# *Diffusion and compensating subsidence as limiting cases of a single flux parameterization*

David Randall

The tendency of the average of a conservative variable $h$ due to a vertical eddy flux is given by

$$\rho \frac{\partial \overline{h}}{\partial t} \sim -\frac{\partial}{\partial z}\left(\rho \overline{w'h'}\right). \tag{1}$$

In this note, we consider two different forms of the vertical eddy flux, and the relationship between them.

A diffusive vertical flux of a conservative variable $h$ satisfies

$$\rho \overline{w'h'} = -K \frac{\partial \overline{h}}{\partial z}, \tag{2}$$

where $K$ is a non-negative diffusion coefficient. For positive $K$, the flux given by (2) is down-gradient. With the use of (2), Eq. (1) becomes *parabolic*:

$$\rho \frac{\partial \overline{h}}{\partial t} \sim \frac{\partial}{\partial z}\left(K \frac{\partial \overline{h}}{\partial z}\right). \tag{3}$$

In contrast, for a single cloud type the tendency of $\overline{h}$ due to deep convection is given by

$$\rho \frac{\partial \overline{h}}{\partial t} \sim M_c \frac{\partial \overline{h}}{\partial z} + \delta\left(h_c - \overline{h}\right), \tag{4}$$

where

$$\boxed{M_c \equiv \rho\sigma_c\left(1-\sigma_c\right)\left(w_c-\tilde{w}\right)} \tag{5}$$

is the positive convective mass flux, $\delta$ is the detrainment mass flux per unit height, $h_c$ is the in-cloud value of $h$, $\sigma_c$ is the fractional area covered by the convective updrafts, and $\tilde{w}$ is the environmental vertical velocity (Arakawa and Schubert, 1974). Eq. (4) is *hyperbolic*. It applies when

$$\sigma_c \ll 1 . \tag{6}$$

We will derive a single formula that reduces to either (3) or (4), in the appropriate limits. The discussion follows Randall et al. (1992) and Lappen and Randall (2001). The starting point is the equation that governs the variance of $h$, i.e.,

$$\rho\frac{\partial\overline{h'^2}}{\partial t} = -2\rho\overline{w'h'}\frac{\partial\overline{h}}{\partial z} - \frac{\partial}{\partial z}\left(\rho\overline{w'h'h'}\right) - \rho\frac{\overline{h'^2}}{\tau_{dis}}, \tag{7}$$

where $\overline{h'^2}/\tau_{dis}$ is the rate at which $\overline{h'^2}$ is dissipated. In writing (7), we have neglected additional terms involving advection by the mean wind, gradient production associated with horizontal fluxes, and horizontal transport by triple moments involving the horizontal wind components.

We now simplify (7) by neglecting the time-rate-or-change term, and use the mass-flux model to rewrite the remaining terms. With the mass flux model, the vertical flux of $h$ is given by

$$\rho\overline{w'h'} = \rho\sigma_c\left(1-\sigma_c\right)\left(w_c-\tilde{w}\right)\left(h_c-\tilde{h}\right), \tag{8}$$

where $\tilde{h}$ is the environmental value of $h$, which satisfies

$$\overline{h} = \sigma_c h_c + \left(1-\sigma_c\right)\tilde{h} . \tag{9}$$

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Similarly, the triple moment is given by

$$\rho\overline{w'h'h'} = \rho\sigma_c\left(1-\sigma_c\right)\left(1-2\sigma_c\right)\left(w_c-\tilde{w}\right)\left(h_c-\tilde{h}\right)^2 . \tag{10}$$

Lappen and Randall (2001) showed that, with the mass-flux model, the dissipation term is given by

$$\rho\frac{\overline{h'^2}}{\tau_{dis}} = \left(\varepsilon+\delta\right)\left(h_c-\tilde{h}\right)^2 , \tag{11}$$

where $\varepsilon \geq 0$ is the rate at which mass is flows from the environment into the updrafts, and $\delta \geq 0$ is the rate at which mass flows from the updrafts into the environment. Both $\varepsilon$ and $\delta$ have dimensions of mass per unit volume per unit time. Conservation of mass for the convective updrafts is expressed by

$$\frac{\partial}{\partial z}\left(\rho\sigma_c w_c\right) = \varepsilon-\delta . \tag{12}$$

