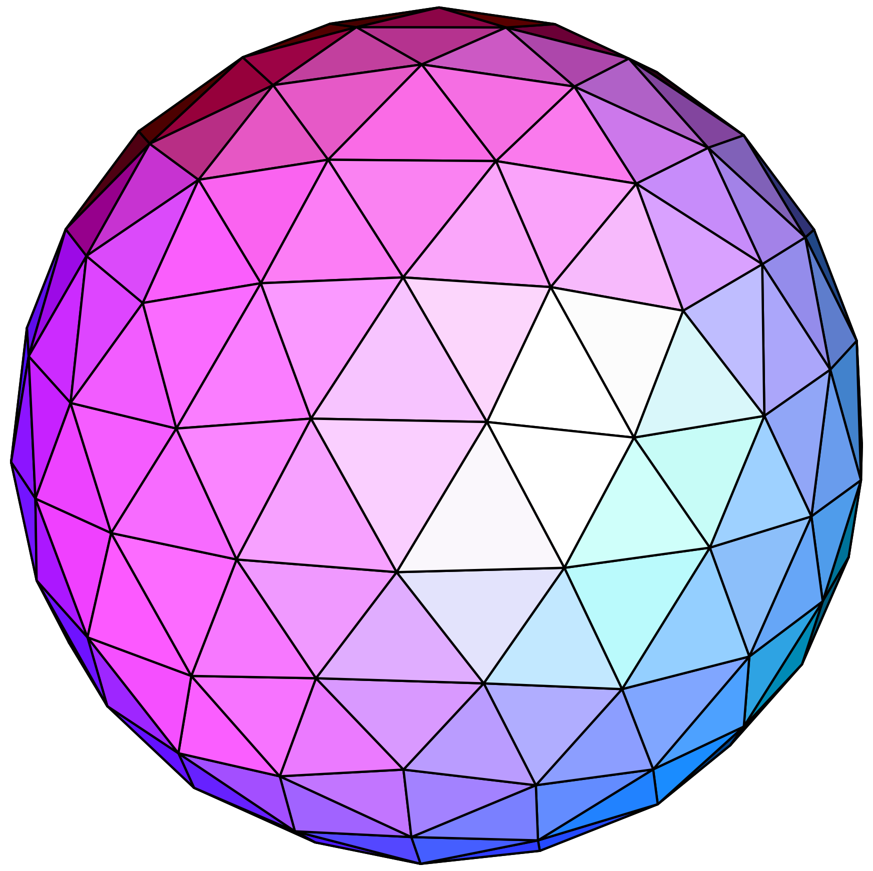
Icosahedron

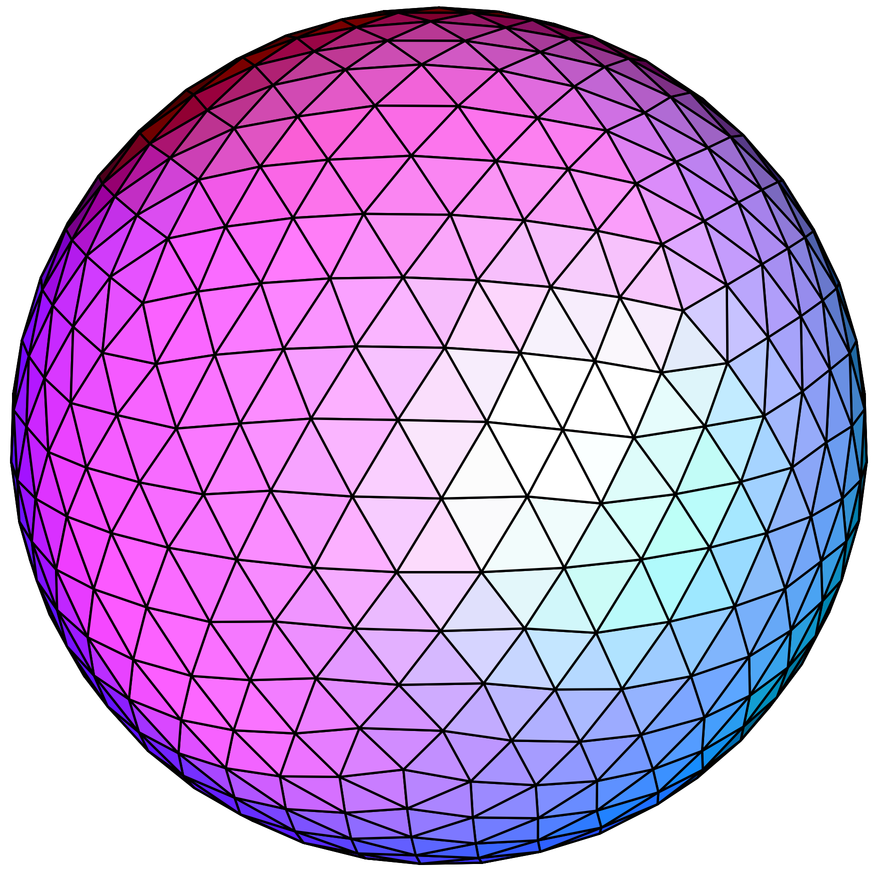


Bisect each edge
and connect the dots



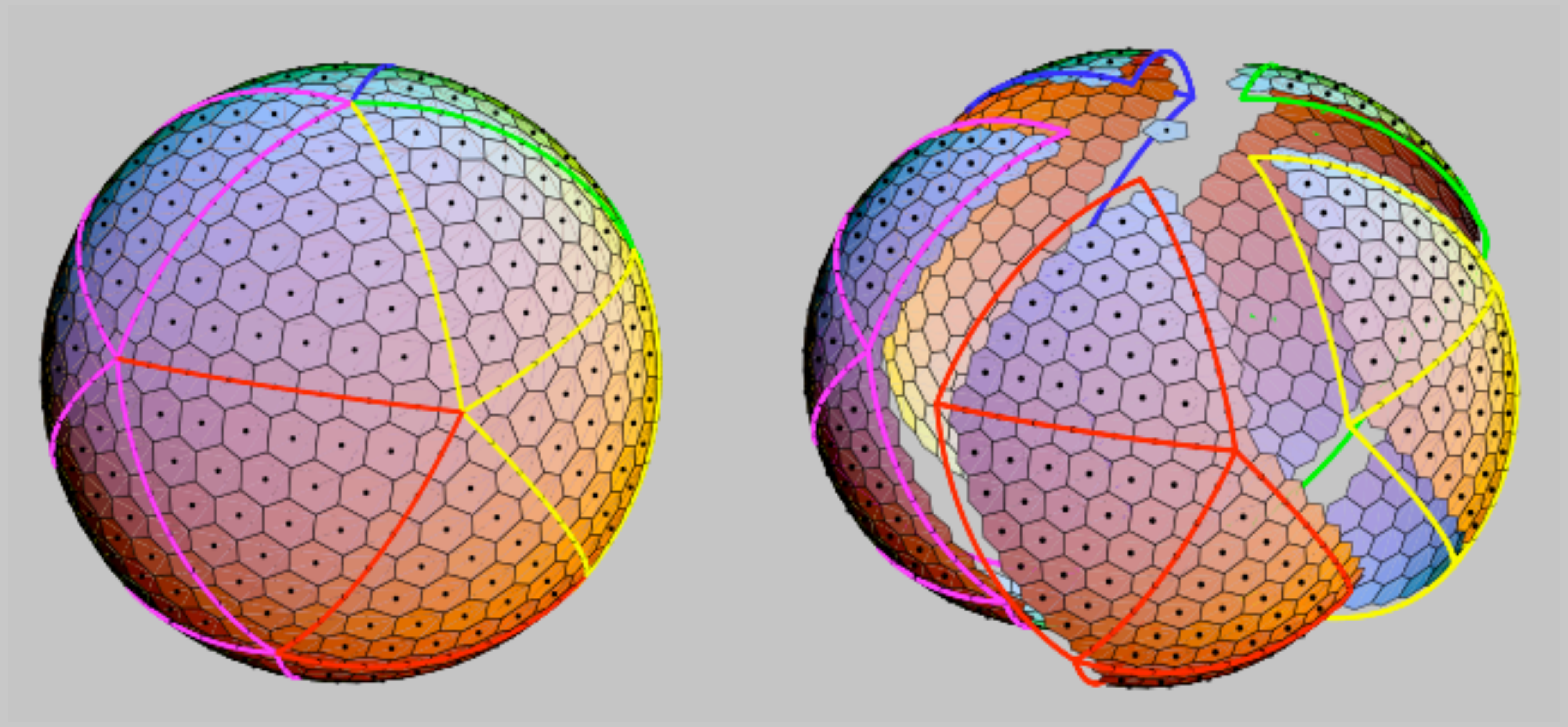
Pop out onto
the unit sphere

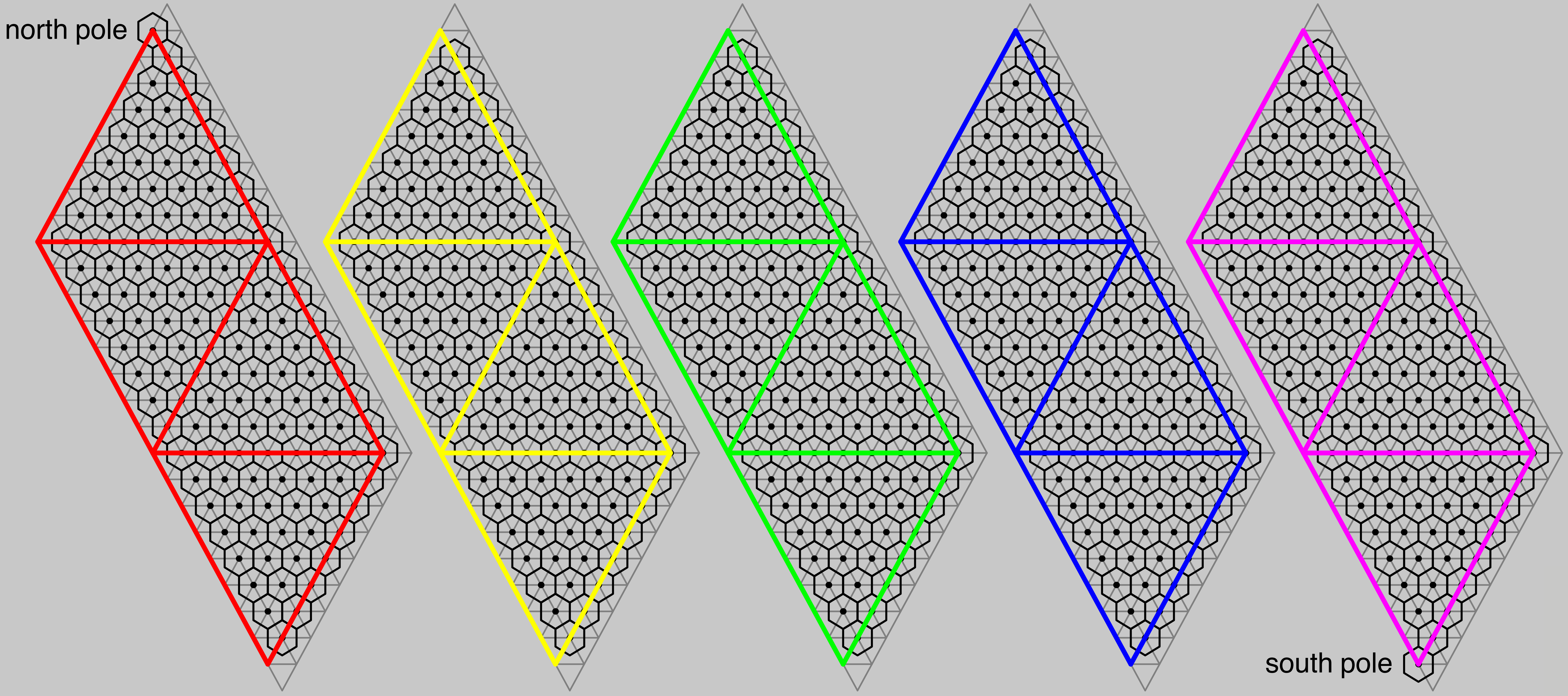




And so on, until we reach our target resolution...

Laying it out





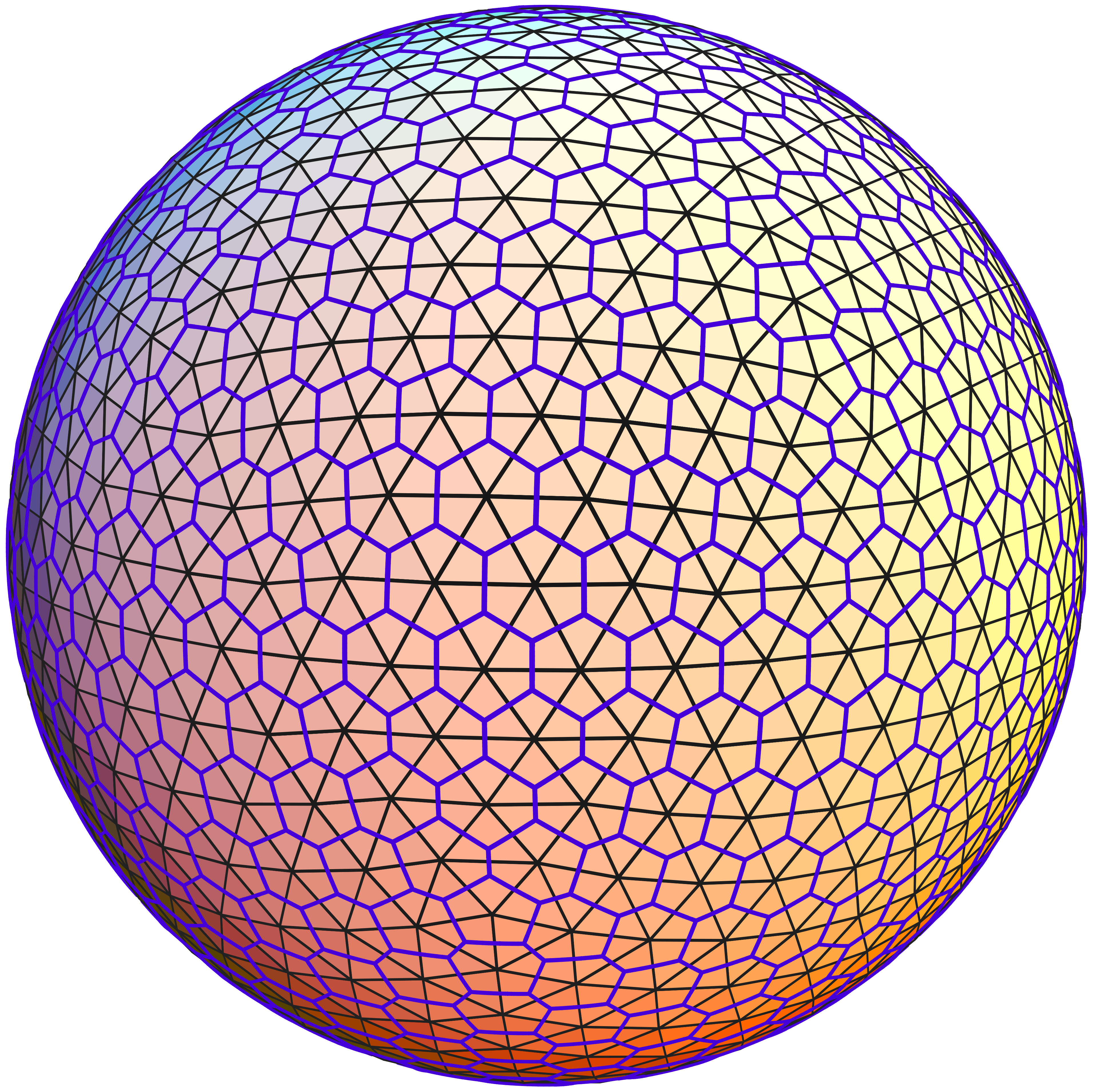
Merits of the geodesic grid

All cells are nearly the same size (within about 5% in terms of area).

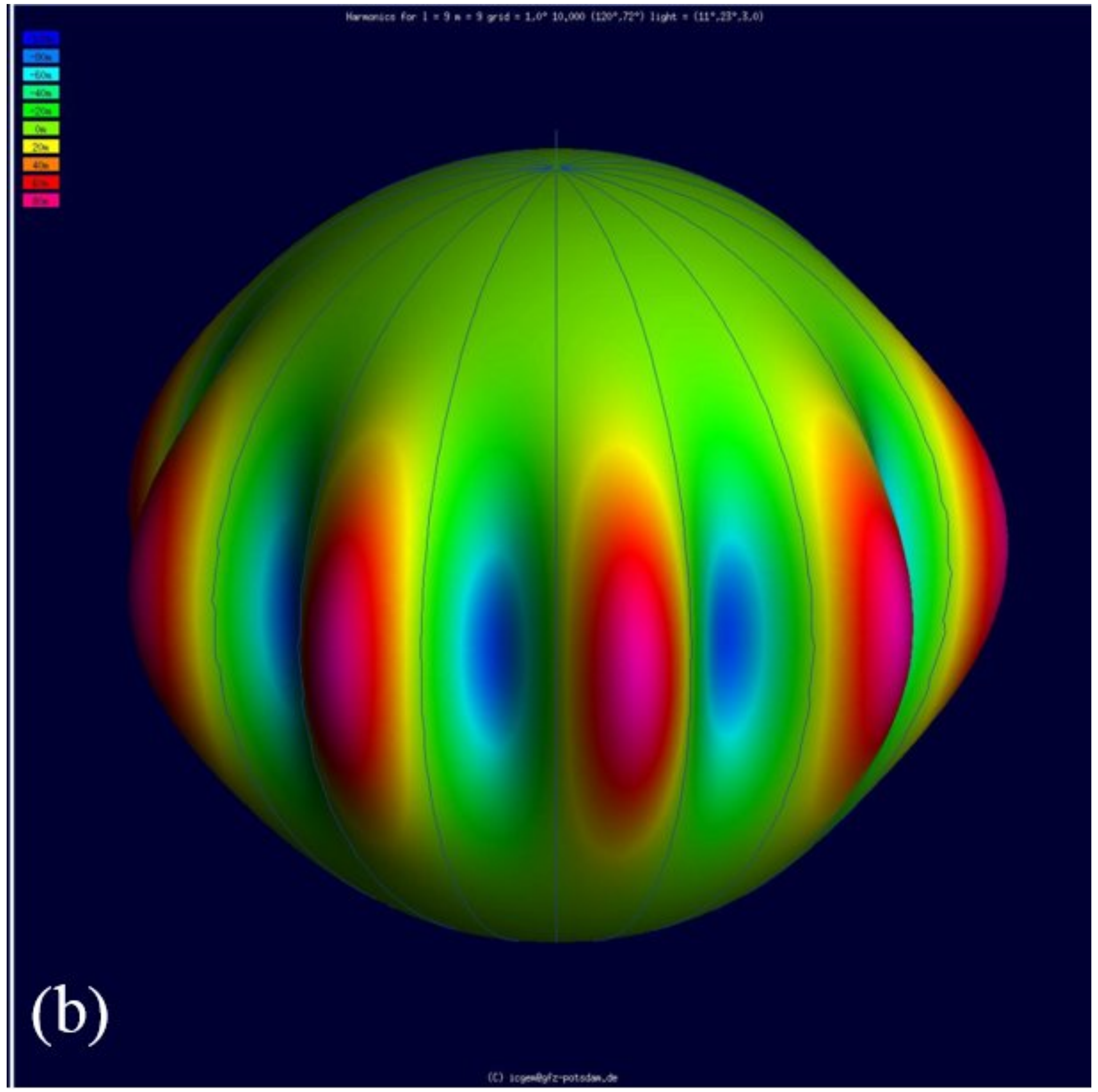
The grid is quasi-isotropic.

CFL for advection is not an issue.

Grid	No. of grid points N	Avg grid distance ℓ (km)
G0	12	6699.1
G1	42	3709.8
G2	162	1908.8
G3	642	961.4
G4	2562	481.6
G5	10 242	240.9
G6	40 962	120.4
G7	163 842	60.2
G8	655 362	30.1
G9	2 621 442	15.0
G10	10 485 762	7.53
G11	41 943 042	3.76
G12	167 772 162	1.88
G13	671 088 642	0.94



Discretizing the sphere with spherical harmonics



The spectral method will be discussed later.

Choice of vertical coordinate system

(Stairways to heaven)

Any vertical coordinate must be monotonic with height.

