Virtual Temperature

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Definition

For dry air, the equation of state is

$$p_d = \rho_d R_d T . \tag{1}$$

Similarly, water vapor obeys its own equation of state with its own gas constant:

$$e = \rho_v R_V T . \tag{2}$$

The total pressure is

$$p = p_d + e . \tag{3}$$

We see that

$$p = \left(\rho_d R_d + \rho_v R_v\right) T , \tag{4}$$

$$\rho = \rho_d + \rho_v + \rho_l . \tag{5}$$

We *define* the virtual temperature as satisfying the ideal gas law with the total pressure and the total density, and the gas constant for dry air:

$$\rho R_d T_v \equiv \rho . \tag{6}$$

Substituting on both sides of (6), we obtain

$$\begin{aligned}(\rho_d + \rho_v + \rho_l) R_d T_v &= (\rho_d R_d + \rho_v R_v) T \\ &= \left(\rho_d + \frac{\rho_v}{\varepsilon}\right) R_d T \ ,\end{aligned} \tag{7}$$

where

$$\varepsilon = \frac{R_d}{R_v} \cong 0.622. \tag{8}$$

This leads to

$$\begin{aligned}T_v &= T\left(\frac{\rho_d + \frac{\rho_v}{\varepsilon}}{\rho_d + \rho_v + \rho_l}\right) \\ &= T\left(\frac{1 + \frac{q}{\varepsilon}}{1 + q + l}\right),\end{aligned} \tag{9}$$

which can be approximated by

$$\begin{aligned}T_v &\cong T\left(1 + \frac{q}{\varepsilon}\right)(1 - q - l) \\ &\cong T\left(1 - q - l + \frac{q}{\varepsilon}\right) \\ &= T(1 + \delta q - l)\ ,\end{aligned} \tag{10}$$

where

$$\delta \equiv \frac{1 - \varepsilon}{\varepsilon} \cong 0.608. \tag{11}$$

When the weight of condensed water is included in the virtual temperature, the name "density temperature" is sometimes used instead.

Buoyancy fluctuations on isobaric surfaces

The following discussion is based on ideas presented by Lilly (1968).

The *virtual* dry static energy is

$$s_v \equiv c_p T_v + gz \ . \tag{12}$$

We use the moist static energy

$$h \equiv c_p T + gz + Lq \ , \tag{13}$$

which is approximately conserved under both moist and dry adiabatic processes, even when precipitation is occurring. The total water mixing ratio, $q + l$, is also approximately conserved under both moist and dry adiabatic processes, although it is of course affected by precipitation.

Consider fluctuations *at constant pressure*, denoted by primes. We can write

$$h' \equiv c_p T' + Lq' \ . \tag{14}$$

Here we neglect height fluctuations on the constant pressure surface. From (10) and (12), we see that

$$\begin{aligned} s_v' &\cong c_p T_v' \\ &\cong c_p T' + c_p \overline{T}\left(\delta q' - l'\right) \\ &= c_p T' + \varepsilon\left(\delta L q' - L l'\right) , \end{aligned} \tag{15}$$

where

$$\varepsilon \equiv \frac{c_p \overline{T}}{L} \cong 0.1 \ . \tag{16}$$

This can be manipulated as follows:

$$\begin{aligned} s_v' &\cong \left(c_p T' + Lq'\right) - \left(1 - \delta\varepsilon\right) Lq' - \varepsilon L l' \\ &= h' - \left(1 - \delta\varepsilon\right) L\left(q' + l'\right) + \left[1 - \left(1 + \delta\varepsilon\right)\right] L l' \ . \end{aligned} \tag{17}$$

Buoyancy fluxes across pressure surfaces

Following Lilly (1968), we can construct an expression for the virtual dry static energy flux, F_{sv} , by multiplying (17) by $\rho w'$ and then averaging. (We neglect the fluctuations of ρ .) The result is

$$\boxed{F_{sv} = F_h - \left(1-\delta\varepsilon\right)LF_{q+l} + \left[1-\left(1+\delta\varepsilon\right)\right]LF_l} \ . \tag{18}$$

Eq. (18) is valid regardless of the cloud amount. It was used by Randall (1987).

Still following Lilly (1968), we consider two cases. First, if there is no cloud, then $LF_l = 0$, and (18) reduces to

$$\boxed{F_{sv} = \left(F_{sv}\right)_{clr} \equiv F_h - \left(1-\delta\varepsilon\right)LF_{q+l}} \text{ for clear air.} \tag{19}$$

If there is a uniform cloud, so that the air is saturated everywhere, we can write

$$\gamma c_p T' \cong Lq' \ , \tag{20}$$

where

$$\gamma \equiv \frac{L}{c_p}\left(\frac{\partial q_*}{\partial T}\right)_p \ . \tag{21}$$

From (20), we get

$$\gamma h' = \left(1+\gamma\right)Lq' \ , \tag{22}$$

or

$$Lq' = \left(\frac{\gamma}{1+\gamma}\right)h' \ . \tag{23}$$

It follows that

$$Ll' = L\left(q'+l'\right) - \left(\frac{\gamma}{1+\gamma}\right)h' \ , \tag{24}$$

or

$$LF_l = LF_{q+l} - \left(\frac{\gamma}{1+\gamma}\right)F_h \ . \tag{25}$$

Substituting (25) into (18), we find that for the fully cloudy case

$$\boxed{F_{sv} = \left(F_{sv}\right)_{cld} \equiv \left[\frac{1+\left(1+\delta\right)\gamma\varepsilon}{1+\gamma}\right]F_h - \varepsilon L F_{q+l}}, \text{ for uniformly cloudy air.} \tag{26}$$

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