Using (8), we can rewrite (10) and (11) as

$$\rho\overline{w'h'h'} = \left(1-2\sigma_c\right)\frac{\left(\rho\overline{w'h'}\right)^2}{M_c} , \tag{13}$$

and

$$\rho\frac{\overline{h'^2}}{\tau_{dis}} = \left(\varepsilon+\delta\right)\left(\frac{\rho\overline{w'h'}}{M_c}\right)^2 , \tag{14}$$

respectively. Substitution of (13) and (14) into the steady-state (or time-averaged) version of (7) gives

$$\boxed{0=-2\rho\overline{w'h'}\frac{\partial\overline{h}}{\partial z}-\frac{\partial}{\partial z}\left[\left(1-2\sigma_c\right)\frac{\left(\rho\overline{w'h'}\right)^2}{M_c}\right]-\left(\varepsilon+\delta\right)\left(\frac{\rho\overline{w'h'}}{M_c}\right)^2}. \tag{15}$$

We now examine two limiting cases. For $\sigma_c \equiv 1/2$, Eq. (15) reduces to

$$0=-2\frac{\partial\overline{h}}{\partial z}-\left(\frac{\varepsilon+\delta}{M^2{}_c}\right)\rho\overline{w'h'}, \tag{16}$$

or

$$\boxed{\rho\overline{w'h'}=-\left(\frac{2M^2{}_c}{\varepsilon+\delta}\right)\frac{\partial\overline{h}}{\partial z}\text{ for }\sigma=1/2}. \tag{17}$$

Eq. (17) has the form of (2), with

$$K=\frac{2M^2{}_c}{\varepsilon+\delta}>0. \tag{18}$$

Eq. (17) is the diffusive limit of (15). The flux given by (17) is always downgradient.

In the limit $\sigma_c \to 0$, Eq. (15) can be expanded and simplified to

$$0=-2M_c\frac{\partial\overline{h}}{\partial z}-2\frac{\partial}{\partial z}\left(\rho\overline{w'h'}\right)+\left(\frac{\rho\overline{w'h'}}{M_c}\right)\frac{\partial M_c}{\partial z}-\left(\varepsilon+\delta\right)\left(\frac{\rho\overline{w'h'}}{M_c}\right). \tag{19}$$

In the same limit, (12) can be written as

$$\frac{\partial M_c}{\partial z}=\varepsilon-\delta. \tag{20}$$

Using (20), Eq. (19) can be further simplified to

$$0 = -M_c \frac{\partial \bar{h}}{\partial z} - \frac{\partial}{\partial z}\left(\rho \overline{w'h'}\right) - \delta\left(\frac{\rho \overline{w'h'}}{M_c}\right). \tag{21}$$

Substitution from (8) gives

$$0 = -M_c \frac{\partial \bar{h}}{\partial z} - \frac{\partial}{\partial z}\left(\rho \overline{w'h'}\right) - \delta\left(h_c - \bar{h}\right), \tag{22}$$

where we have used $\tilde{h} \cong \bar{h}$ , which is appropriate in the limit $\sigma_c \to 0$ . Finally, (22) can be rearranged to

$$\boxed{-\frac{\partial}{\partial z}\left(\rho \overline{w'h'}\right) = M_c \frac{\partial \bar{h}}{\partial z} + \delta\left(h_c - \bar{h}\right) \text{ for } \sigma \to 0}, \tag{23}$$

which is consistent with (1) and (4). A flux that satisfies (23) can be be either upgradient or downgradient.

We have shown that a parameterization based on (15) can produce either (local) diffusion or (non-local) penetrative convection, depending on the value of $\sigma_c$ .Wyngaard and Weil (1991) obtained a similar result. The diffusive, downgradient flux satisfying (17) applies when production is balanced by dissipation, and (1) is parabolic. The (possibly upgradient) flux satisfying (23) is obtained when production is balanced by transport, and (1) is hyperbolic.

## References

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