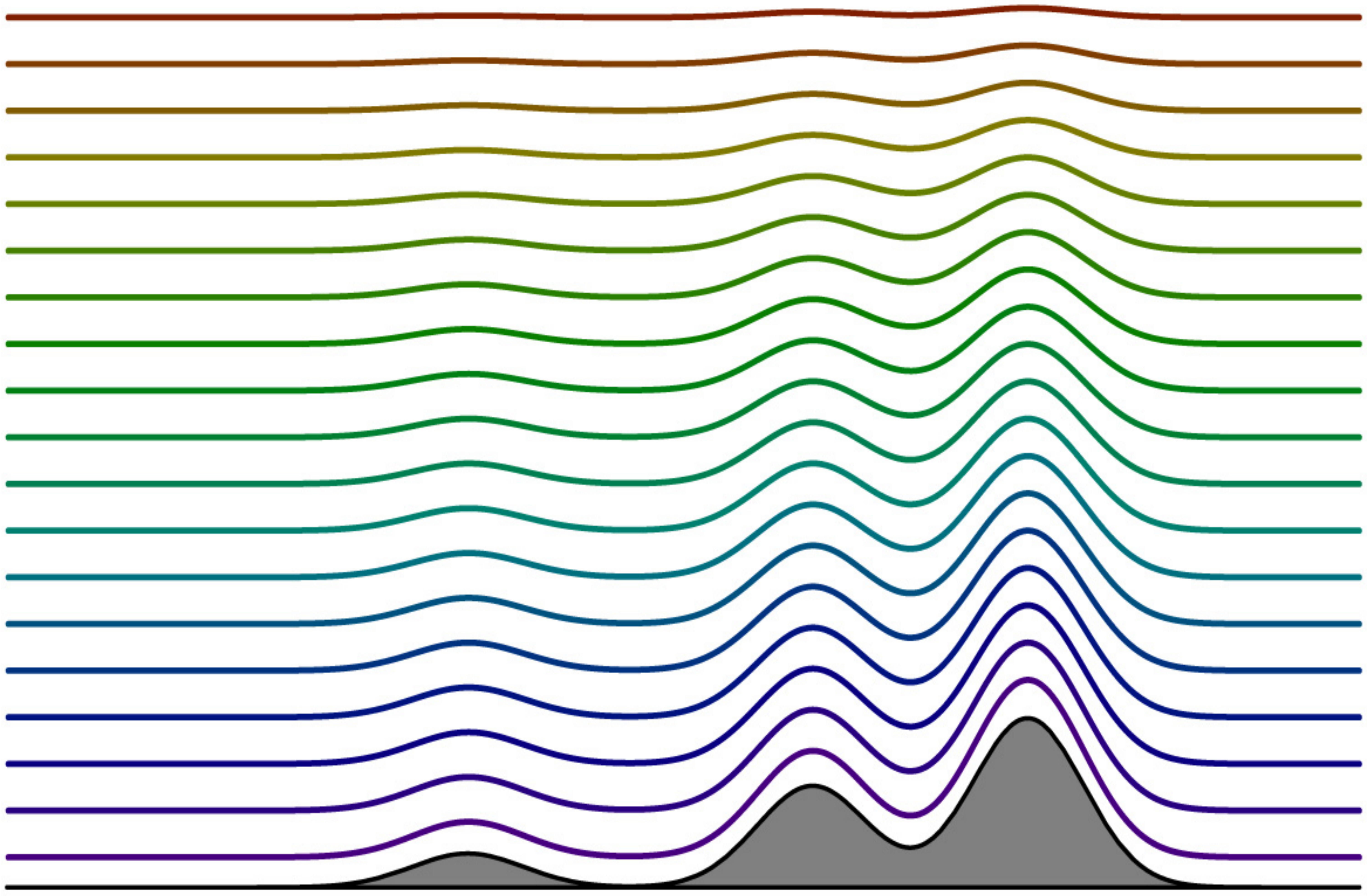
Coordinate	Strengths	Weaknesses
Height	Most intuititive	Intersects topography; Complicated equation for vertical velocity if quasi-static approximation is used
Pressure	Simple continuity equation; Simple computation of vertical velocity Pressure-gradient force is a gradient;	Intersects topography in different places on different days
Sigma	Follows the lower boundary; Constant at the top and bottom of the model	Issue with the pressure-gradient force
Hybrid sigma-pressure	Follows the lower boundary; Constant at the top and bottom of the model	Smaller issue with the pressure-gradient force
Theta	Minimizes vertical advection; Pressure-gradient force is a gradient; Convenient for computing PV	Not strictly monotonic with height; Low vertical resolution in mixed layers; Intersects topography in different places on different days
Hybrid sigma-theta	Follows the lower boundary; Behaves like theta away from the lower boundary; Can work even when theta is not monotonic with height	Complicated equation for vertical velocity if quasi-static approximation is used

Sigma coordinates

$$\sigma \equiv \frac{p - p_T}{p_S - p_T}$$

$$\sigma_S = 1$$

$$\sigma_T = 0$$



Example of a finite-difference scheme:

The "upstream scheme" for advection of a scalar

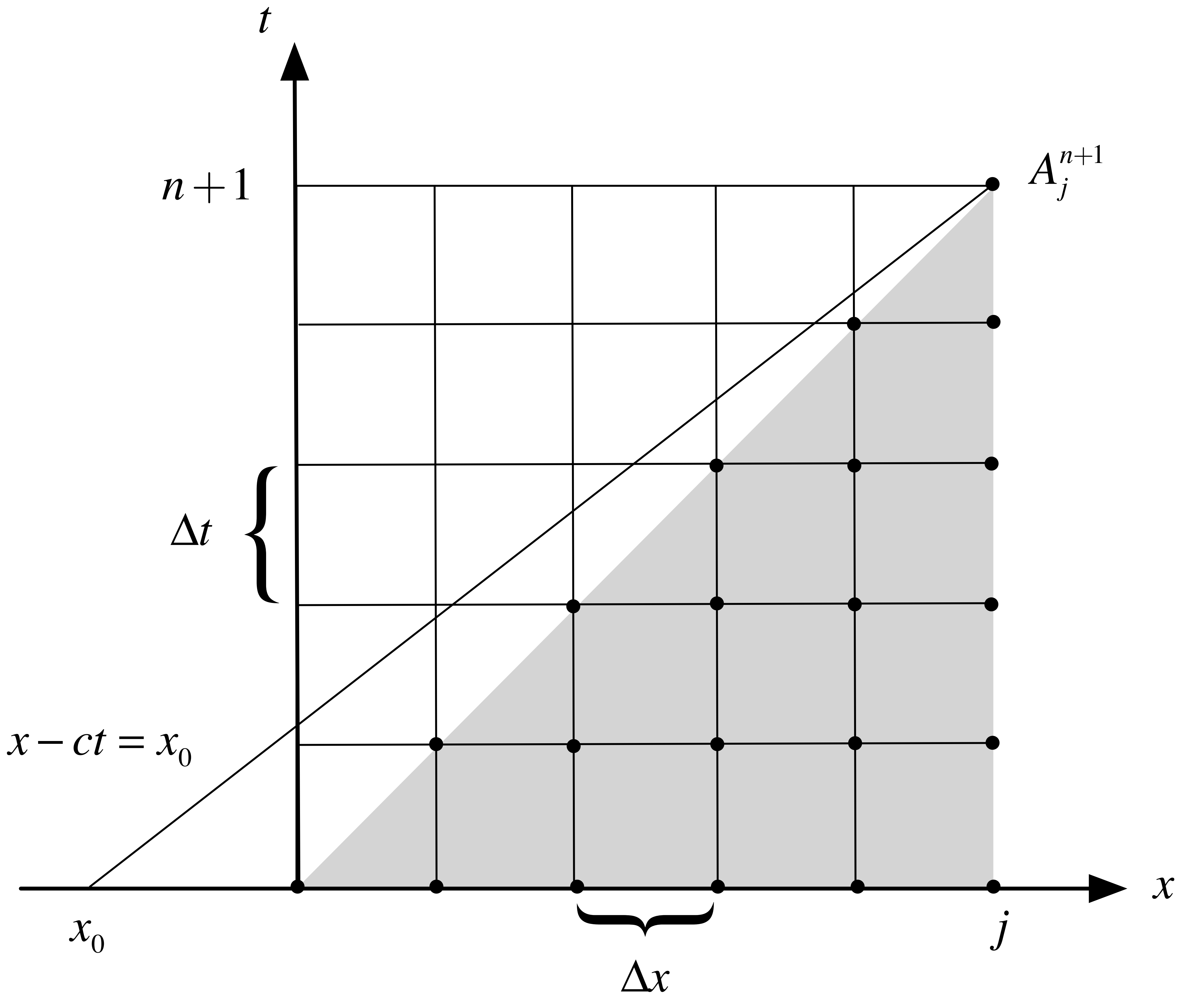
$$\left(\frac{\partial A}{\partial t}\right)_x + c\left(\frac{\partial A}{\partial x}\right)_t = 0$$

$$\frac{A_j^{n+1} - A_j^n}{\Delta t} + c\left(\frac{A_j^n - A_{j-1}^n}{\Delta x}\right) = 0$$

This is for written $c > 0$. For $c < 0$ we would use j and $j+1$.

$$A_j^{n+1} = A_j^n(1-\mu) + A_{j-1}^n \mu$$

Here $\mu \equiv c\Delta t/\Delta x$.



The shaded area is the *domain of dependence* for A_j^{n+1}.

Computational instability

Computational instability is the unbounded growth of discretization errors.

Instability leads to noise, but a noisy scheme is not necessarily unstable.

There are multiple types of numerical instability, just as there are multiple types of physical instability.

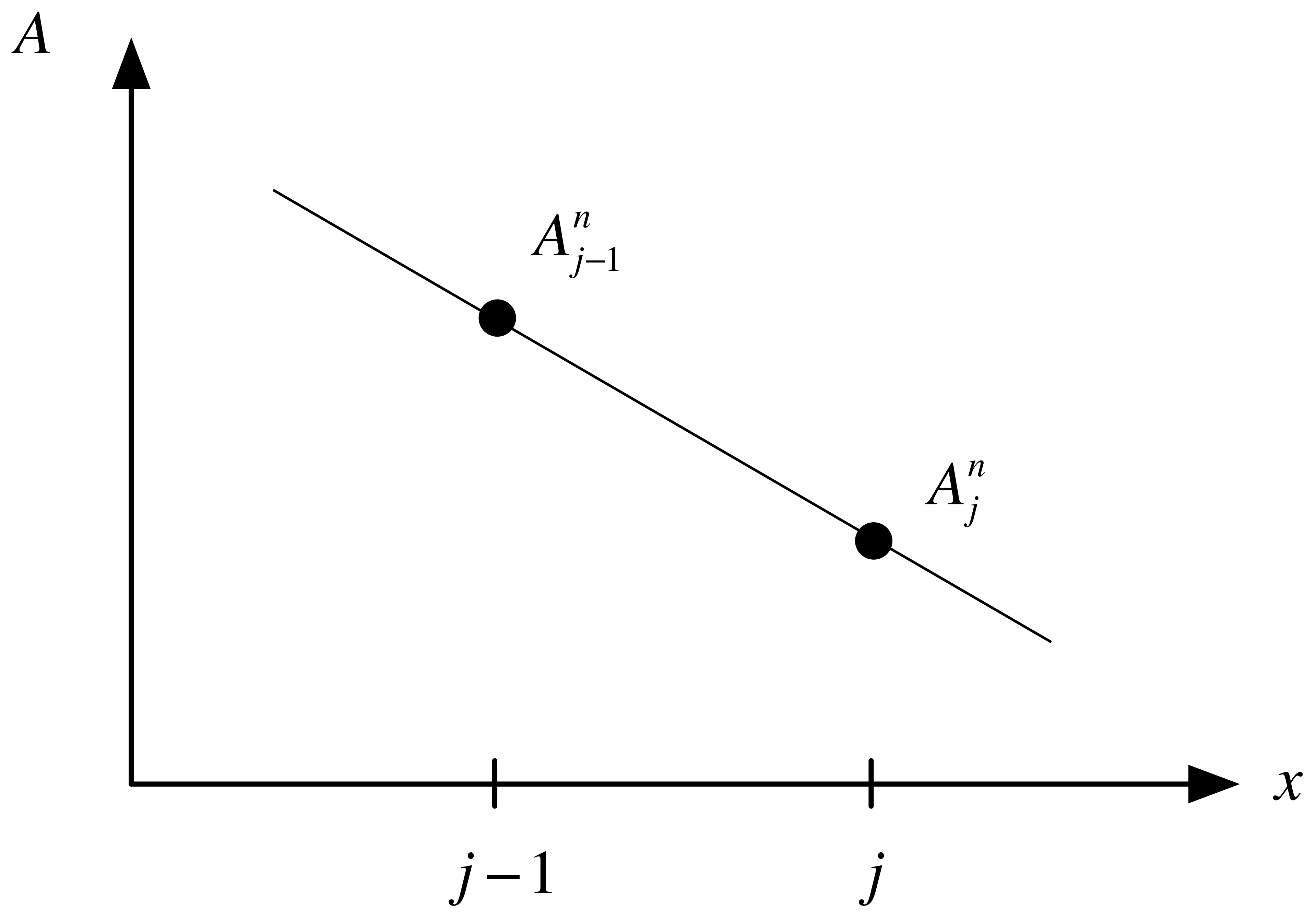
- "Linear" instability can sometimes be avoided by using a sufficiently small time step.
- "Nonlinear" instability can occur even with continuous time derivatives.

Instability usually occurs on the smallest scales represented in a model.

Unstable schemes can often be identified by a pencil-and-paper analysis, before a single line of code is written.

Is the upstream scheme stable?

$$A_j^{n+1} = A_j^n(1-\mu) + A_{j-1}^n \mu \qquad \text{where } \mu \equiv c\Delta t/\Delta x$$

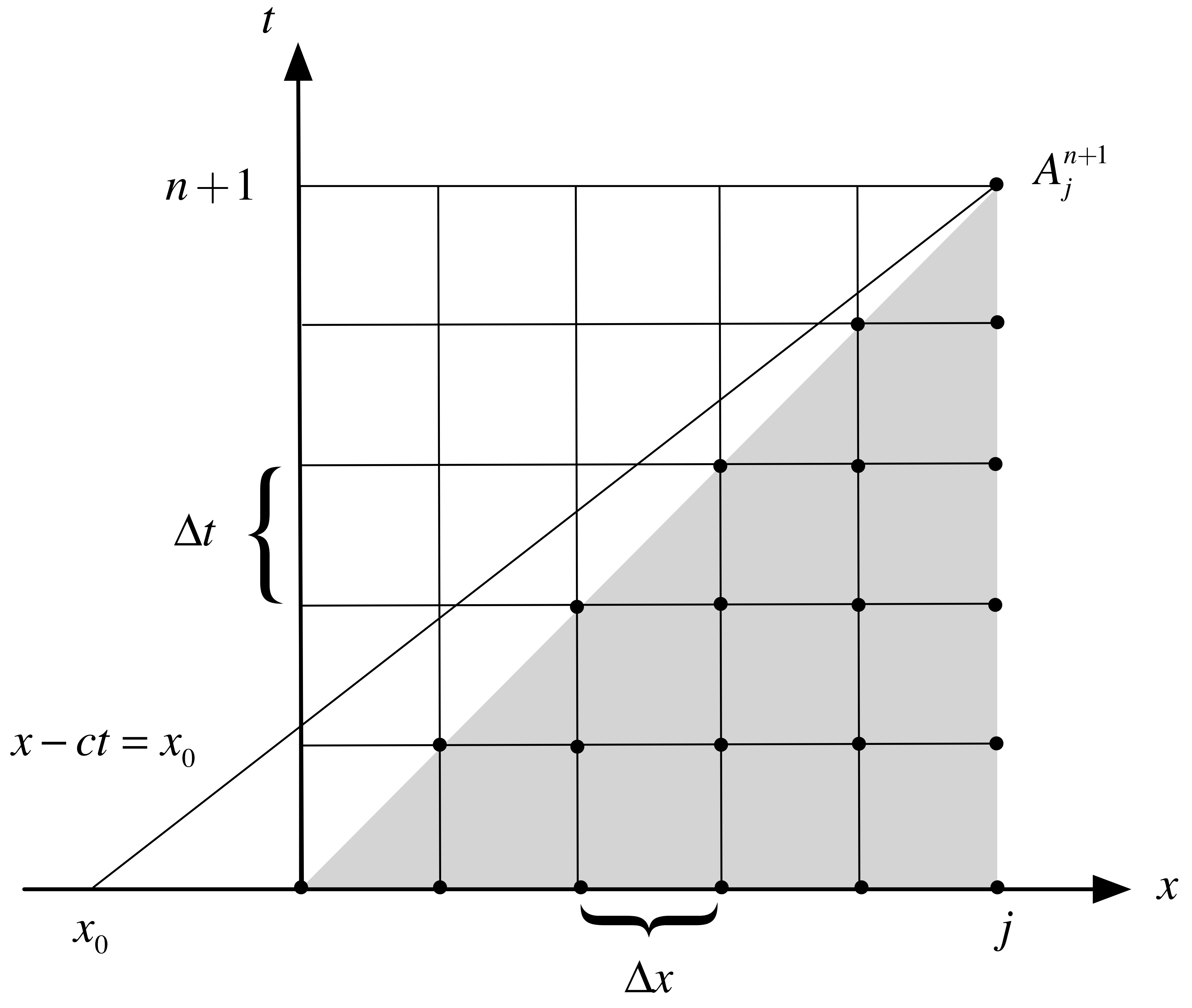


For $0 \le \mu \le 1$, the scheme interpolates, so that A_j^{n+1} falls in between A_j^n and A_{j-1}^n. Blow-up is therefore impossible, but damping can smooth out the solution excessively.

For $\mu < 0$ or $\mu > 1$, the scheme extrapolates, so that A_j^{n+1} lies outside the range of A_j^n and A_{j-1}^n. With more and more time steps, the solution will run away.

Conclusion: The scheme is stable for $0 \le \mu \le 1$, and unstable for $\mu < 0$ or $\mu > 1$.

Circling back



For $0 \leq \mu \leq 1$, the solution lies within the domain of dependence.

The scheme is stable when the solution lies within the domain of dependence.

Wave propagation can also cause numerical instability

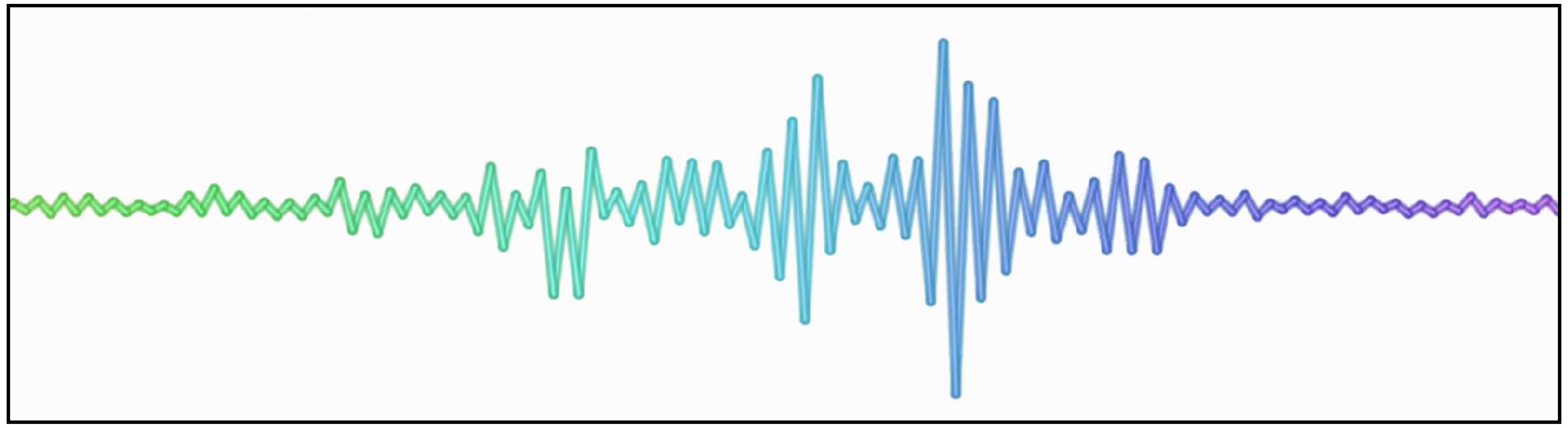
For wave propagation, the requirement for stability has almost the same form as for advection, i.e.,

$$\mu \equiv c\Delta t / \Delta x,$$

where c is the phase speed of the wave.

Sound waves, including the Lamb wave, have phase speeds ~ 300 m s^{-1}, much larger than typical wind speeds, so the time-step limit tends to be set by wave propagation rather than advection.

Vertically propagating sound waves are troublesome.



In the Earth's atmosphere, the speed of sound is in the range 280 - 350 m s^{-1}, depending on temperature.

If vertically propagating sound waves are simulated with explicit time differencing, the time step required for computational stability can be as small as one second, because the vertical grid spacing can be on the order of 300 m.

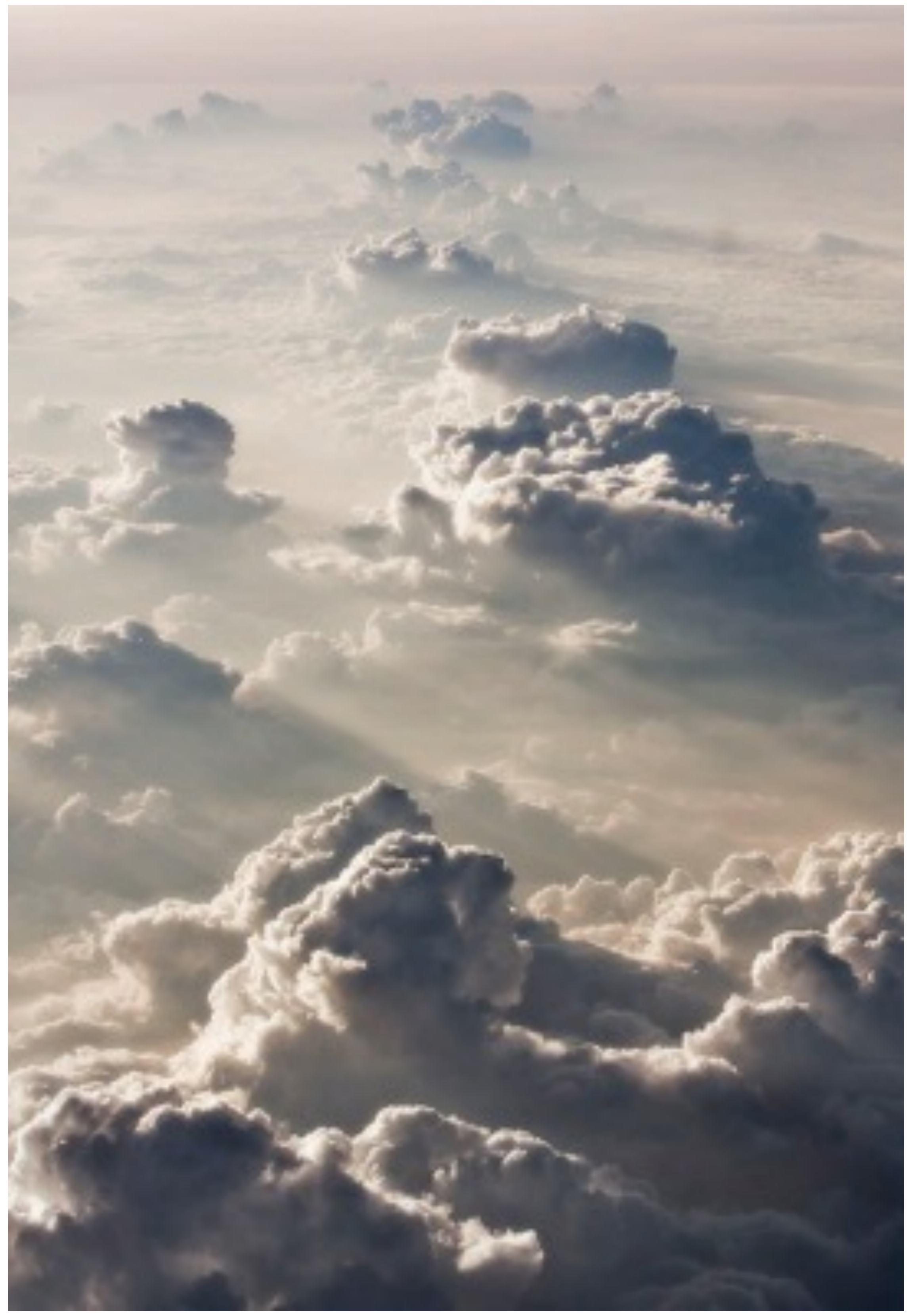
Horizontally propagating sound waves are not so bad if the grid cells are much wider than they are tall.

300 m

100 km

Four ways to deal with vertically propagating sound waves

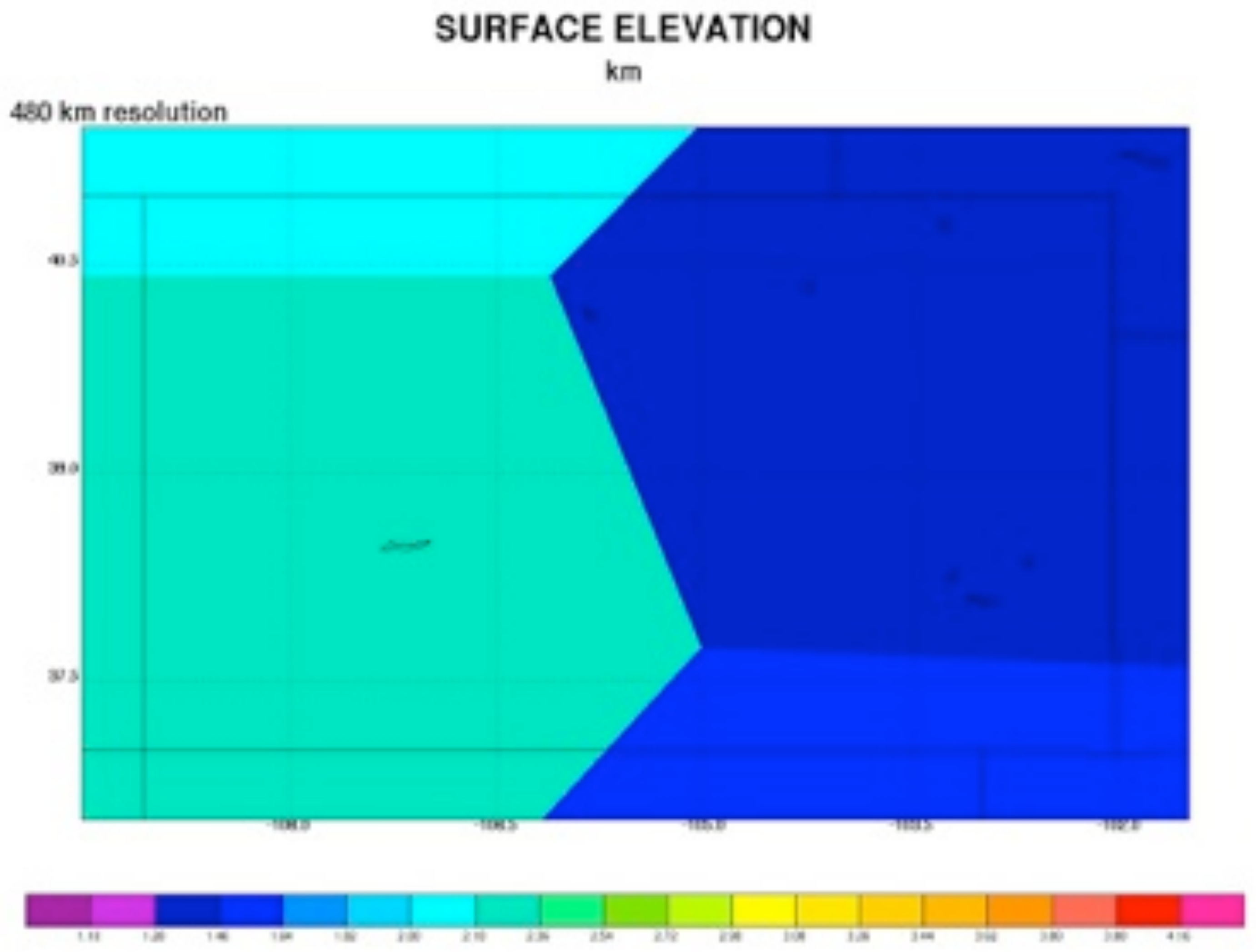
- Take small time steps (probably too expensive).
- Sub-cycle (used in some mesoscale models).
- Use partially implicit time-differencing methods (keeps sound waves, but slows them down).
- Filter the vertically propagating sound waves from the continuous equations.
  - Quasi-static approximation
  - Anelastic approximation
  - Unified System

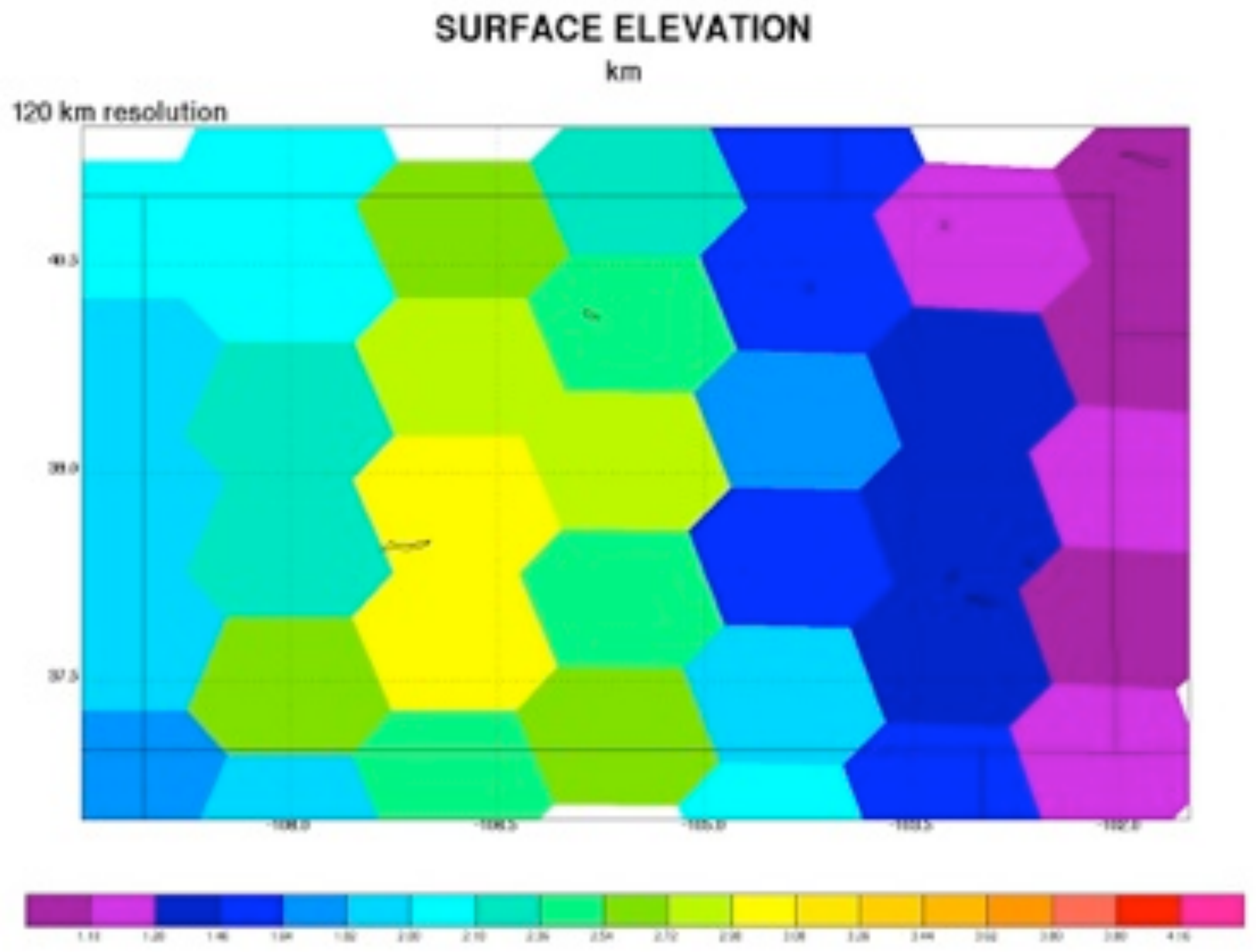


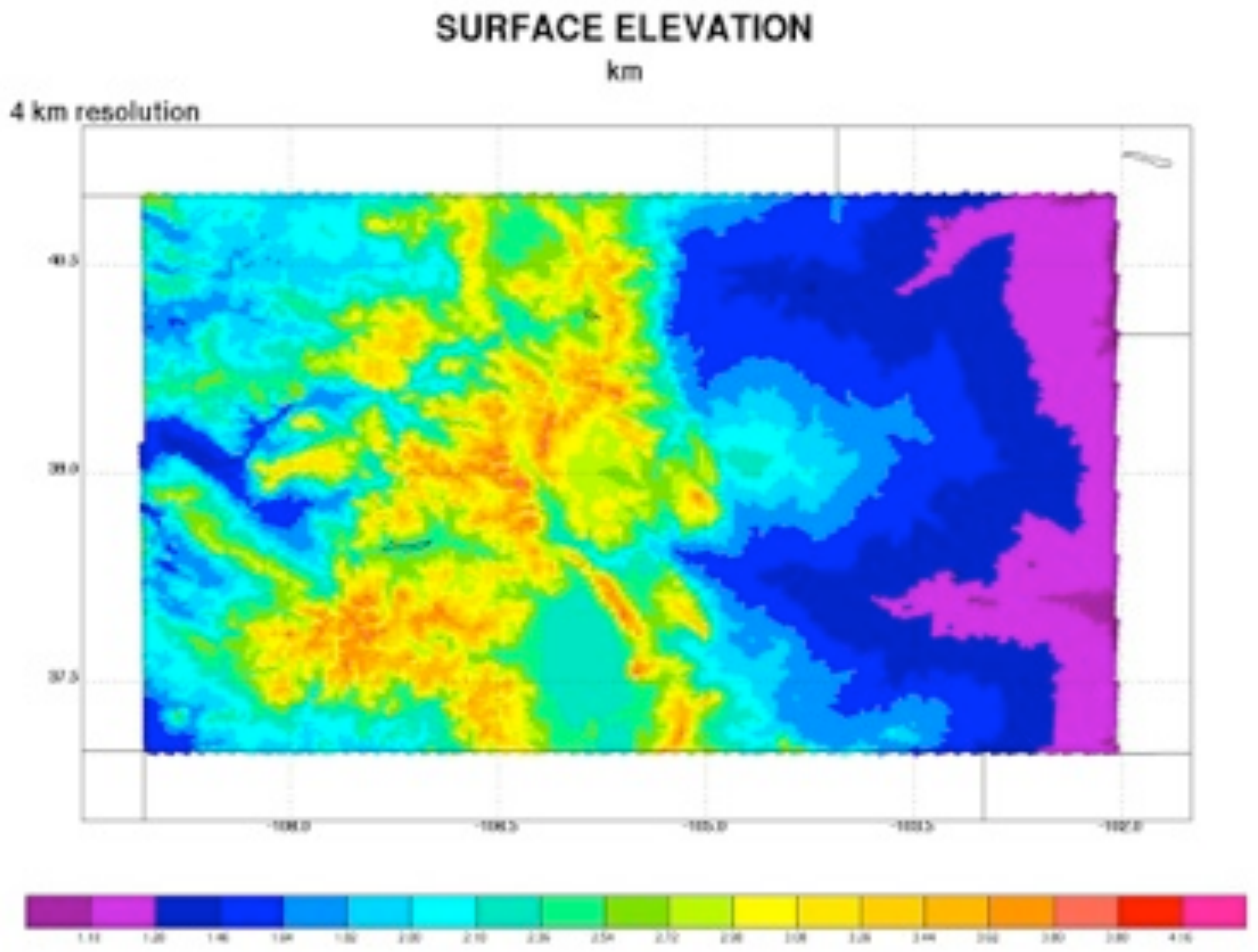
The first three approaches are numerical.

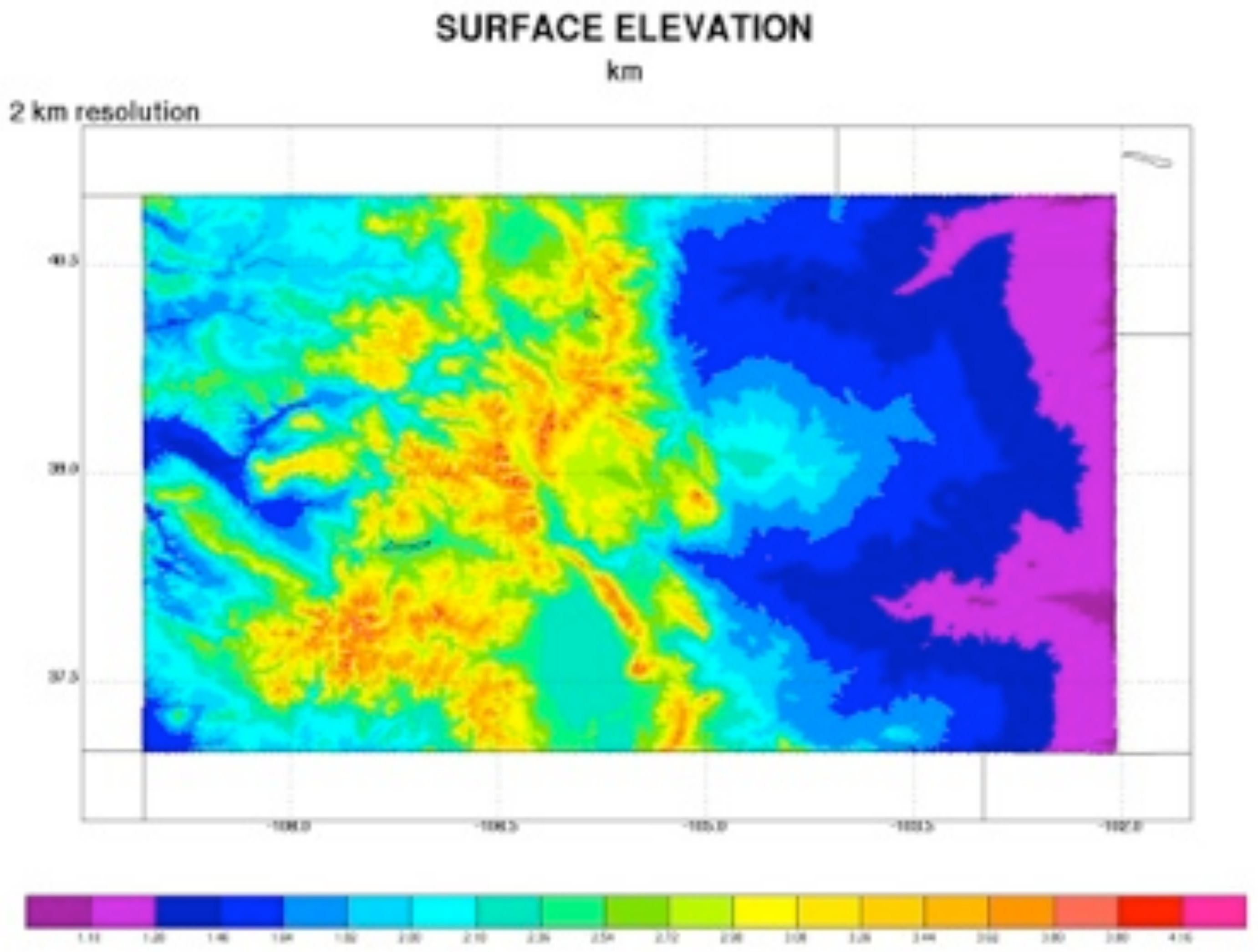
The fourth approach is based on physical approximation.

Colorado topography



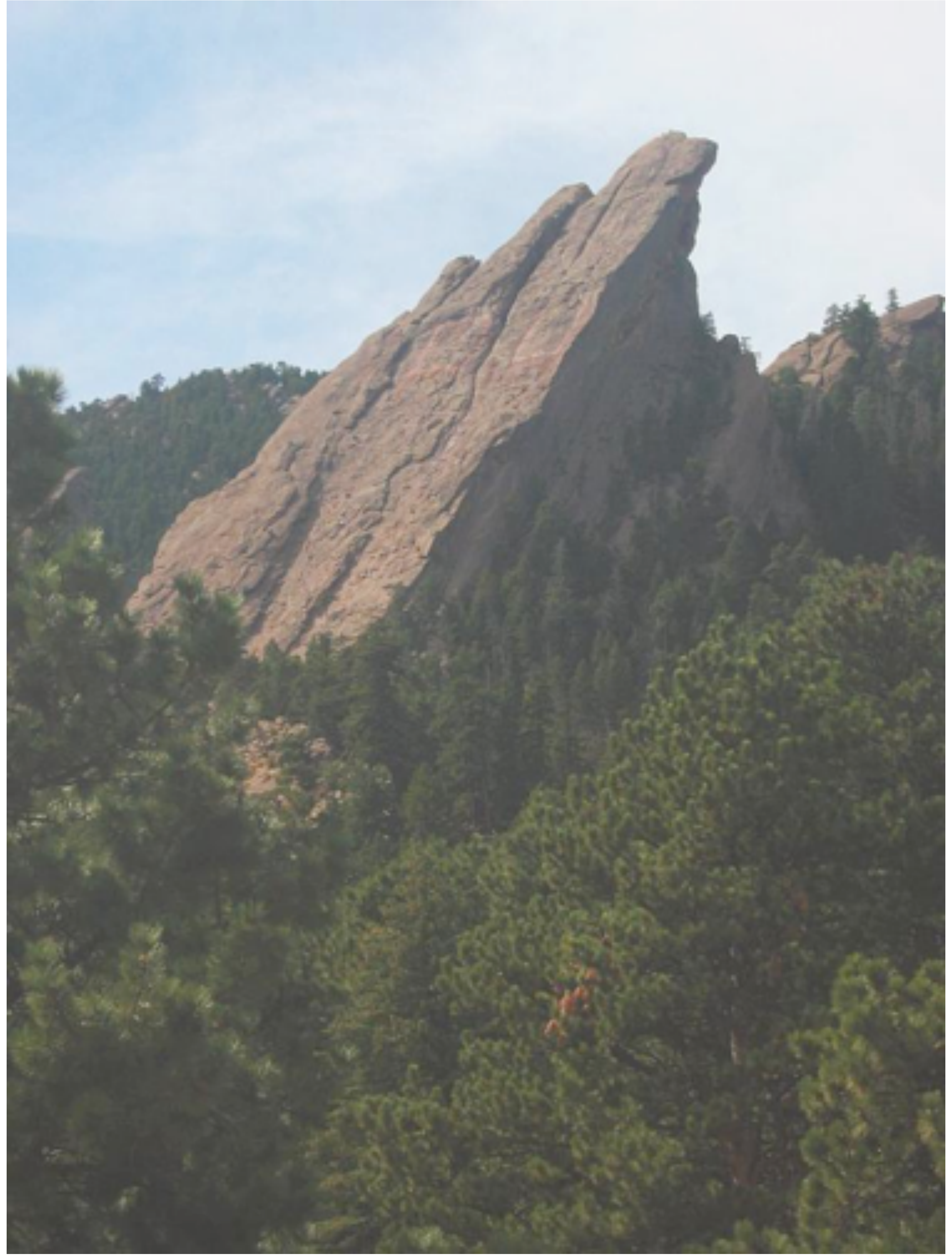






Topography

- With high resolution, terrain slopes become very large.
- Terrain-following coordinates do not work well under those conditions.



Filtering sound waves

There are two ways to filter sound waves.

Quasi-hydrostatic system :

$$\cancel{\frac{Dw}{Dt}} + \frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

Vertical momentum equation becomes diagnostic.

To satisfy this for all t, vertical velocity must be passive to other variables.

Anelastic system :

$$\cancel{\frac{\partial \rho}{\partial t}} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Continuity equation becomes diagnostic.

To satisfy this for all t, pressure gradient force must be passive to other forces.

↑ For cloud-resolving models, filtering must be this type.

WHAT IS THE MINIMUM REQUIREMENT FOR FILTERING VERTICALLY PROPAGATING SOUND WAVES?

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho_{qs}}{\partial t} + \frac{\partial \delta\rho}{\partial t}$$

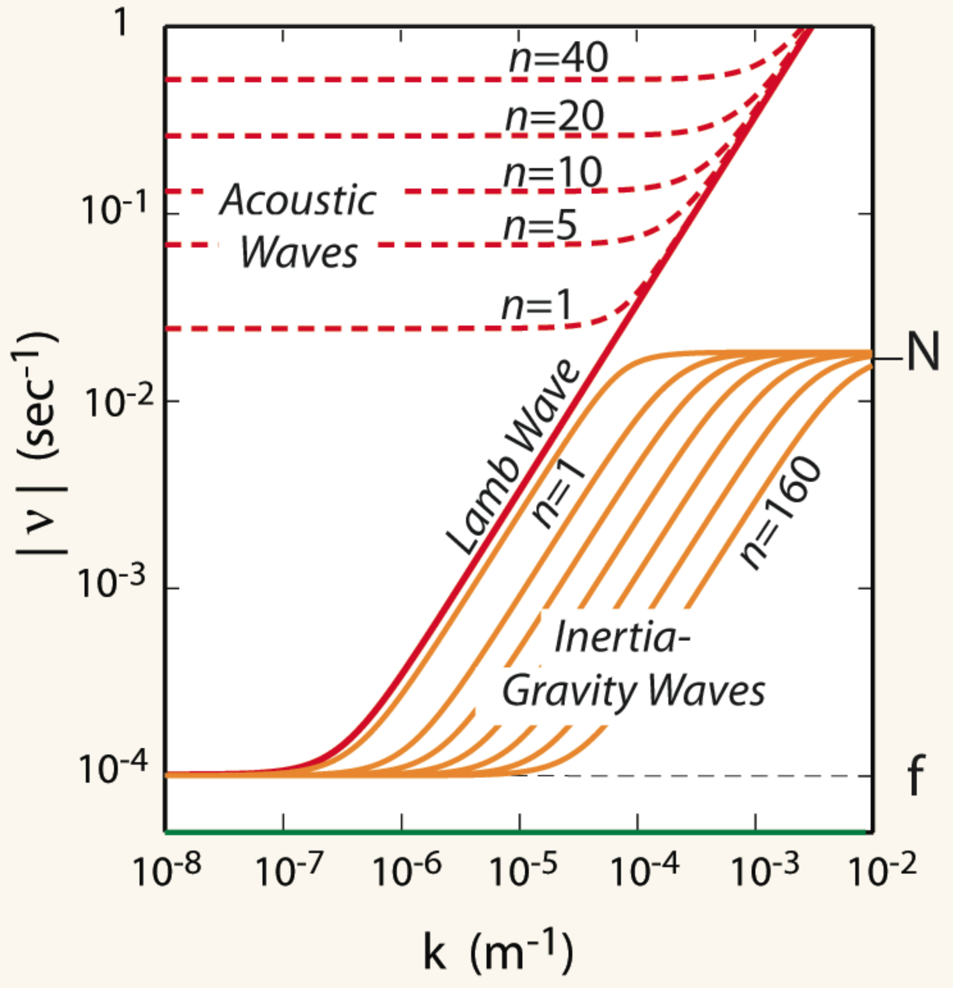
- ρ_{qs} : "quasi-hydrostatic density" that satisfies $\partial p_{qs} / \partial z = -\rho_{qs} g$
- $\delta\rho$: "non-hydrostatic density" defined by $\delta\rho \equiv \rho - \rho_{qs}$

Since vertically propagating sound waves are non-hydrostatic, it is sufficient to drop only the $\partial \delta\rho / \partial t$ term for filtering those waves.

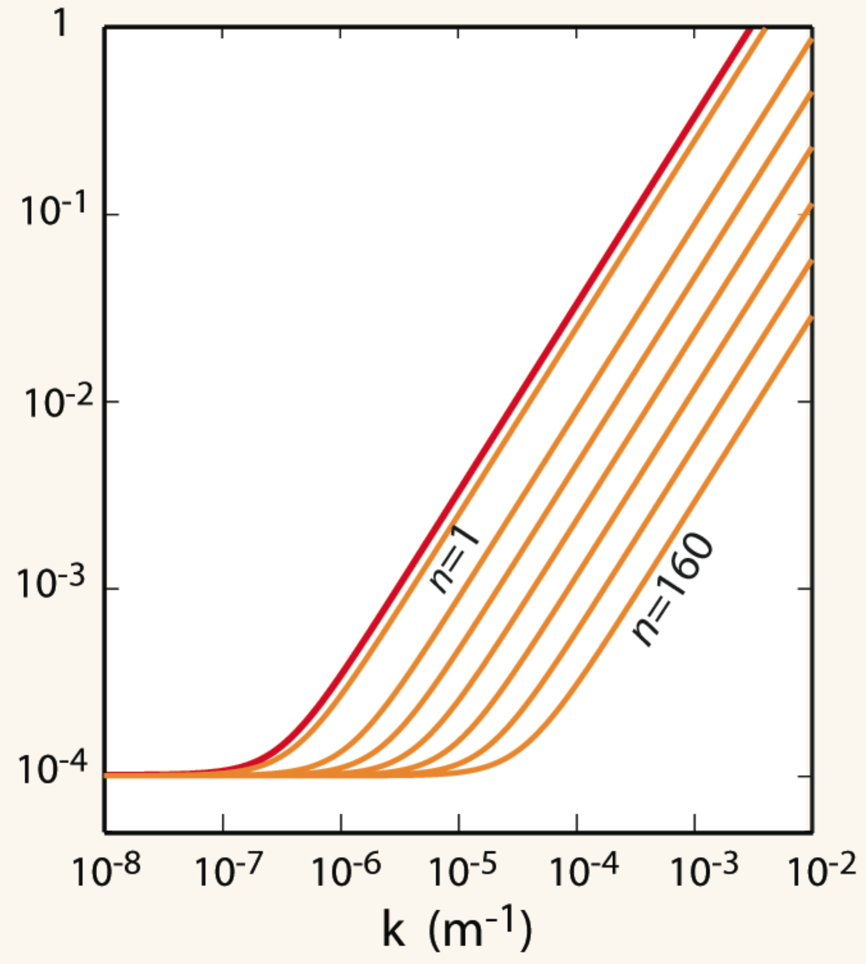
DISPERSION RELATION FOR PERTURBATIONS ON A RESTING ISOTHERMAL ATMOSPHERE ON A f-PLANE

(WITHOUT QUASI-GEOSTROPHIC APPROXIMATION)

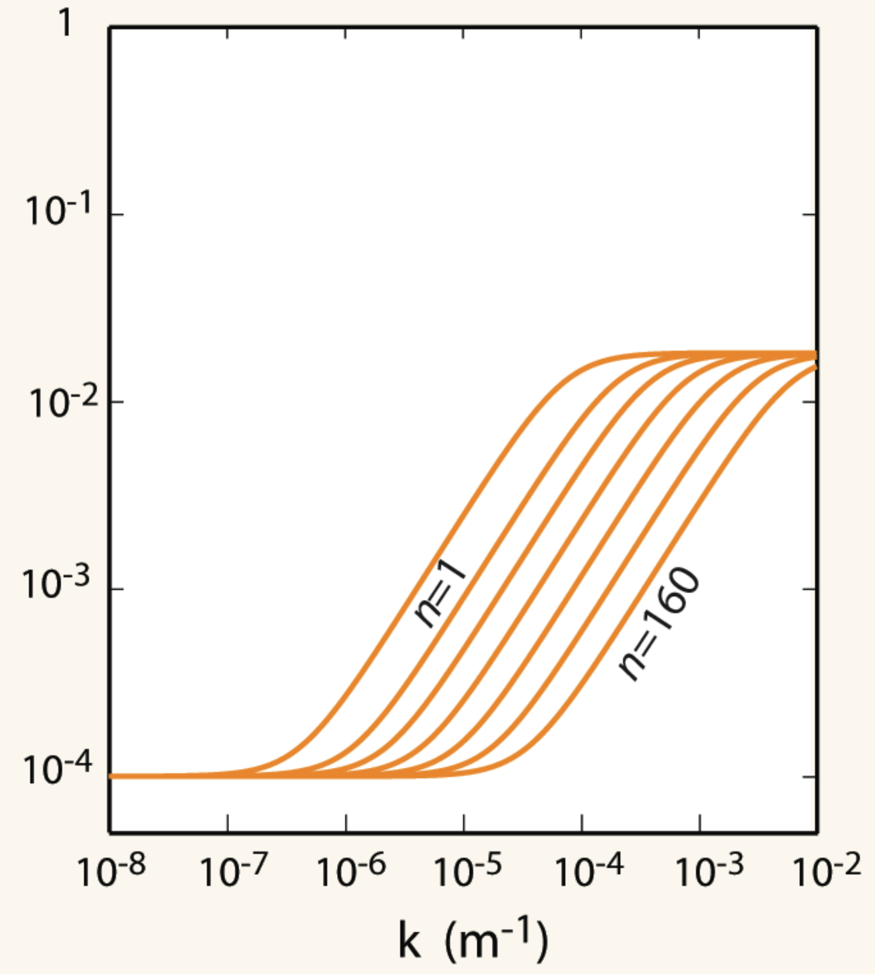
(a) Compressible Non-Nydrostaic



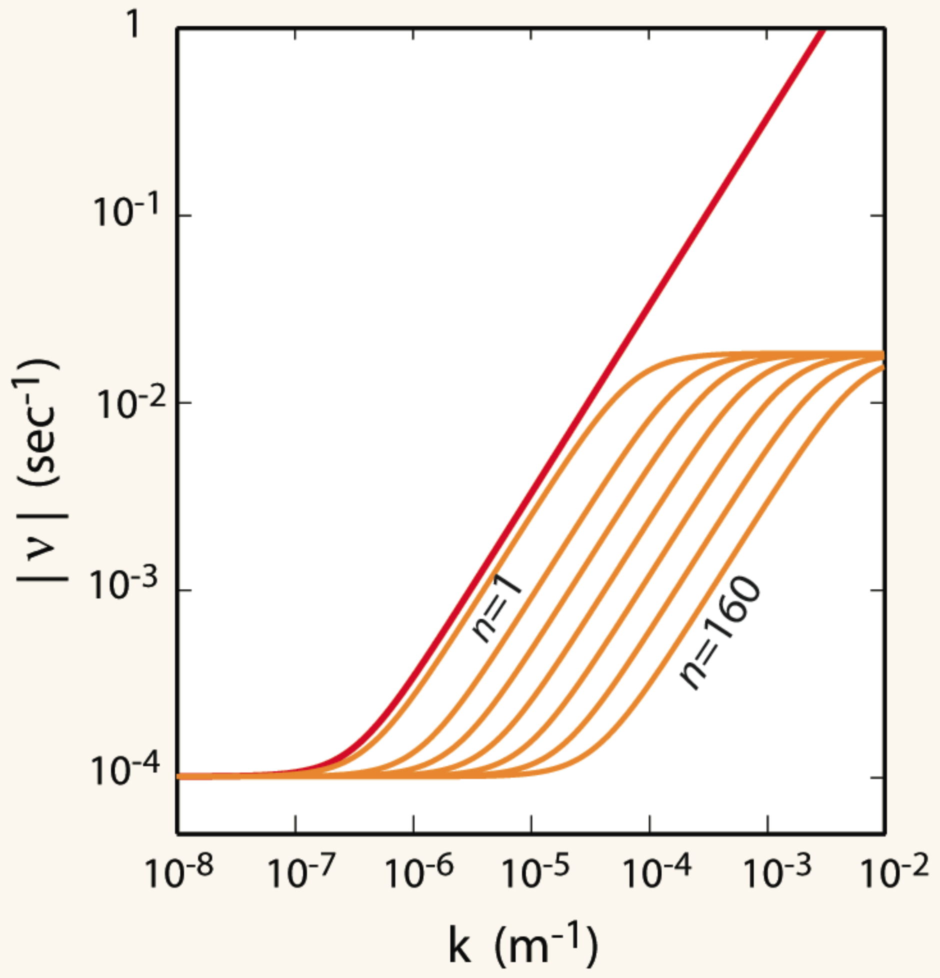
(b) Quasi-Hydrostatic



(c) Anelastic Non-Hydrostatic



(d) Unified



The real problem of the anelastic system
is distortion of vertical structure,
not in this dispersion relation.

Computational modes

Computational modes are solutions of the discrete equations that do not correspond to any solutions of the continuous equations.

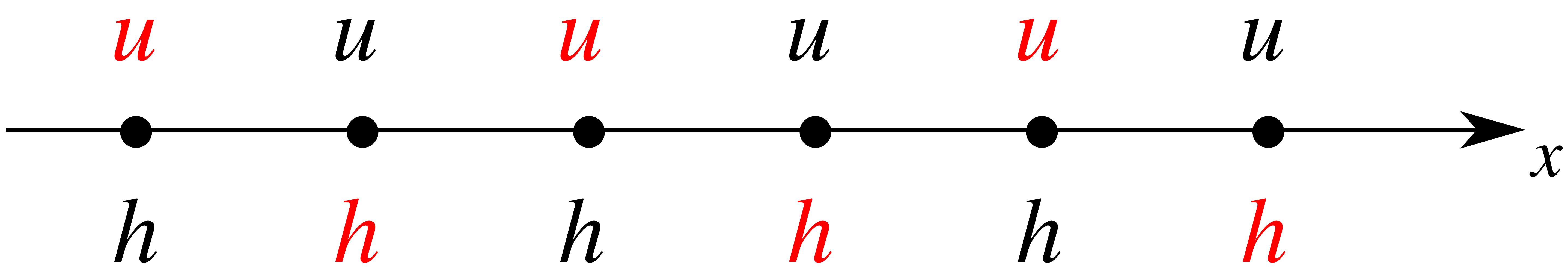
In other words, computational modes are spurious or bogus "extra" solutions.

Computational modes generally manifest as "noise."

An example of a computational mode in space

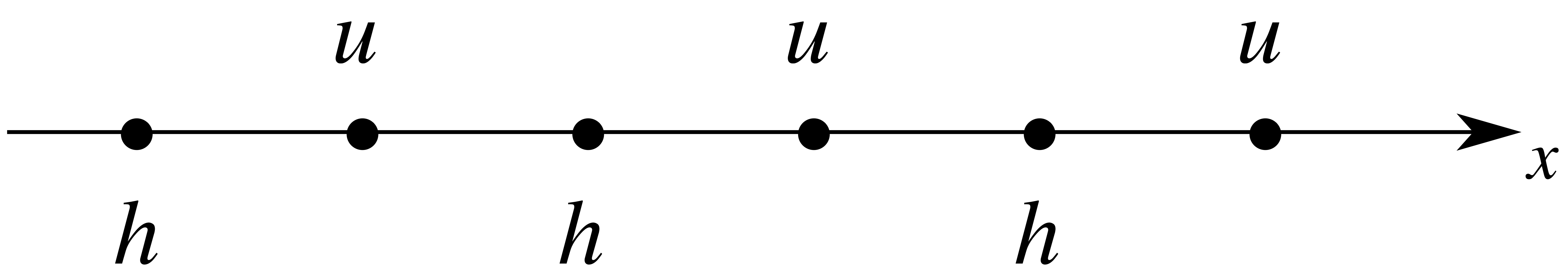
$$\frac{du_j}{dt} + g\left(\frac{h_{j+1} - h_{j-1}}{2d}\right) = 0,$$

$$\frac{dh_j}{dt} + H\left(\frac{u_{j+1} - u_{j-1}}{2d}\right) = 0,$$



Here the computational mode comes from (accidental) redundancy.

A staggered grid

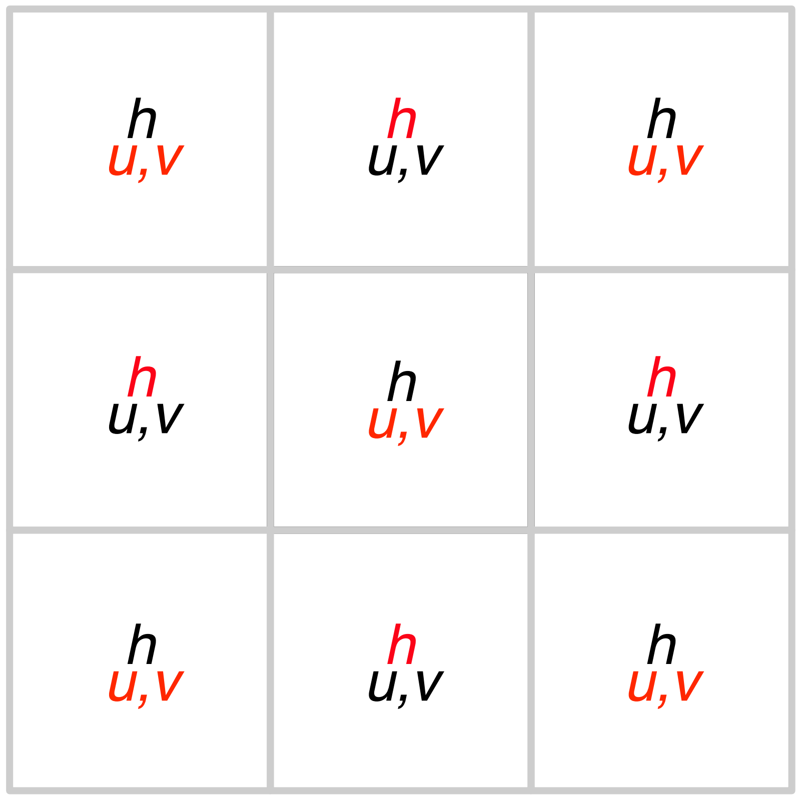


If we "erase" the red variables, the computational mode is eliminated, but now u and h are defined in different places. This is an example of a "staggered" grid.

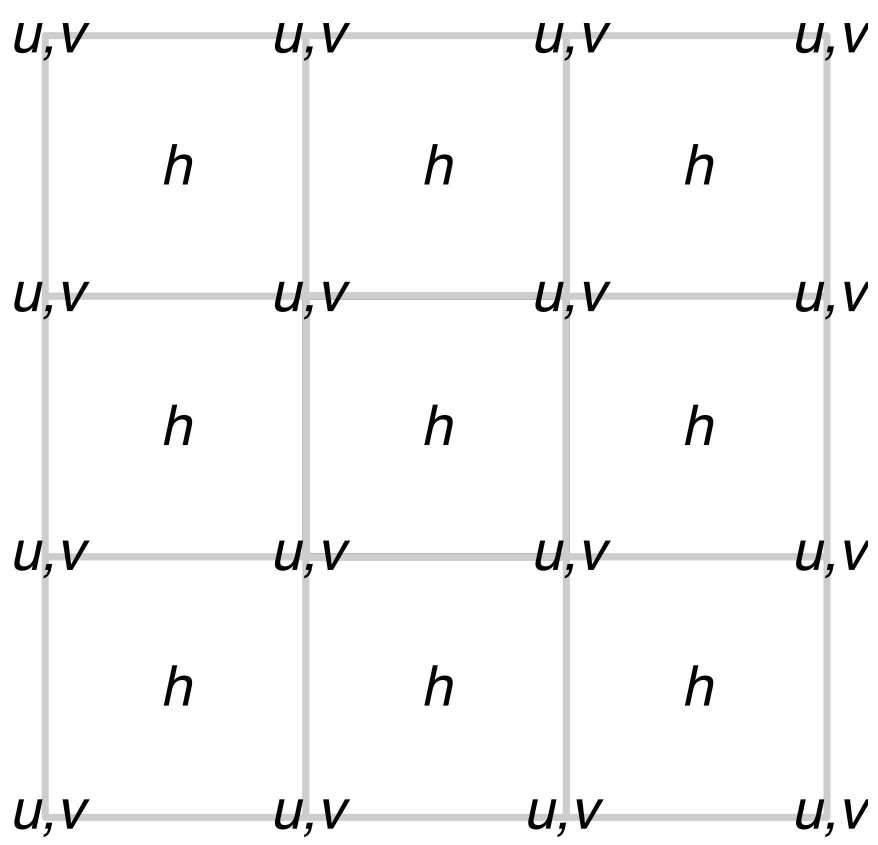
The density of u and h variables is cut in half, but we can fix this by cutting the grid spacing in half.

With the smaller grid spacing, we have to take a shorter time step to maintain computational stability.

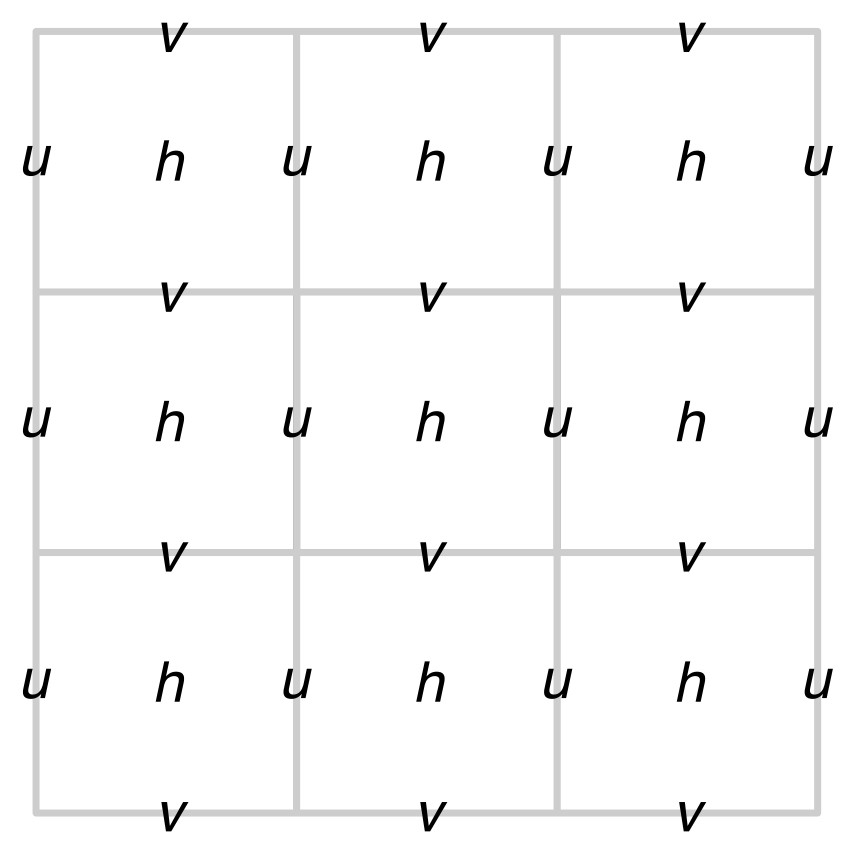
Various staggered grids in two dimensions



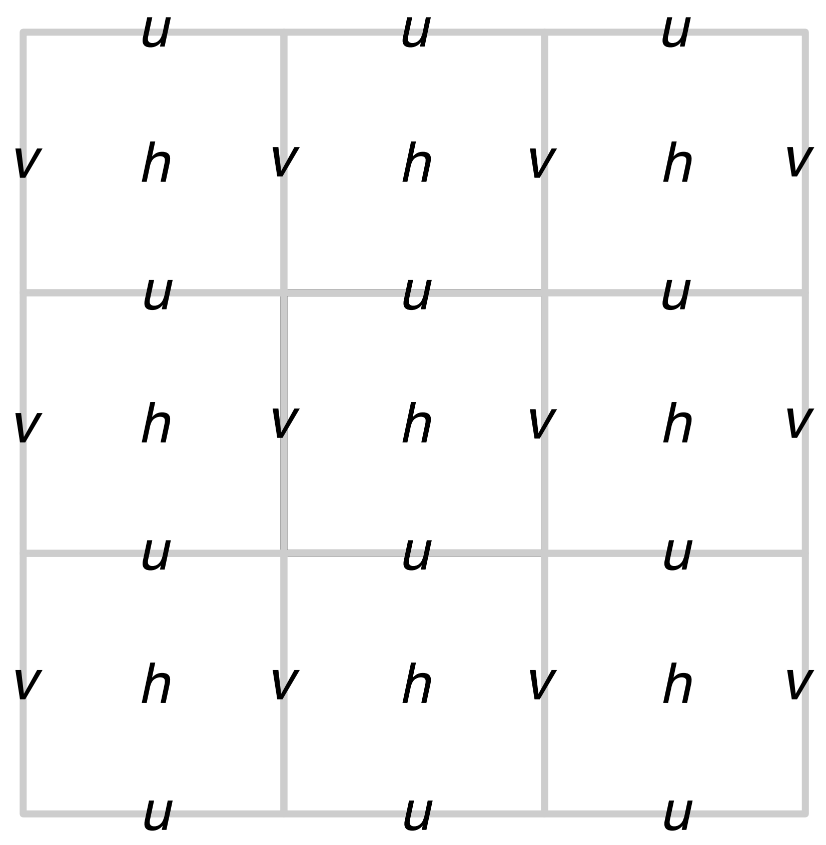
A grid



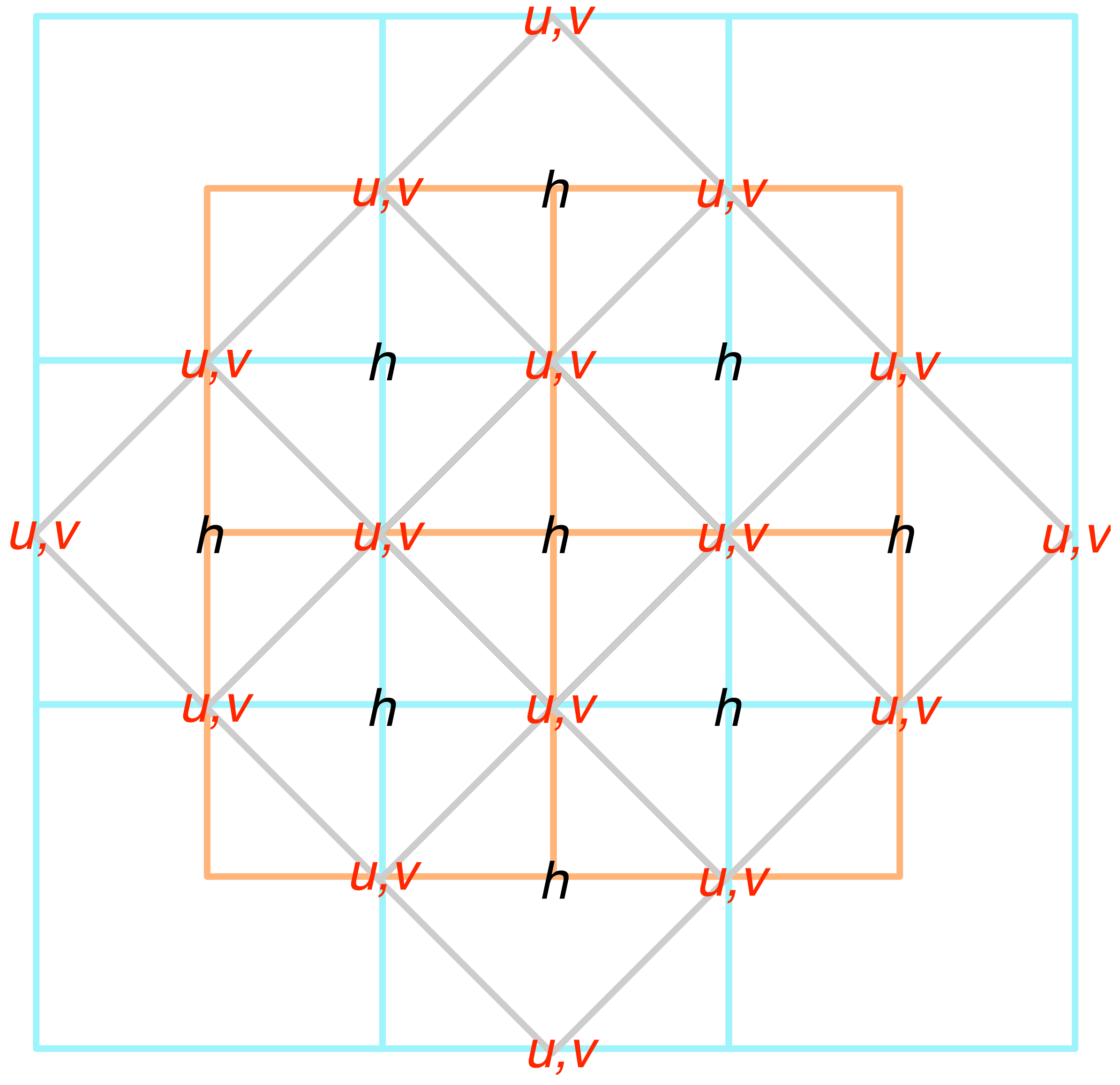
B grid



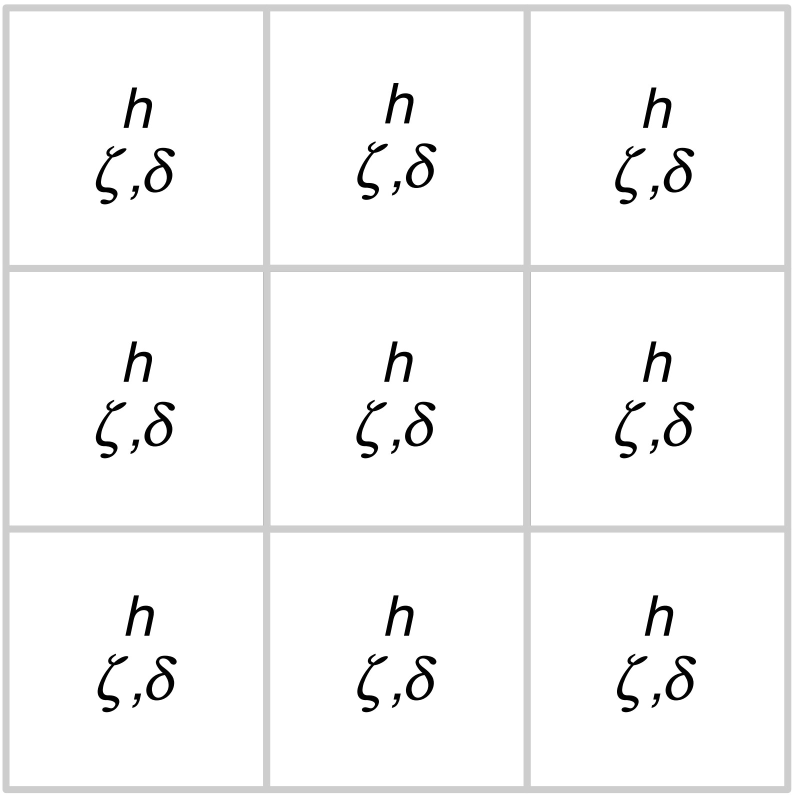
C grid



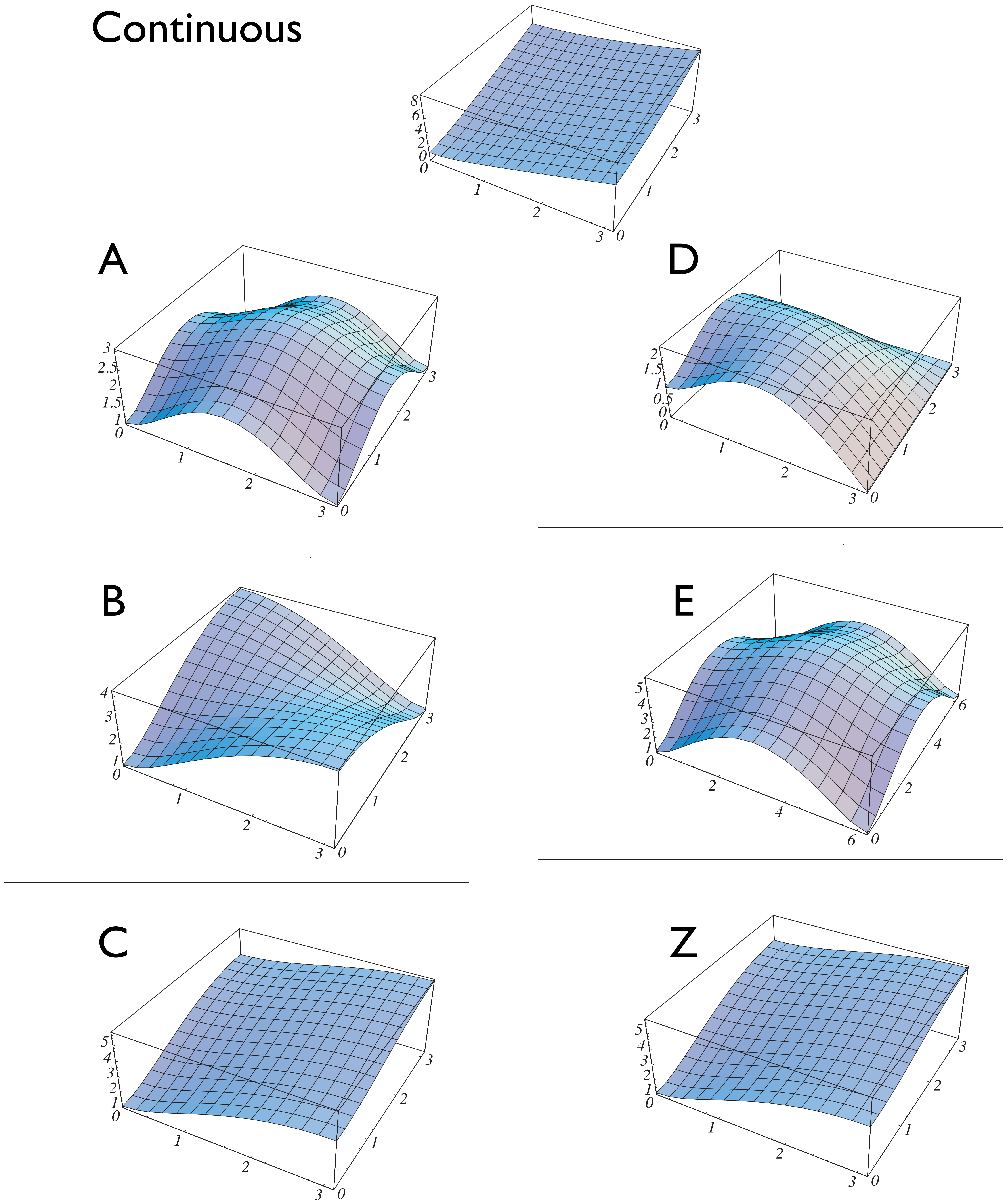
D grid



E grid



Z grid

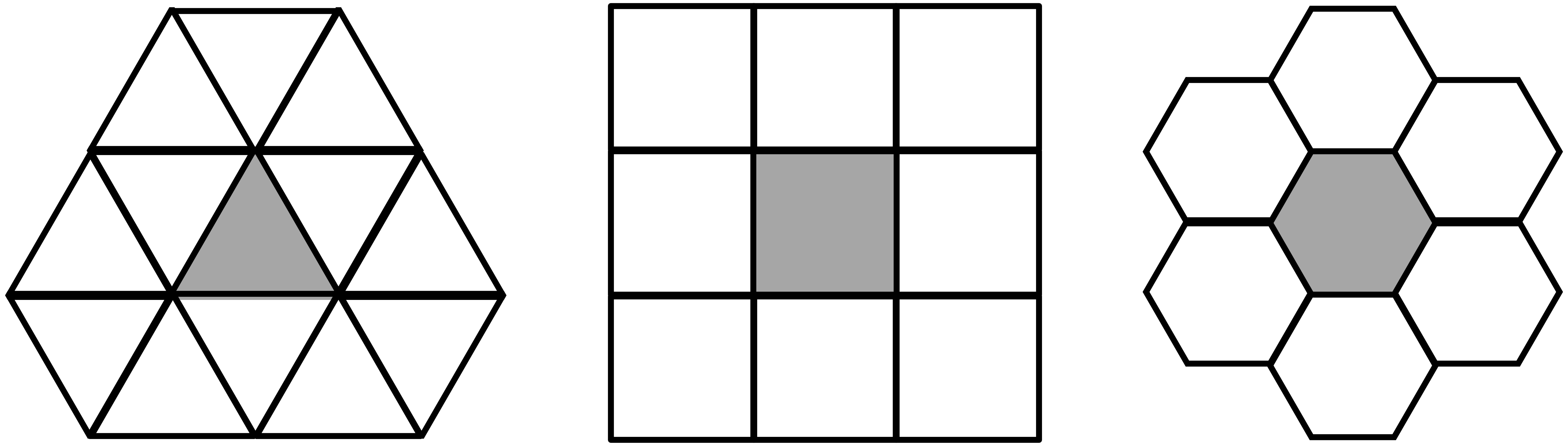


These dispersion plots show how well (or how badly) wave propagation is simulated on the various grids.

The two best choices are C and Z.

Generalization

A "C" grid places the normal components of the winds on the walls of mass cells.



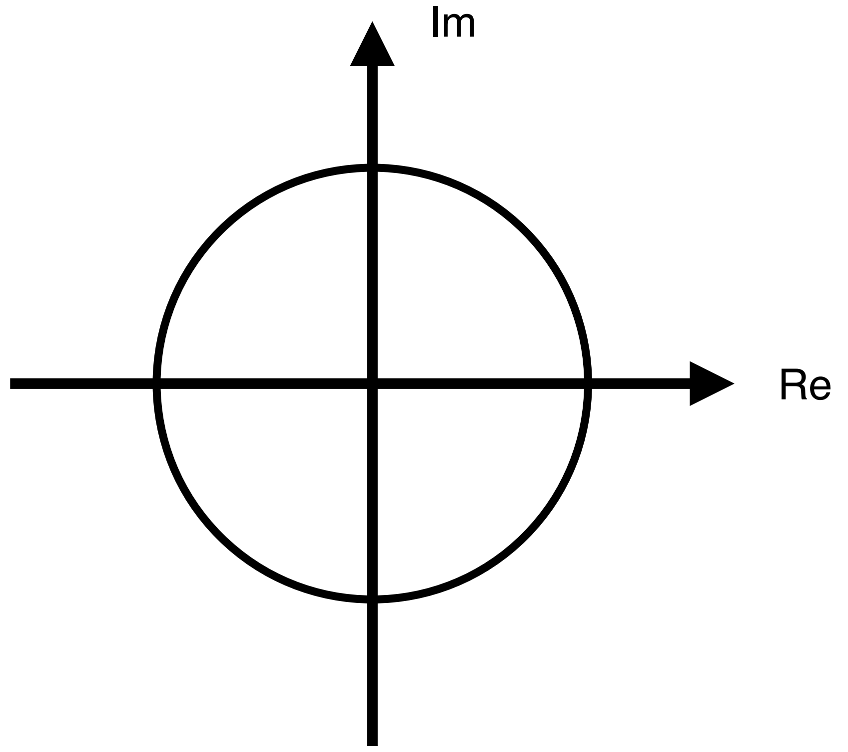
A C grid has computational modes on triangles and hexagons, but not on squares.

The right answer is “2”.

Grid	Triangles	Squares	Hexagons
A	2	2	2
B	1	2	4
C	3/2	2	3
D	3/2	2	3
E	Does not exist	2	3
Z	2	2	2

The number of prognostic degrees of freedom in the horizontal wind field, per mass point, on grids A-E and Z, and for triangular, square, and hexagonal meshes. For the Z-grid, the vorticity and divergence carry the information about the wind field.

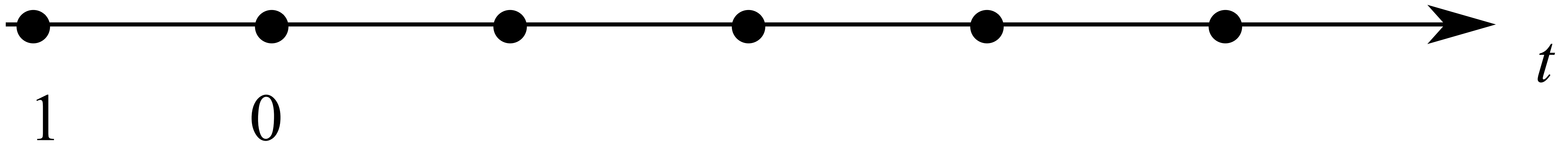
An example of a computational mode in time



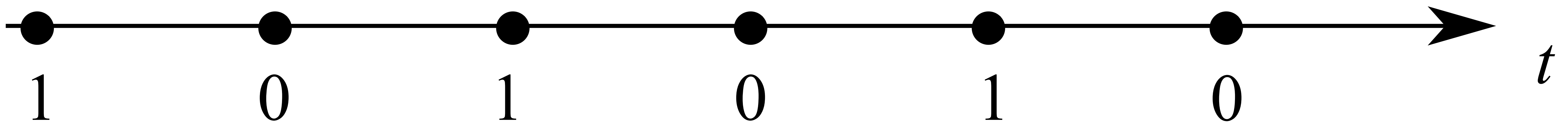
Oscillation equation $$\frac{dq}{dt} = i\omega q$$

Leapfrog scheme $$\frac{q^{n+1} - q^{n-1}}{2\Delta t} = i\omega q^n$$

Two initial conditions needed

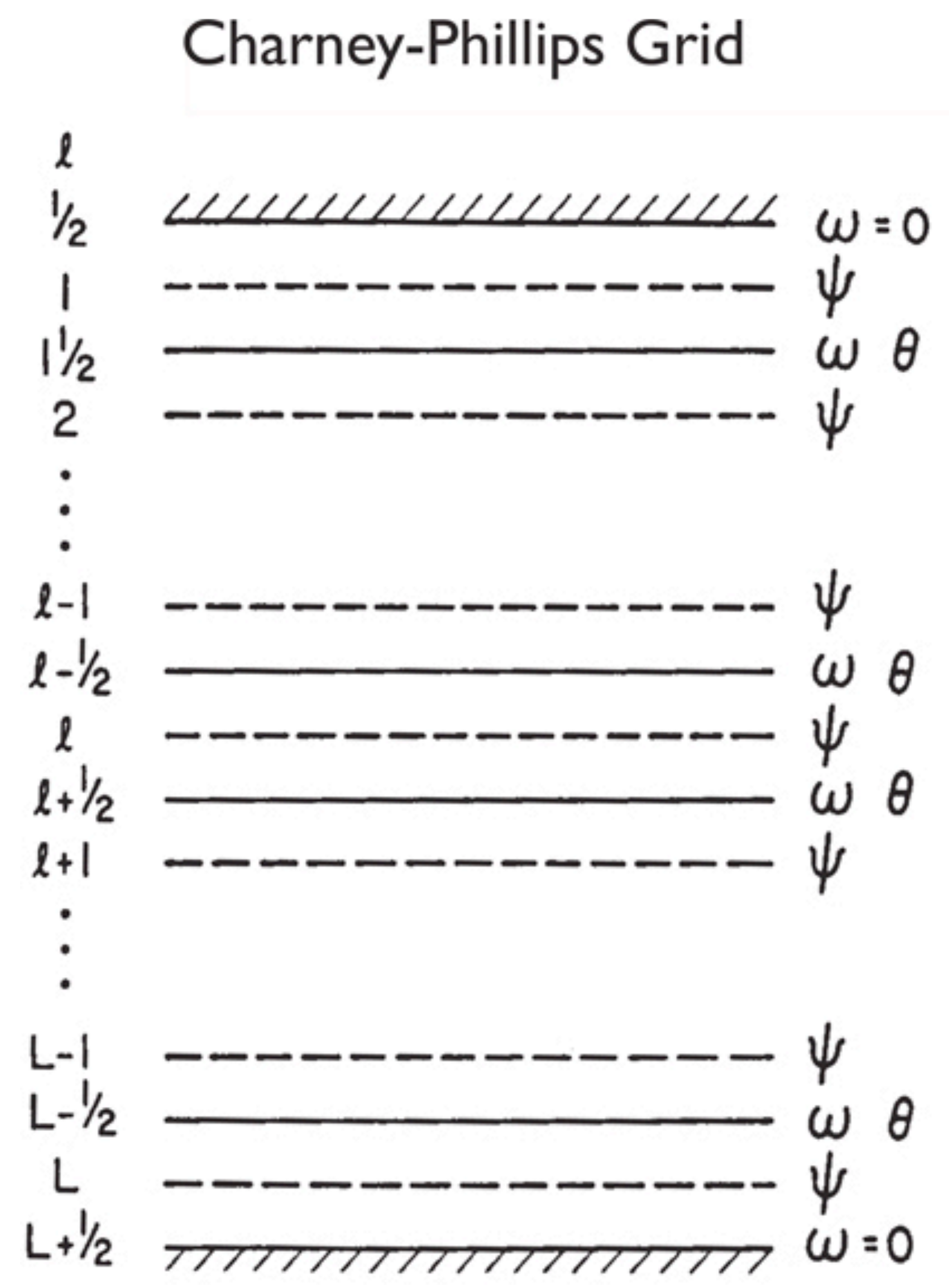


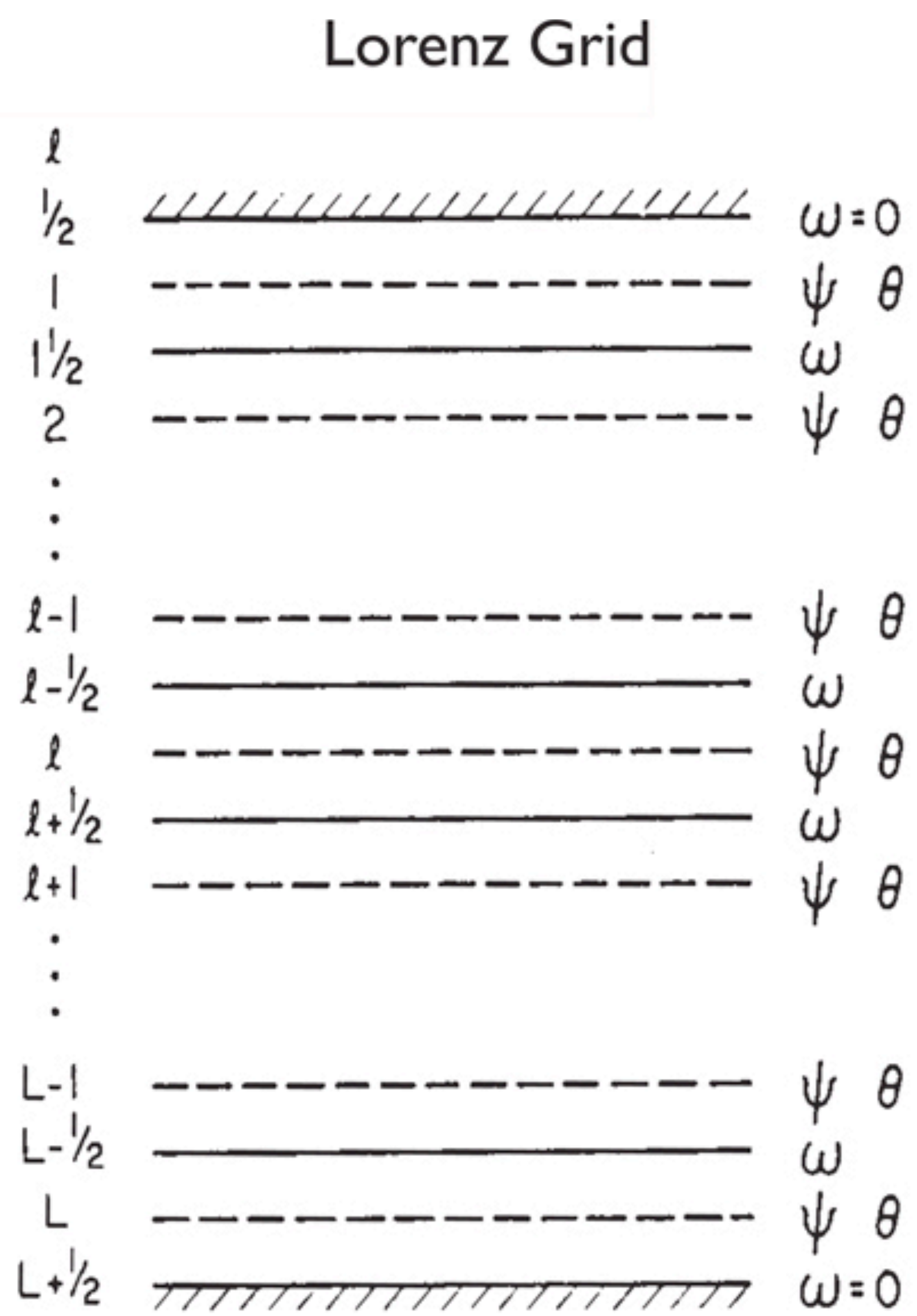
Solution for $\omega = 0$



Computational modes in time can be avoided by using schemes that involve only two time levels. The upstream scheme is an example.

Vertical staggering





$$\phi_l - \phi_{l+1} \sim \theta_{l+\frac{1}{2}}$$

$$\phi_l - \phi_{l+1} \sim \frac{1}{2}\left(\theta_l + \theta_{l+1}\right)$$

With the L grid, a zig-zag in θ has no effect on the thickness between neighboring wind levels.

This illustrates that *averaging* can give rise to computational modes.

Conservative schemes

Physical laws have the form "*X does not change, except…*"

Examples of things that obey equations like that:

- Mass of dry air
- Mass of water and other "minor" species
- Momentum (or angular momentum)
- Thermodynamic energy
- Kinetic energy
- Total energy
- Potential vorticity

Conservation with the continuous equations

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{V})$$

$$\int_{\mathrm{WA}} \nabla \cdot (\rho \mathbf{V}) dx^3 = 0$$

$$\frac{d}{dt}\left(\int_{\mathrm{WA}} \rho dx^3\right) = 0$$

Conservation with finite differences

$$\sum_j \rho_j^{n+1} dR_j = \sum_j \rho_j^n dR_j$$

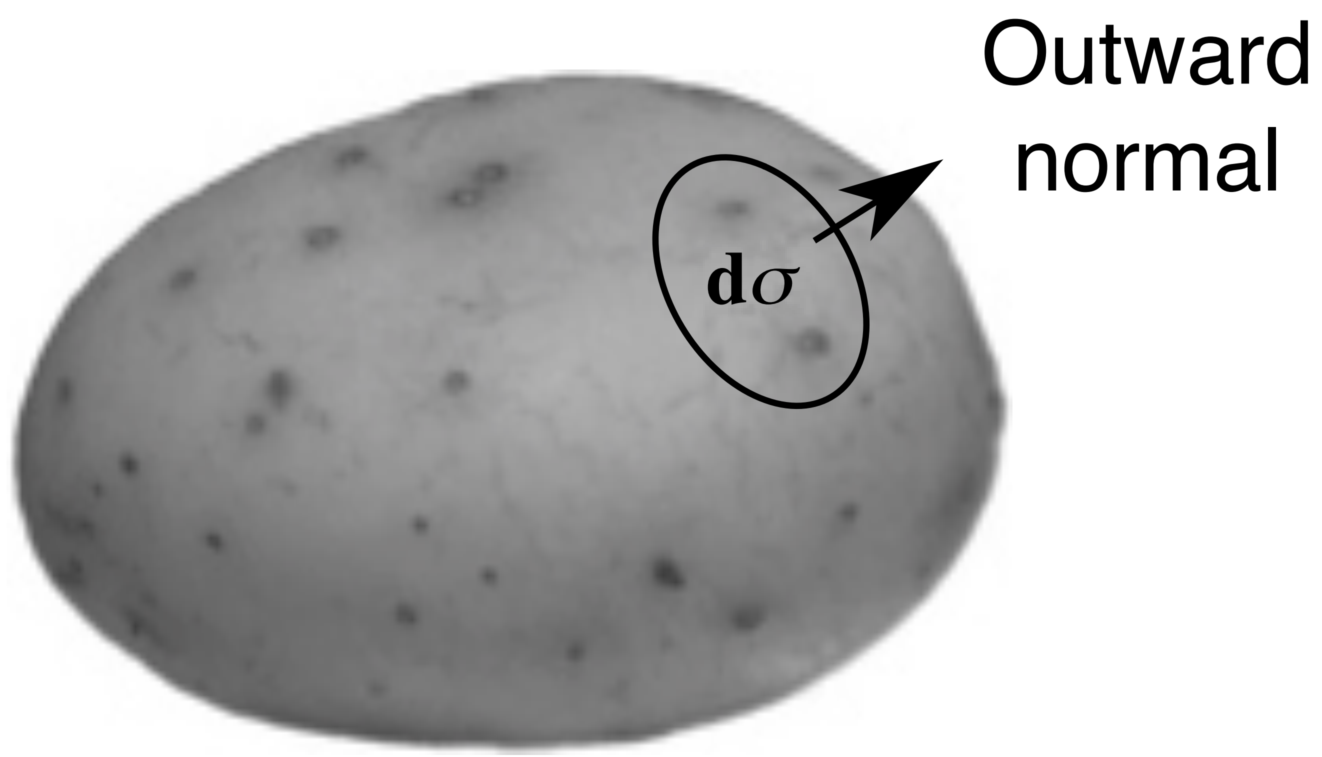
$$\sum_j (\rho A)_j^{n+1} dR_j = \sum_j (\rho A)_j^n dR_j + \Delta t \sum_j (\rho S)_j^n dR_j$$

Finite-volume methods

Divergence $\nabla \cdot \mathbf{Q} \equiv \lim_{S \to 0} \left[\frac{1}{V} \oint_S \mathbf{n} \cdot \mathbf{Q}\, dS \right]$ Advection

Curl $\nabla \times \mathbf{Q} \equiv \lim_{S \to 0} \left[\frac{1}{V} \oint_S \mathbf{n} \times \mathbf{Q}\, dS \right]$ Vorticity

Gradient $\nabla A \equiv \lim_{S \to 0} \left[\frac{1}{V} \oint_S \mathbf{n} A\, dS \right]$ Pressure gradient



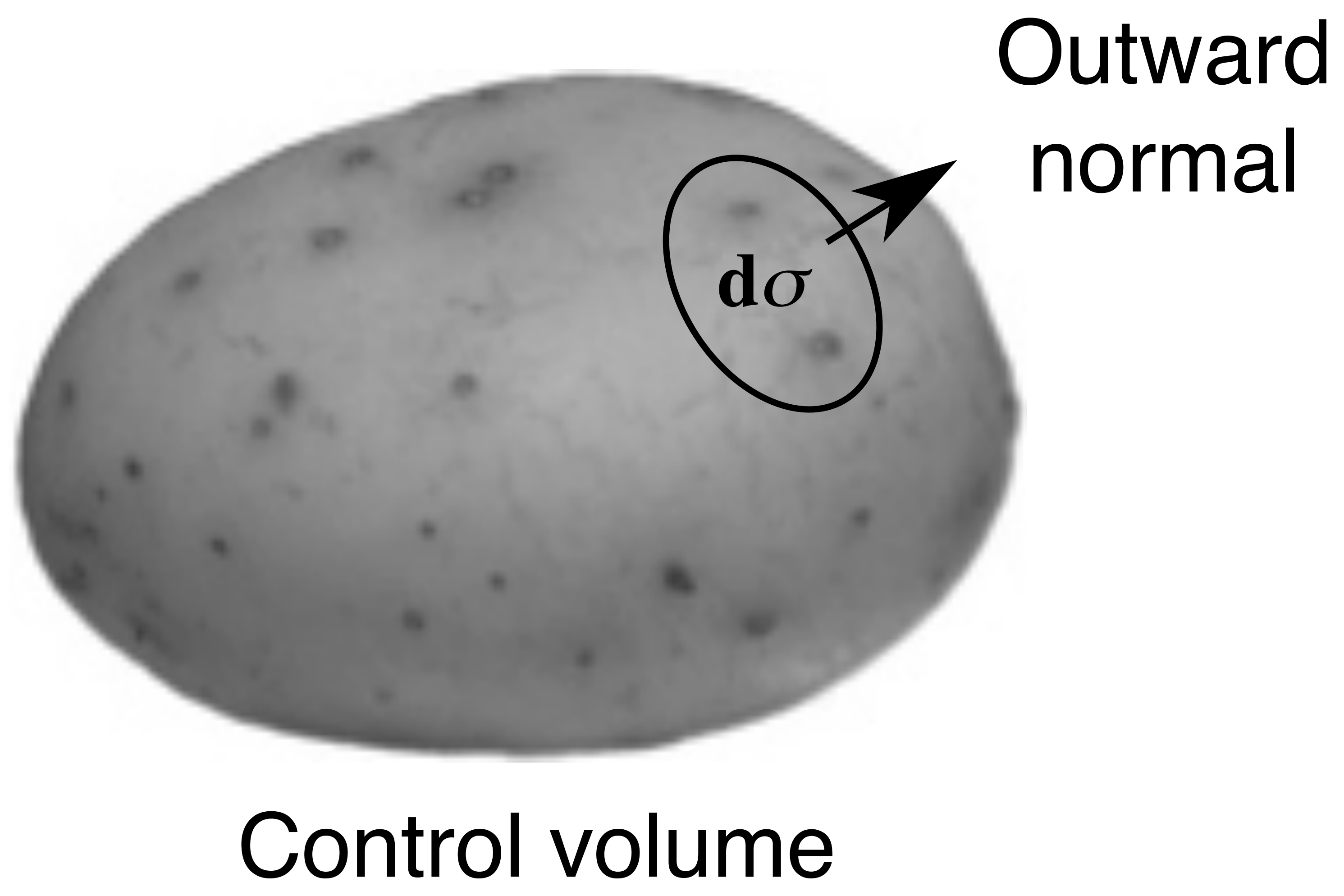
Control volume

Works in any number of dimensions

3D: Volume and surface

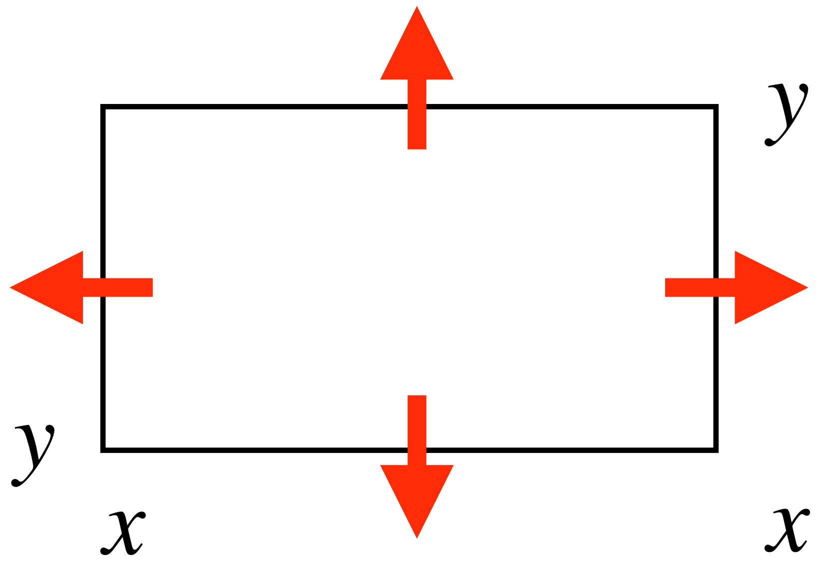
2D: Area and perimeter

1D: Curve and end points



Example

In 2D, we can write $\nabla \cdot \mathbf{Q} = \frac{1}{A} \oint_C \mathbf{Q} \cdot \mathbf{n} \, ds$.



$$\nabla \cdot \mathbf{Q} \cong \frac{1}{\delta x \delta y} \left[Q_x \left(x + \delta x, y + \frac{\delta y}{2} \right) \delta y + Q_y \left(x + \frac{\delta x}{2}, y + \delta y \right) \delta x - Q_x \left(x, y + \frac{\delta y}{2} \right) \delta y - Q_y \left(x + \frac{\delta x}{2}, y \right) \delta x \right]$$

Simplifying, we find that

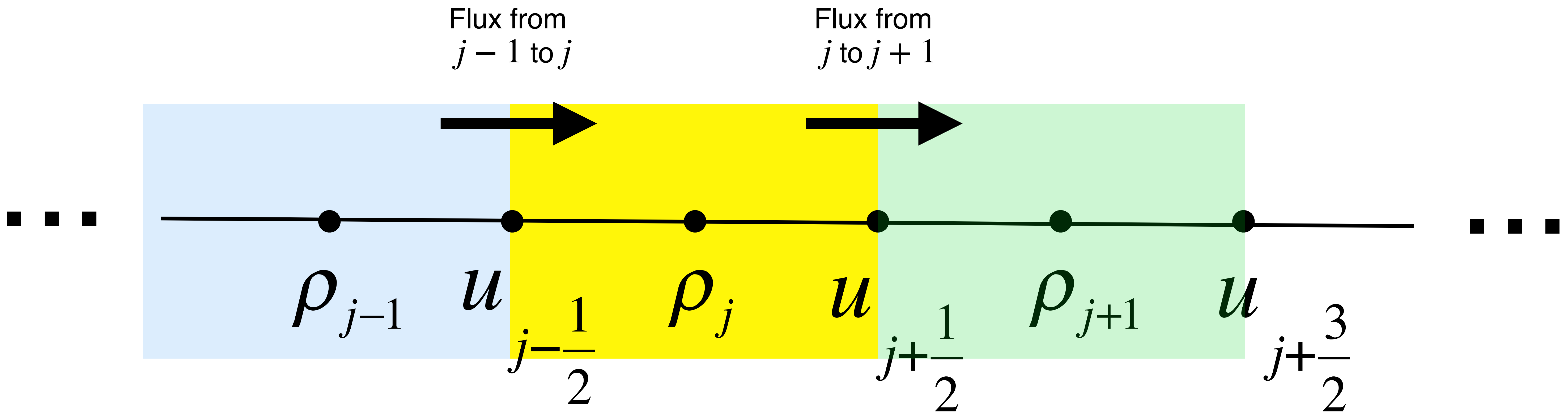
$$\nabla \cdot \mathbf{Q} \cong \frac{1}{\delta x} \left[Q_x \left(x + \delta x, y + \frac{\delta y}{2} \right) - Q_x \left(x, y + \frac{\delta y}{2} \right) \right] + \frac{1}{\delta y} \left[Q_y \left(x + \frac{\delta x}{2}, y + \delta y \right) - Q_y \left(x + \frac{\delta x}{2}, y \right) \right]$$

In the limit, this reduces to the expected result:

$$\nabla \cdot \mathbf{Q} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}$$

Discrete mass conservation in one dimension

$$\frac{d\rho_j}{dt} + \frac{(\rho u)_{j+1/2} - (\rho u)_{j-1/2}}{\Delta x_j} = 0$$



$$\frac{d}{dt}\sum_{j=0}^{J}\left(\rho_j \Delta x_j\right) + (\rho u)_{J+1/2} - (\rho u)_{1/2}$$

Conserving an intensive scalar: Continuous case

$$\frac{\partial}{\partial t}\left(\rho A\right) + \frac{\partial}{\partial x}\left(\rho u A\right) = 0$$

Flux form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\right) = 0$$

Continuity

Subtract A times continuity from the flux form to obtain

$$\rho\left(\frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x}\right) = 0$$

Advective form

Multiply the advective form by A :

$$\rho\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\frac{A^2}{2} = 0$$

Use continuity to go back to flux form:

$$\frac{\partial}{\partial t}\left(\rho\frac{A^2}{2}\right) + \frac{\partial}{\partial x}\left(\rho u\frac{A^2}{2}\right) = 0$$

Conservation of A^2

Conserving an intensive scalar: Discrete case

Flux from j to $j+1$

Flux from $j-1$ to j

$$\frac{d}{dt}\left(\rho_j A_j\right) + \frac{(\rho u)_{j+1/2} A_{j+1/2} - (\rho u)_{j-1/2} A_{j-1/2}}{\Delta x_j} = 0$$

Flux form

We have to interpolate somehow to get $A_{j+1/2}$ and $A_{j-1/2}$ from the prognostic values of A_j.

Flux from j to $j+1$

Flux from $j-1$ to j

$$\frac{d\rho_j}{dt} + \frac{(\rho u)_{j+1/2} - (\rho u)_{j-1/2}}{\Delta x_j} = 0$$

Continuity

Reduces to flux form above if $A \equiv 1$.

$$\rho_j \frac{dA_j}{dt} + \frac{(\rho u)_{j+1/2}\left(A_{j+1/2} - A_j\right) + (\rho u)_{j-1/2}\left(A_j - A_{j-1/2}\right)}{\Delta x_j} = 0$$

Advective form

Does this result make sense?

Can we also conserve the square of A_j ?

$$\rho_j \frac{dA_j}{dt} + \frac{(\rho u)_{j+1/2}\left(A_{j+1/2} - A_j\right) + (\rho u)_{j-1/2}\left(A_j - A_{j-1/2}\right)}{\Delta x_j} = 0$$

Advective form

Multiply each term by A_j :

$$\rho_j \frac{d}{dt}\left(\frac{A_j^2}{2}\right) + \frac{(\rho u)_{j+1/2} A_j \left(A_{j+1/2} - A_j\right) + (\rho u)_{j-1/2} A_j \left(A_j - A_{j-1/2}\right)}{\Delta x_j} = 0$$

Multiply continuity by $A_j^2/2$:

$$\left(\frac{A_j^2}{2}\right)\frac{d\rho_j}{dt} + \left(\frac{A_j^2}{2}\right)\left[\frac{(\rho u)_{j+1/2} - (\rho u)_{j-1/2}}{\Delta x_j}\right] = 0$$

Add the two equations above:

$$\frac{d}{dt}\left(\rho_j \frac{A_j^2}{2}\right) + \frac{(\rho u)_{j+1/2}\left[A_j\left(A_{j+1/2} - A_j\right) + A_j^2/2\right] - (\rho u)_{j-1/2}\left[A_j\left(A_{j-1/2} - A_j\right) + A_j^2/2\right]}{\Delta x_j} = 0$$

Finishing up

$$\frac{d}{dt}\left(\rho_j \frac{A_j^2}{2}\right) + \frac{(\rho u)_{j+1/2}\left[A_j\left(A_{j+1/2} - A_j\right) + A_j^2/2\right] - (\rho u)_{j-1/2}\left[A_j\left(A_{j-1/2} - A_j\right) + A_j^2/2\right]}{\Delta x_j} = 0$$

Repeated from previous slide

Simplify:

$$\frac{d}{dt}\left(\rho_j \frac{A_j^2}{2}\right) + \frac{(\rho u)_{j+1/2}\left[A_j\left(A_{j+1/2} - A_j/2\right)\right] - (\rho u)_{j-1/2}\left[A_j\left(A_{j-1/2} - A_j/2\right)\right]}{\Delta x_j} = 0$$

Is this really a flux form? Yes, if the flux into $j+1$ is equal to the flux out of j:

$$(\rho u)_{j+1/2}\left[A_j\left(A_{j+1/2} - A_j/2\right)\right] = (\rho u)_{j+1/2}\left[A_{j+1}\left(A_{j+1/2} - A_{j+1}/2\right)\right]$$

We can solve for the unknown in this equation, which is $A_{j+1/2}$. The result is $A_{j+1/2} = \left(A_j + A_{j+1}\right)/2$.

Conclusion: Conservation of A_j is consistent with conservation of $A_j^2/2$ if we choose $A_{j+1/2} = \left(A_j + A_{j+1}\right)/2$.

The equations tell us what to do.

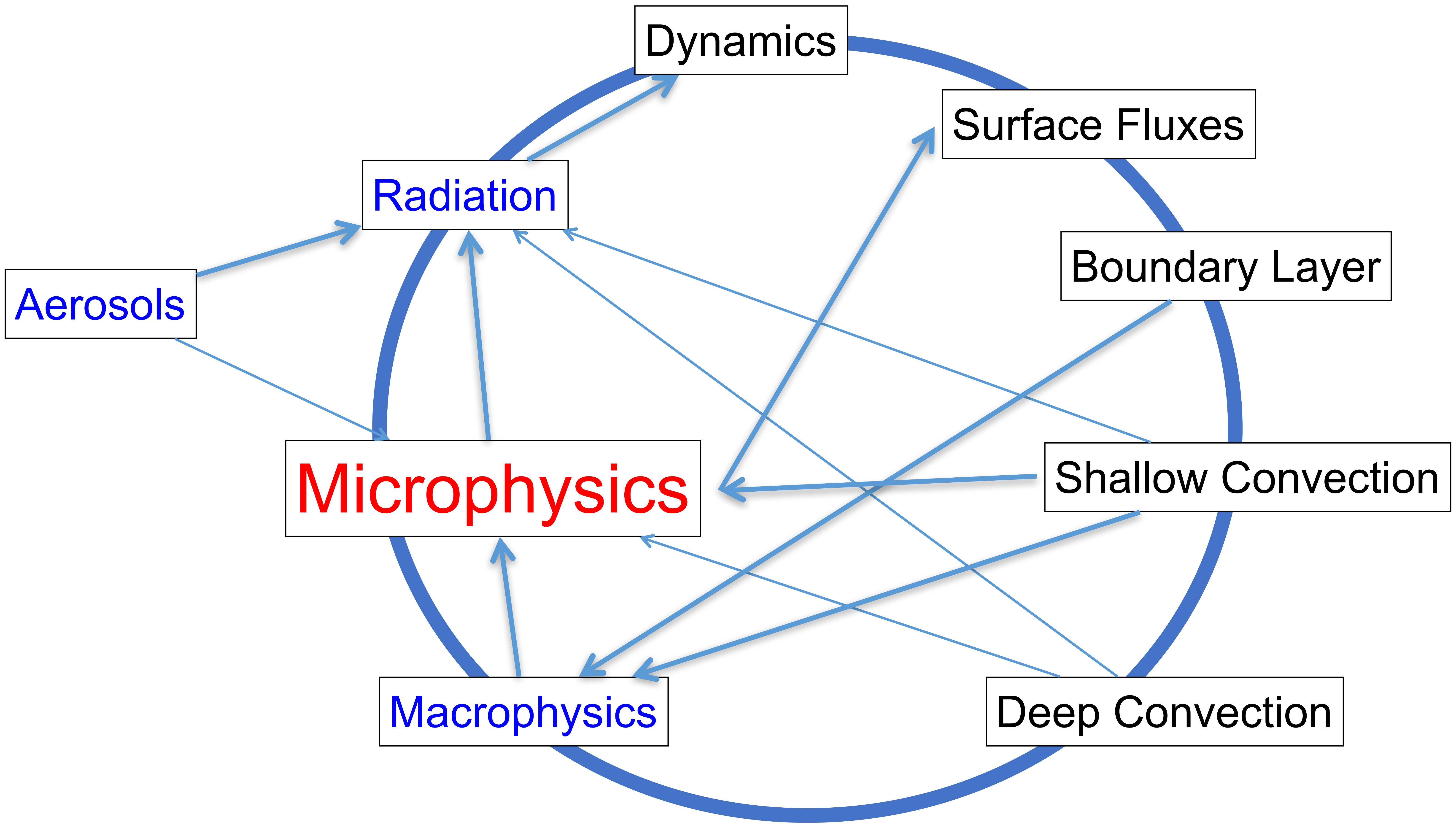
Earlier I said that:

"We have to interpolate somehow to get $A_{j+1/2}$ and $A_{j-1/2}$ from the prognostic values of A_j."

The requirement that $A_j^2/2$ must be conserved has told us how to choose $A_{j+1/2}$.

This choice is "accurate" if the grid is uniform.

The finite-volume approach is a natural way to include parameterizations.



The spectral method for 1-D advection

$$\frac{\partial q}{\partial t} = -c\frac{\partial q}{\partial x}$$

$$q\left(x_j,t\right) \cong \sum_{k=-n}^{n} \hat{q}_k\left(t\right) e^{ikx_j}$$

$$\frac{\partial q}{\partial x}\left(x_j,t\right) \cong \sum_{k=-n}^{n} ik\hat{q}_k\left(t\right) e^{ikx_j}$$

$$\sum_{k=-n}^{n} \frac{d\hat{q}_k}{dt} e^{ikx} = -c\sum_{k=-n}^{n} ik\hat{q}_k e^{ikx}$$

$$\frac{d\hat{q}_k}{dt} = -ikc\hat{q}_k \text{ for } -n \le k \le n$$

Discretizing the sphere with spherical harmonics

$$Y_n^m(\mu,\lambda) = P_n^m(\mu)\exp(im\lambda)$$

Spherical harmonics have this form.

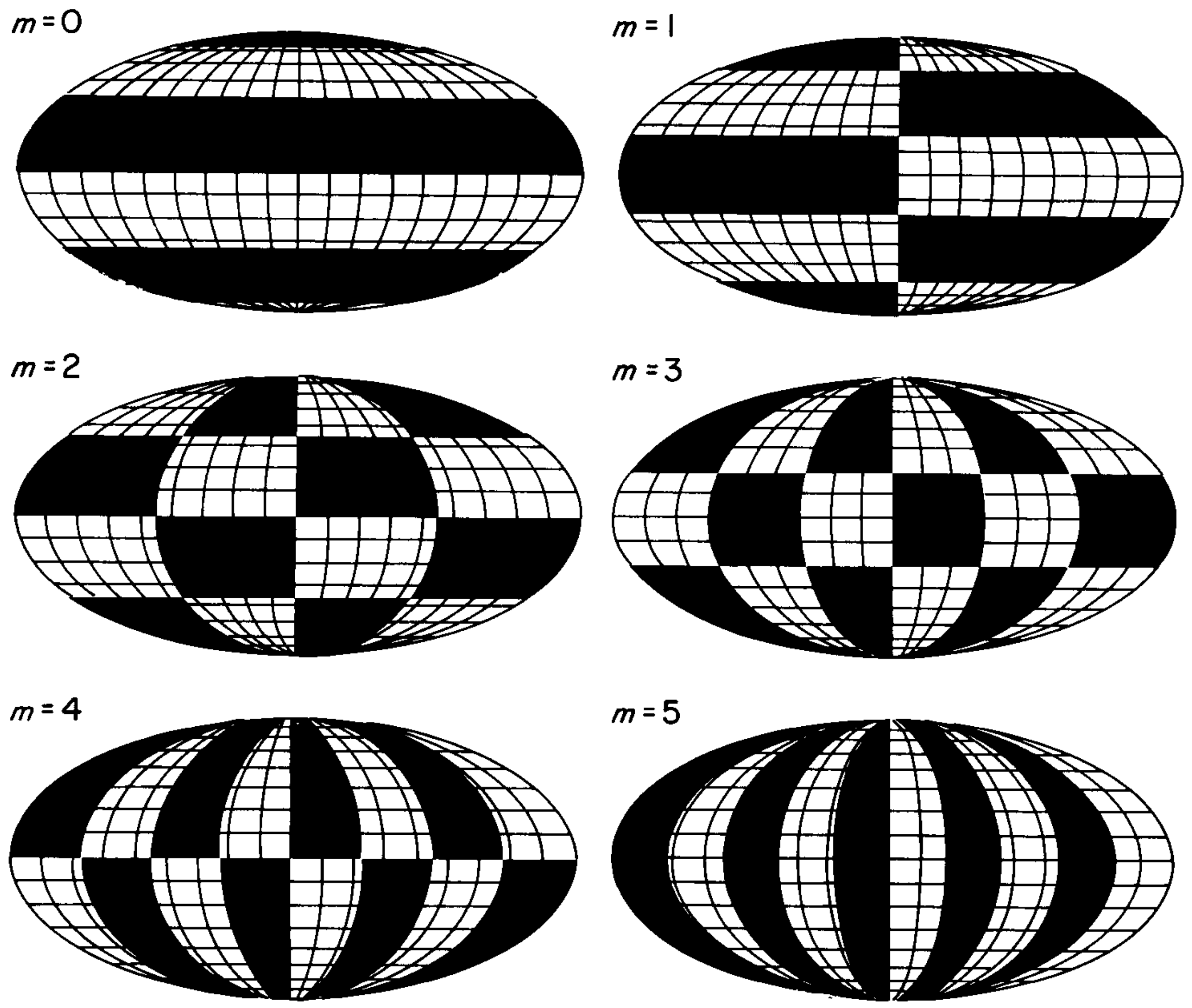


Figure C.3: Alternating patterns of positives and negatives for spherical harmonics with $n = 5$ and $m = 0, 1, 2, \ldots, 5$. From Baer (1972).

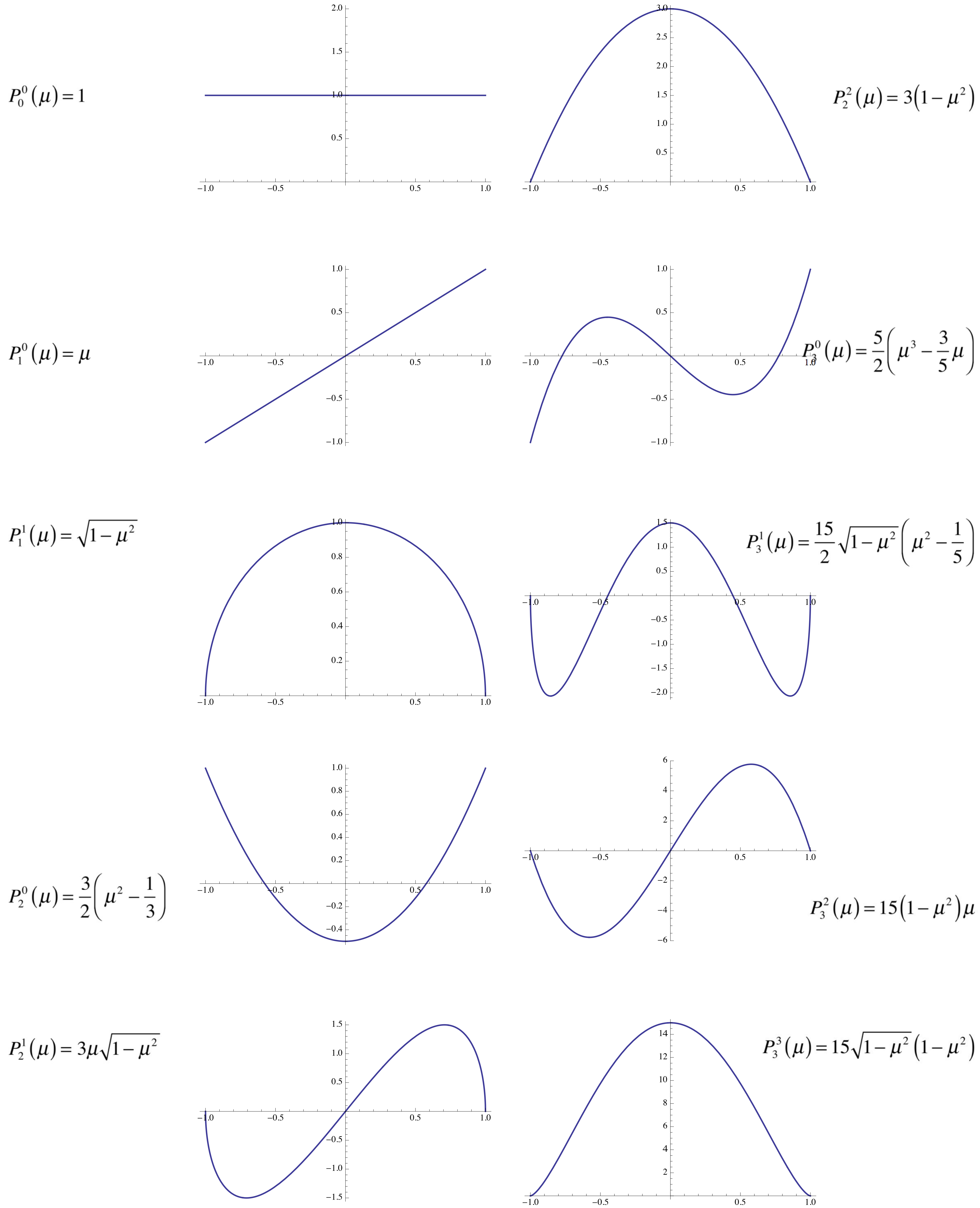
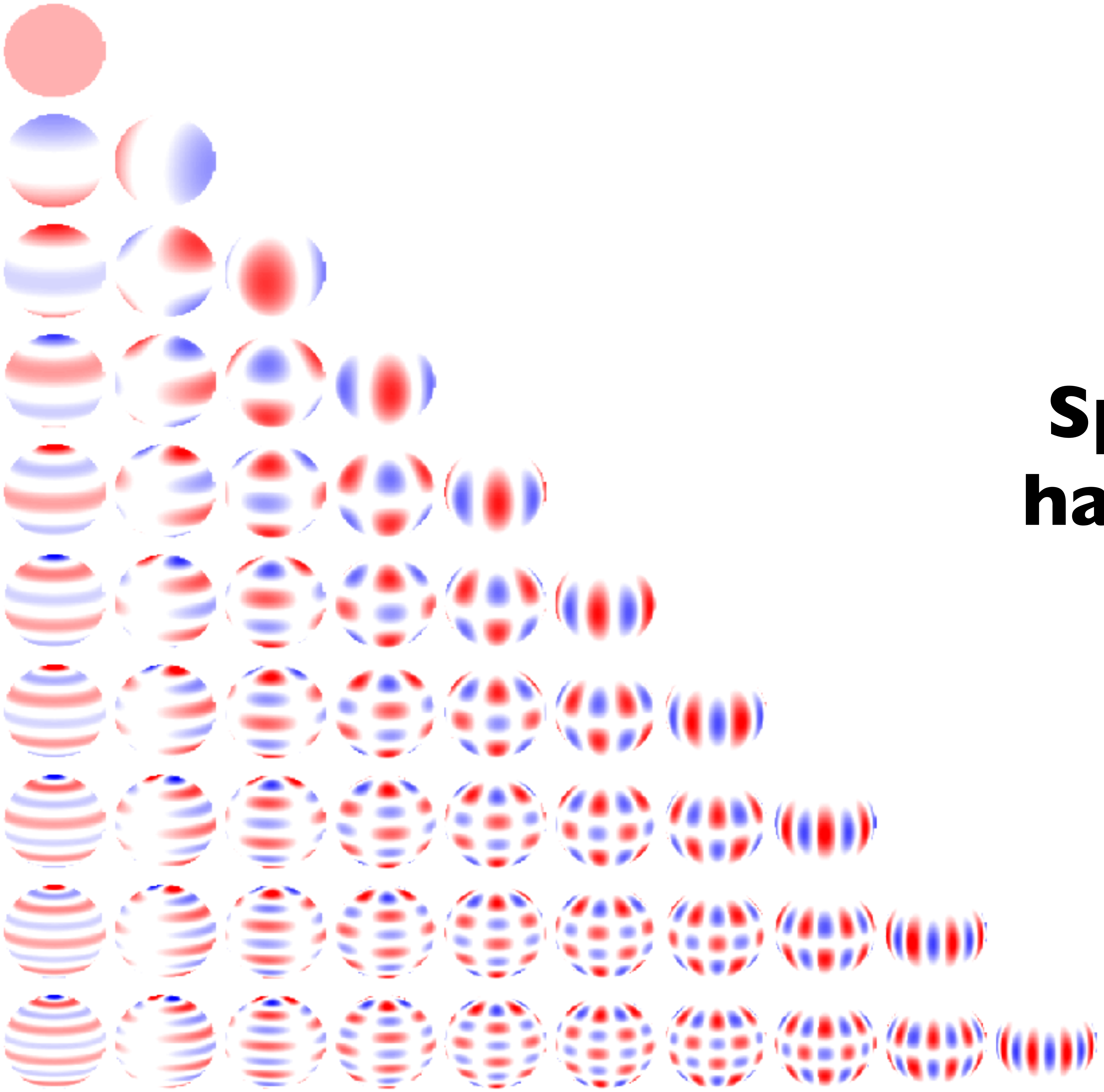
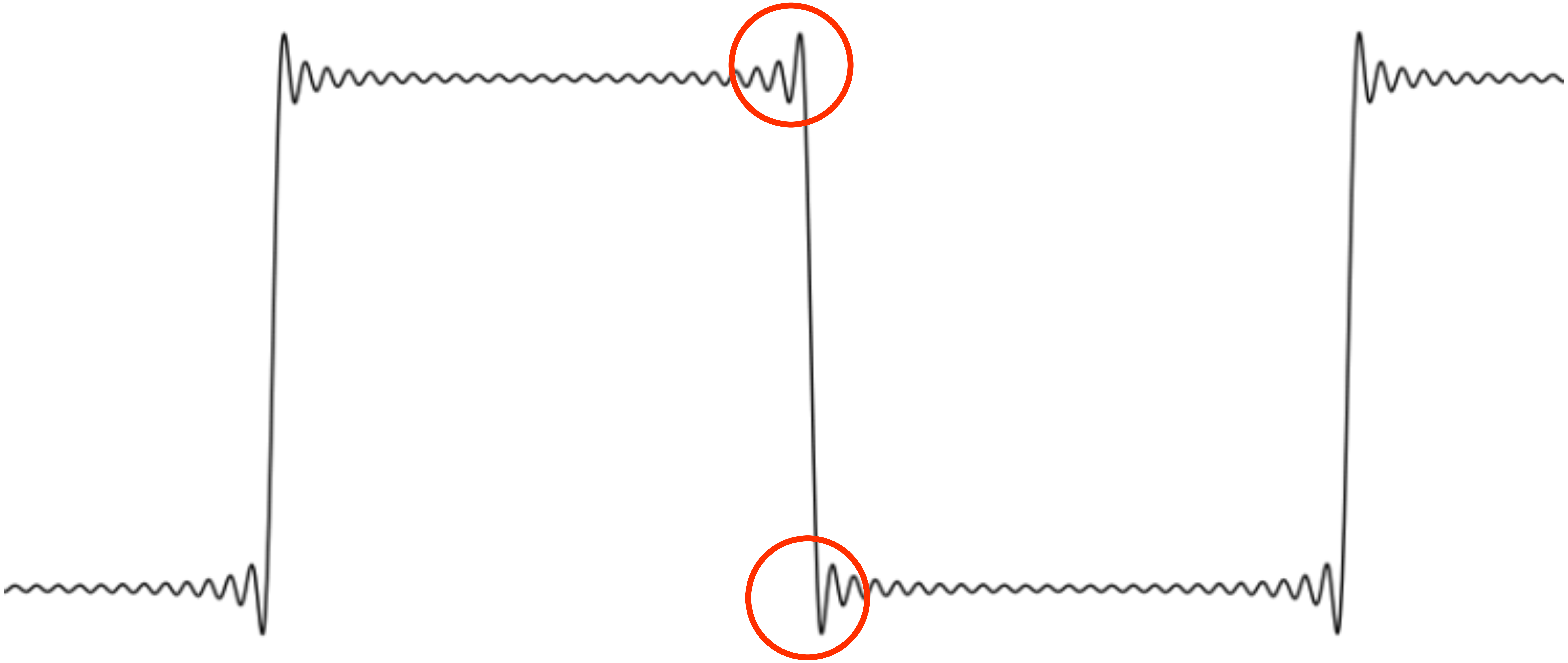


Figure C.1: Algebraic forms and plots of selected associated Legendre functions.



Spherical harmonics

The Gibbs Phenomenon



As the resolution increases, the RMS error goes to zero, but finite errors persist locally, near discontinuities, *no matter how many basis functions are used*.

Sign preservation and monotonicity

Sign-preserving: No negative water.

Monotone: No excursions outside the initial range.

Monotone schemes are always sign-preserving, but sign-preserving schemes are not necessarily monotone.

Schemes that are sign-preserving or monotone tend to damp.

The upstream scheme is a good example.

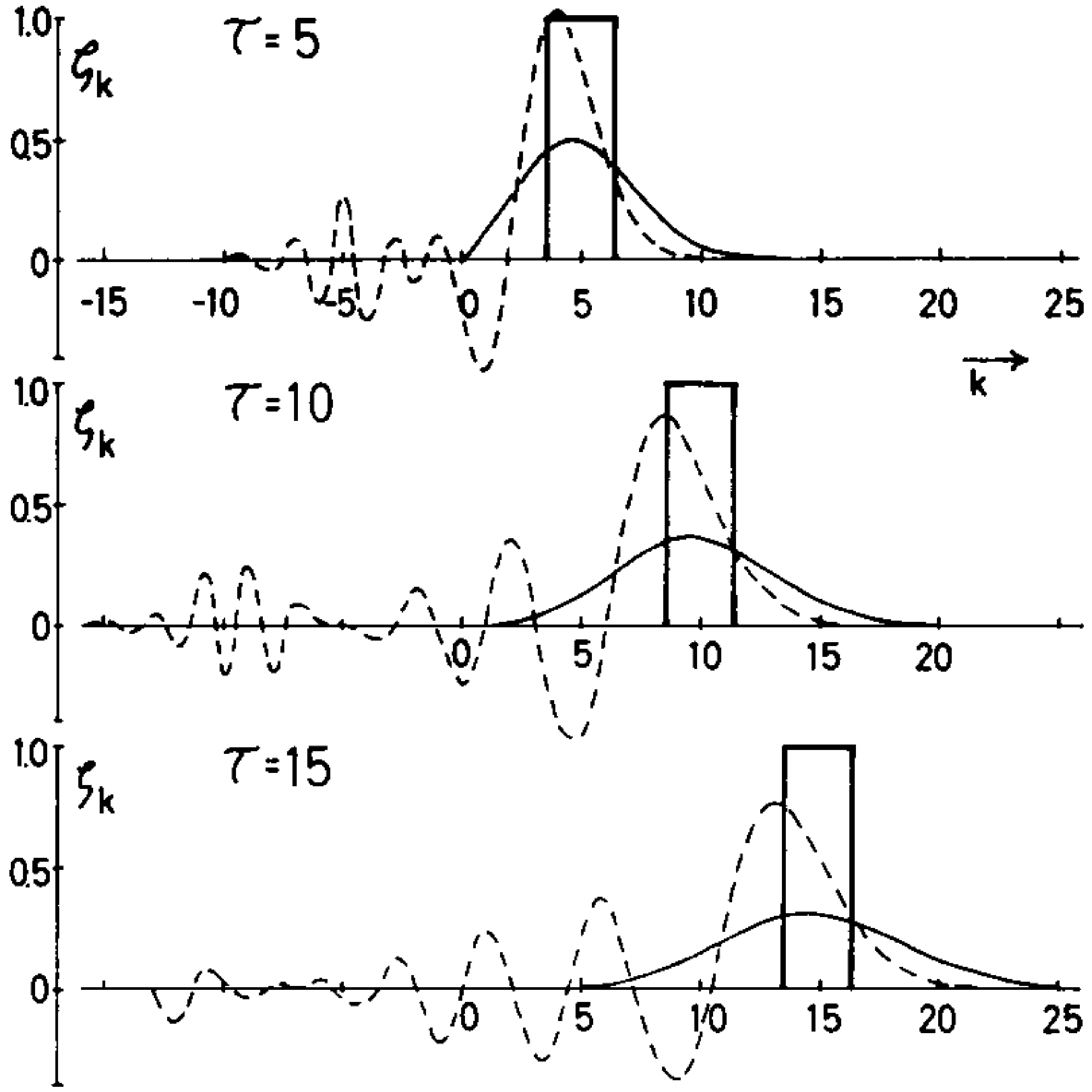
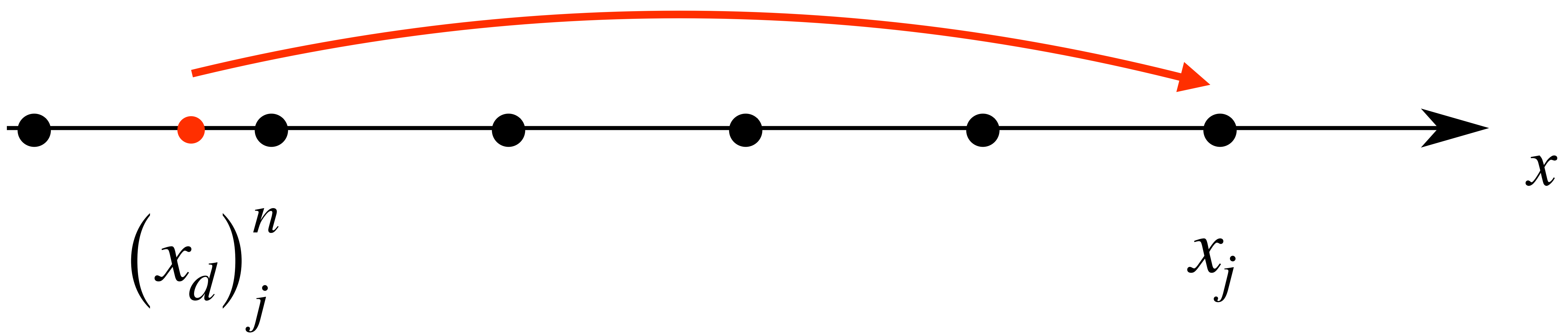


Fig. 1. Three solutions of the advection equation for (non-dimensional) times τ = 5, 10, 15.
——— (exact) solution of continuous equation (4)
– – – – solution (8) of centered differential-difference equation
——— solution (11) of backward differential-difference equation
For typical meteorological values, ten units of non-dimensional time correspond to about 42 hours.

Semi-Lagrangian Advection

And you may ask yourself, “Well, how did I get here?”



Find the departure point $(x_d)_j^n = x_j - c\Delta t$

$$A_j^{n+1} = \hat{\mu} A_{j-a}^n + (1-\hat{\mu}) A_{j-a+1}^n$$

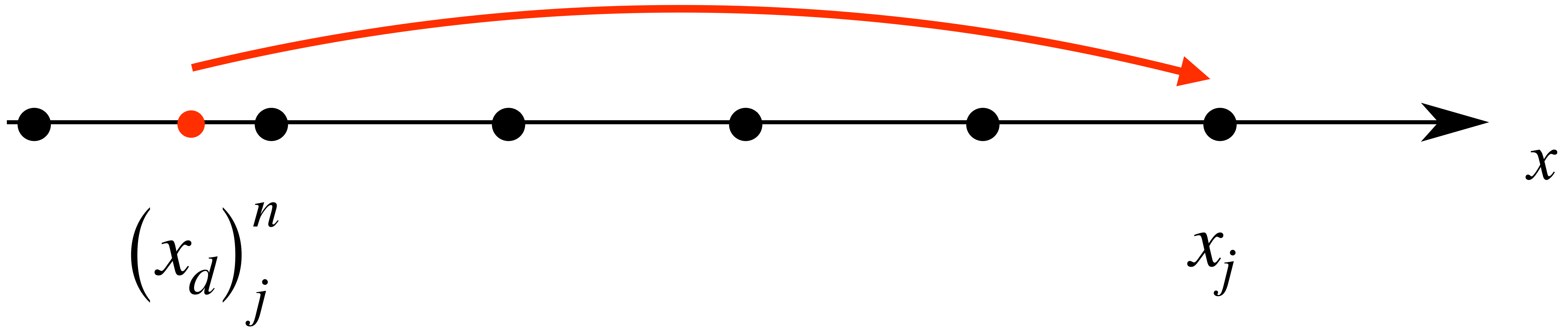
$$\hat{\mu} \equiv 1 + (\mu - a)$$

Note that $0 \leq \hat{\mu} \leq 1$ “by construction”.

The scheme is an interpolation, so it is stable and sign-preserving, regardless of Δt.

In the sketch, $a = 5$. For $a = 1$, $\hat{\mu} = \mu$ and we get the upstream scheme.

What's not to like?



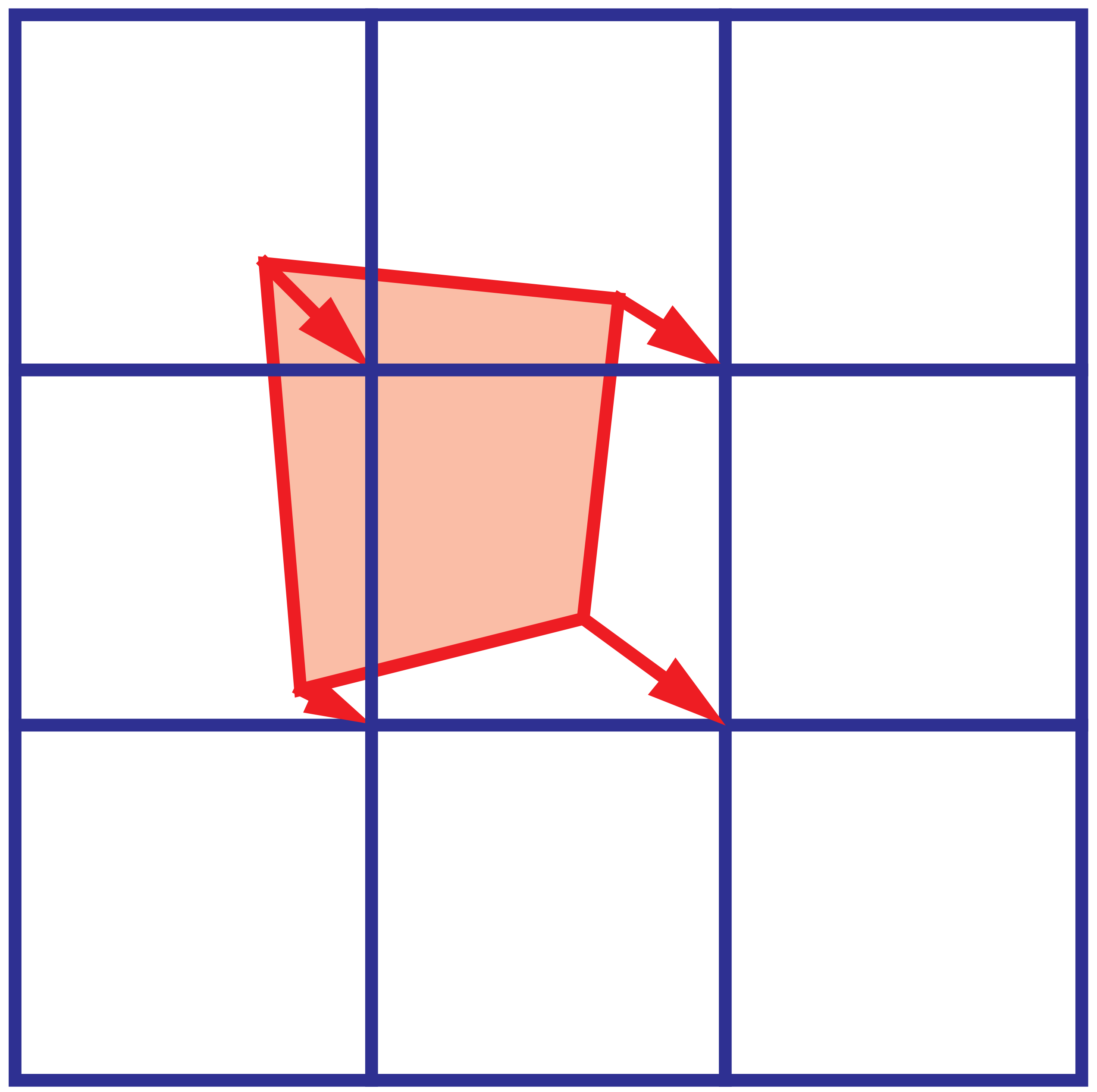
Semi-Lagrangian schemes are unconditionally stable and sign-preserving.

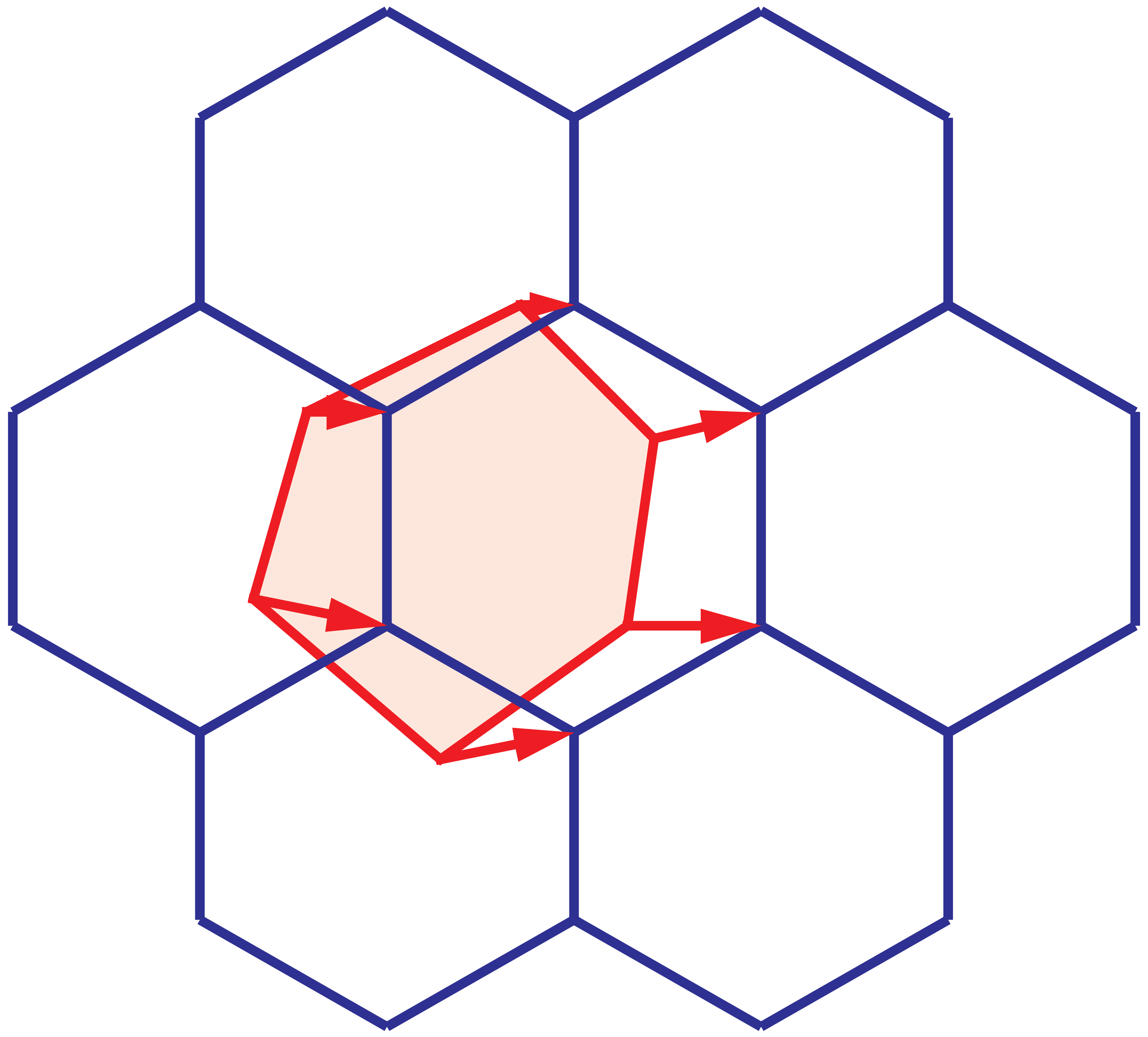
It is possible to make them conservative, e.g., by remapping.

Remapping:

A conservative type of semi-Lagrangian scheme

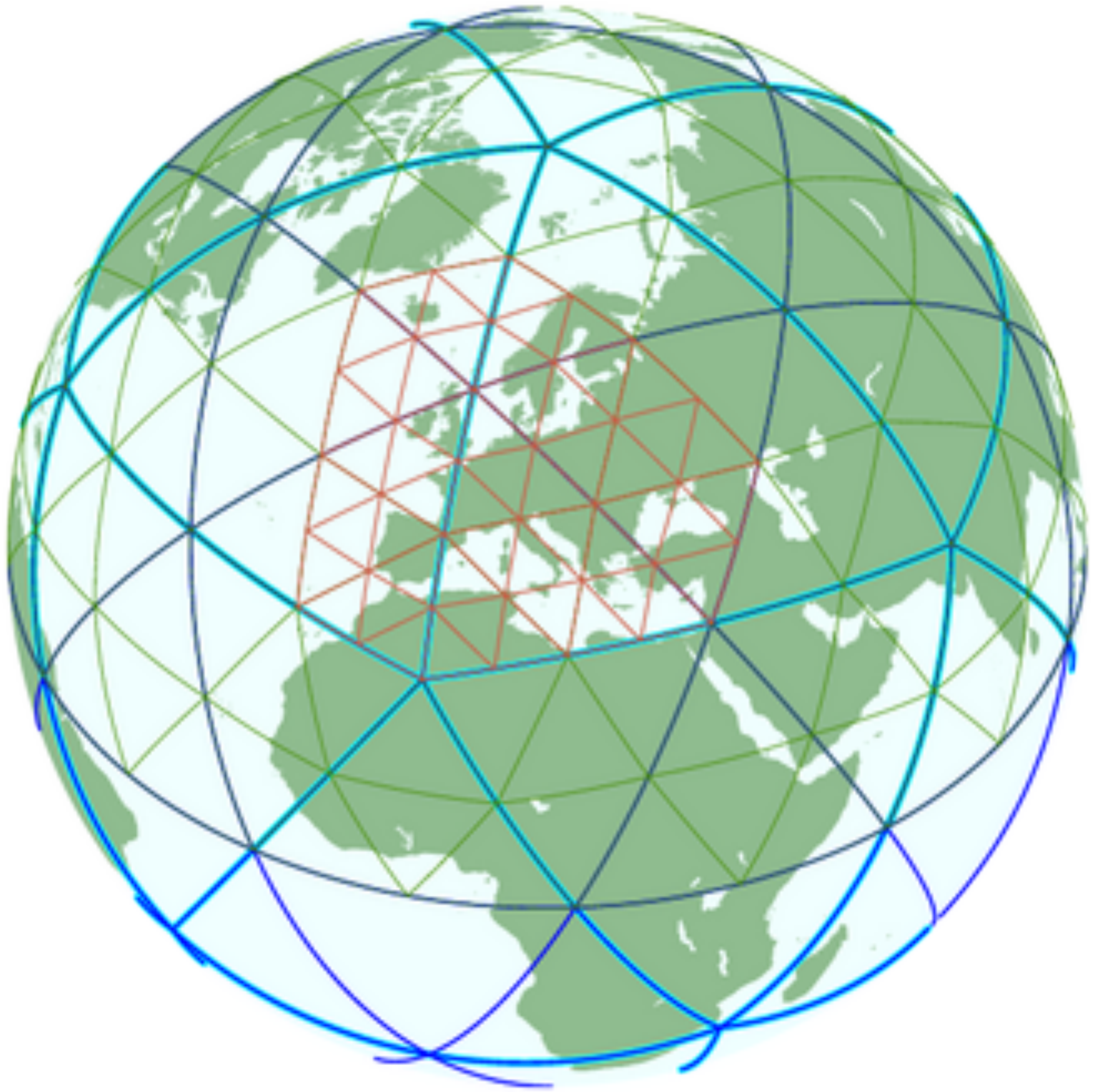
Find the departure points.



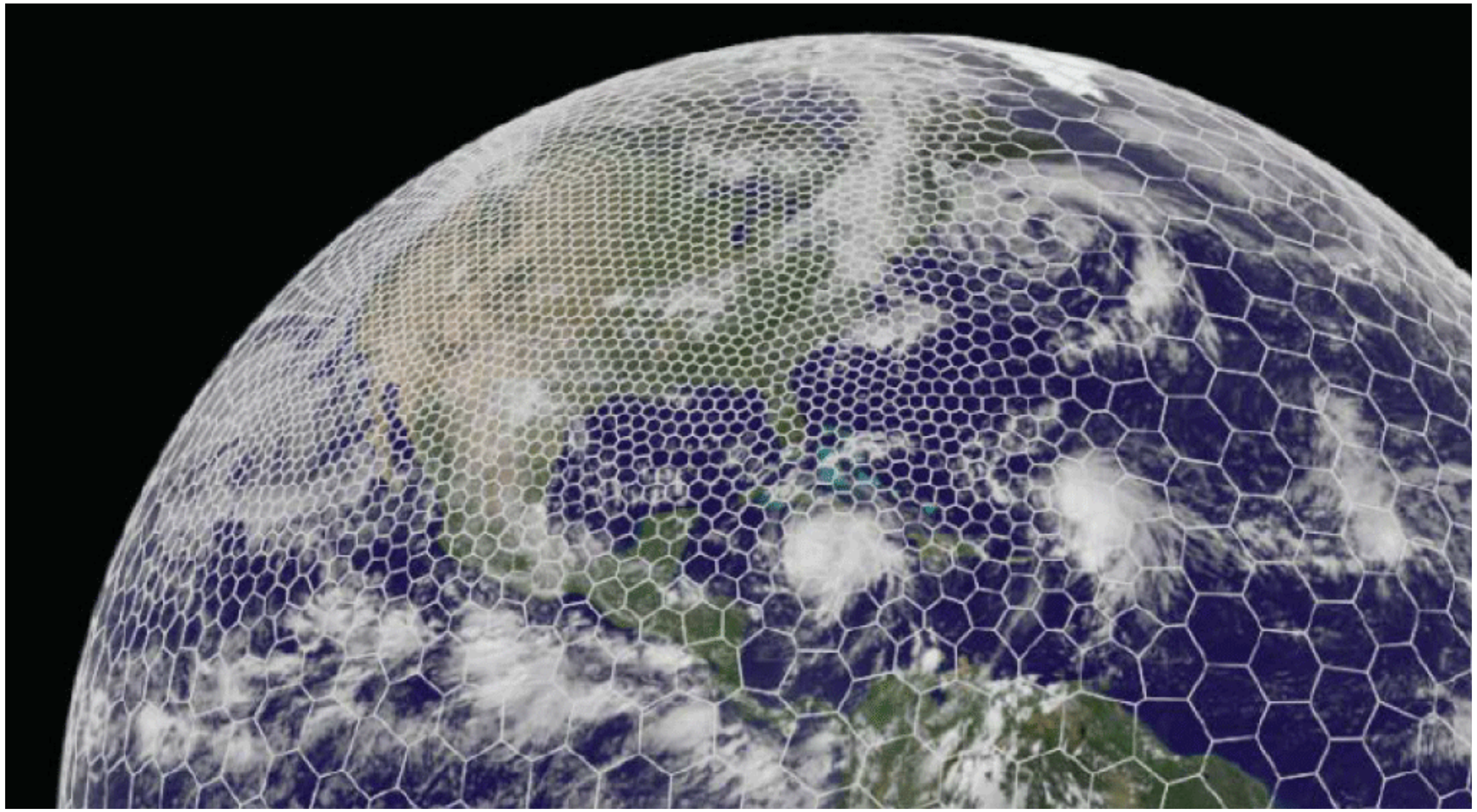


This idea was proposed by Dukowicz and Baumgardner (2000).
It’s conservative because it just slides the mass around.
It’s complicated but very attractive.
(It’s even more complicated in 3D.)

Regional refinement



Nesting



Stretching

The same physics can be used with different dynamical cores.

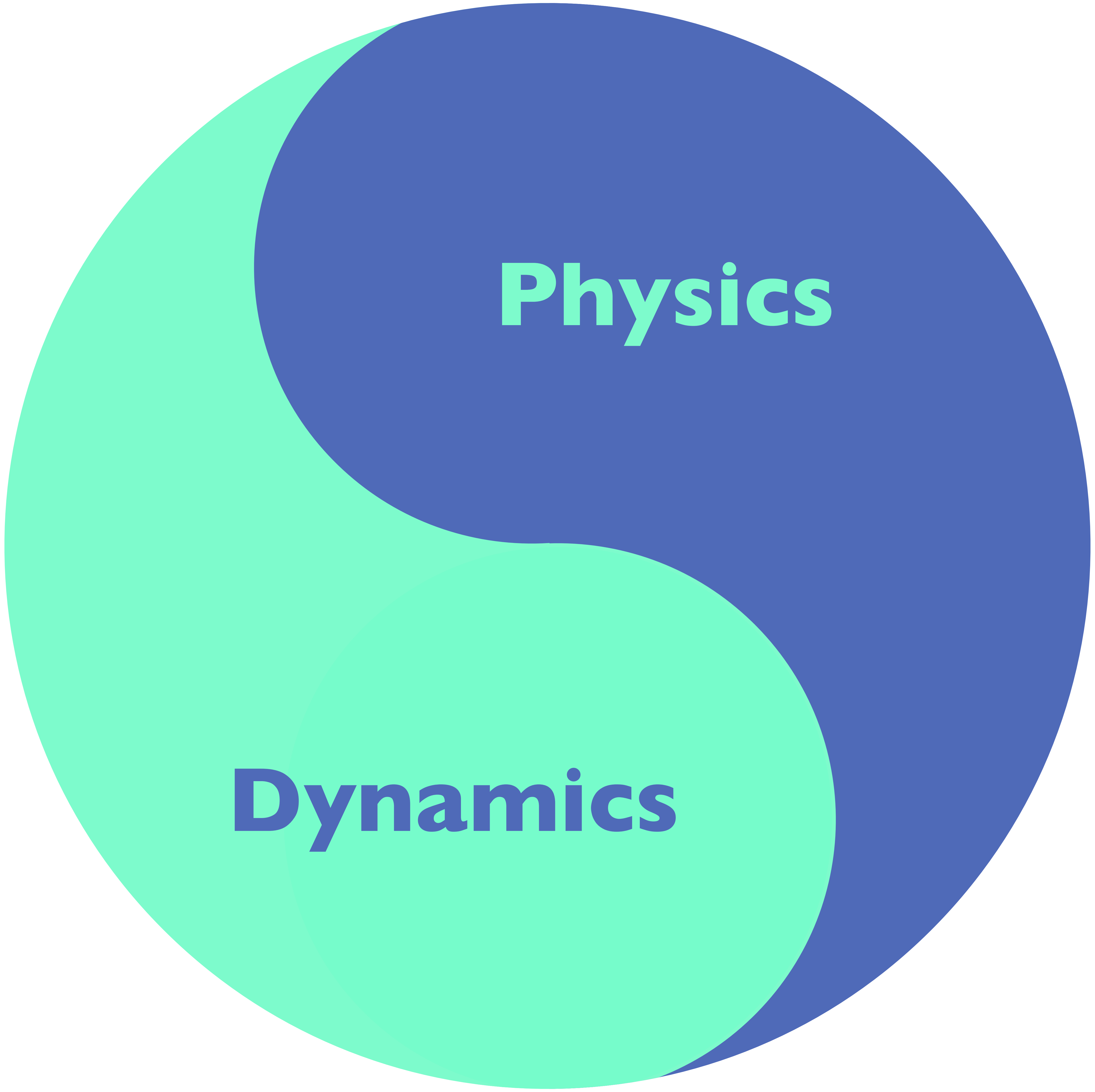
Example: The dynamical cores currently available for use with the CAM are:

- Spectral Eulerian
- FV1
- SE
- MPAS
- FV3

Discretizing the equations:

What do we want?

- Good accuracy, or more precisely rapid convergence
- Computational stability — no blowing up
- No computational modes
- Conservation of various things
- Sign preservation and monotonicity
- The possibility of regional refinement
- Simplicity
- Speed


Physics
Dynamics