THE SPECTRAL TECHNIQUE

(adapted from Jarraud and Simmons, 1983)

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☞ The equations used in numerical weather prediction can be written:

$$\frac{\partial F_i}{\partial t} = A_i(F_j), \qquad j = 1, \ldots, J \tag{1}$$

where J is the number of prognostic variables $F_i = F_i(x, t)$, with x representing the 3-dimensional space coordinate and t the time coordinate. F_i are suppose to be smooth (i.e. differentiable) as needed, at least away from the boundaries. The A_i are operators, generally non linear and involve usually partial derivatives.

☞ When finite-difference methods are used, an ensemble of grid points (x_p, t_q) is chosen in both space and time and the continuous

operators $\partial / \partial t$ and A_i are replaced by discrete analogues, the complexity of which depends in part on the accuracy required.

☞ As an alternative the fields F_i at any time are smooth functions of x and as such can be considered as elements of some vector space H, a so called Hilbert space.

☞ Let us assume that we know an Hilbertian basis of H, $\{e_m(x)\}$, orthonormal (i.e. $(e_m, e_n) = \delta_{m,n}$) which is such that any element $F \in H$ can be expressed as,

$$F = \sum_{m=-\infty}^{+\infty} F^m e_m \qquad (2)$$

The F^m are the orthogonal projection of F on the subspace generated by the e_m, and they are defined by the scalar product,

$$F^m = (F, e_m) = \int_S F\, e_m^*\, dx \qquad (3)$$

☞ Since we are dealing with global or hemispheric problems on the earth, it is desirable to chose expansion functions suited to spherical geometry. It is also desirable to choose functions simplifying the operators A_i. Such a simplification is achieved if the expansion functions are eigenfunctions of some operators of A_i.

This is the case of spherical harmonics,

$$Y_n^m(\lambda, \theta) = e^{im\lambda} P_n^m(\sin\theta) \tag{4}$$

where λ is longitude and θ is latitude. m is the zonal wavenumber and n is the total wavenumber. P_n^m is the associated Legendre function of order m and degree n. There are several analytical forms for the P_n^m. A classical one is given by Rodrigues formula,

$$P_n^m(\mu) = \sqrt{2n+1\frac{(n-m)!}{(n+m)!}}\frac{(1-\mu^2)^{\frac{|m|}{2}}}{2^n n!}\frac{d^{n+|m|}(1-\mu^2)^n}{d\mu^{n+|m|}} \tag{5}$$

$$P_n^m(\mu) = P_n^{-m}(\mu) \qquad P_n^m(\mu) \equiv 0 \text{ if } |m| > n \tag{6}$$

where $\mu = \sin\theta$.

☞ The spherical harmonics Y_n^m are eigenfunctions of the 2-D Laplacian operator on the sphere.

$$\Delta Y_n^m = -\frac{n(n+1)}{r^2} Y_n^m \tag{7}$$

where r is the radius of the sphere.

☞ Every continuous function $F(\lambda, \mu, \eta, t)$ on the sphere can be expressed as,

$$F(\lambda, \mu, \eta, t) = \sum_{m=-\infty}^{+\infty} \sum_{n=|m|}^{+\infty} F_n^m(\eta, t) Y_n^m(\lambda, \mu) \tag{8}$$

where the spectral components, F_n^m are computed as,

$$F_n^m(\eta, t) = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} F(\lambda, \mu, \eta, t)\, Y_n^{m*}(\lambda, \mu)\, d\lambda\, d\mu$$

The Truncation Problem

☞ Since we cannot cope with an infinite number of components, we have to project F on a finite dimensional subspace $\bar{H}$ of H and F is approximated by,

$$\bar{F} = \sum_{m=-M}^{M}\sum_{n=|m|}^{N(m)} F_n^m(\eta, t)\, Y_n^m(\lambda, \mu) \qquad (9)$$

How quickly $\bar{F}$ converges to F ?

$$F_n^m n^{2p} \to 0 \text{ when} n \to \infty \text{ for any } p \qquad (10)$$

which shows that a smooth field $\bar{F}$ converges uniformly to F faster than any finite power of $1/n$ where n tends to infinity.

☞ This extremely rapid convergence is an important argument in favor of spherical harmonics expansion since it explains why they can achieve an accuracy comparable with grid point methods with substantially fewer degrees of freedom.

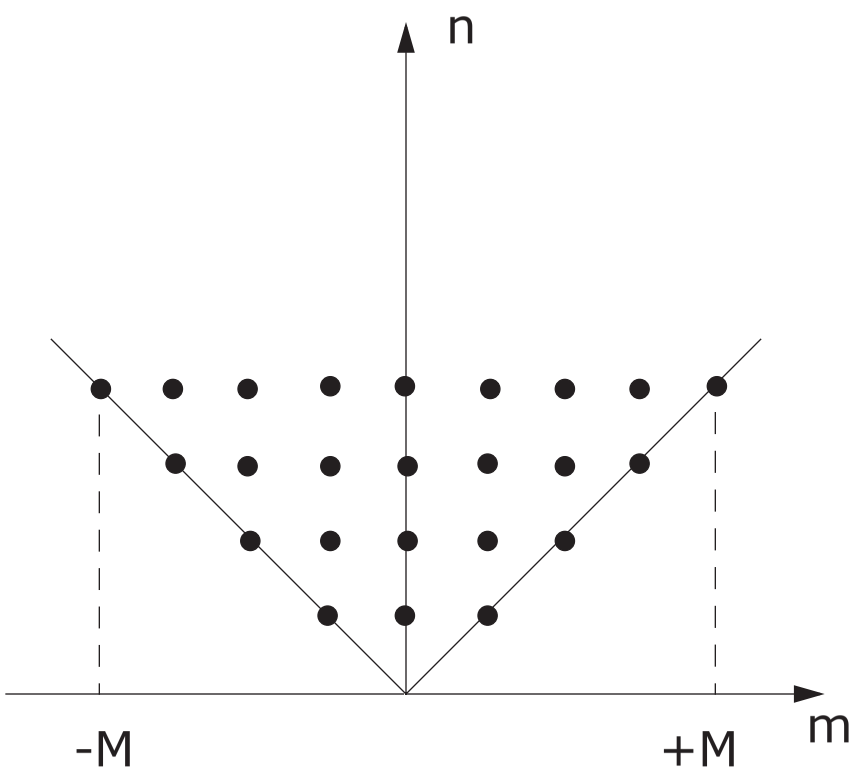
How to choose *N(m)?*

☞ The reason for choosing a truncation should be to make the best usage of a given number of degrees of freedom (i.e. of a given number of retained spectral components) which means either getting the most accurate results for a given cost or the most economical result for a given accuracy.

☞ The two most common truncations used are the so called

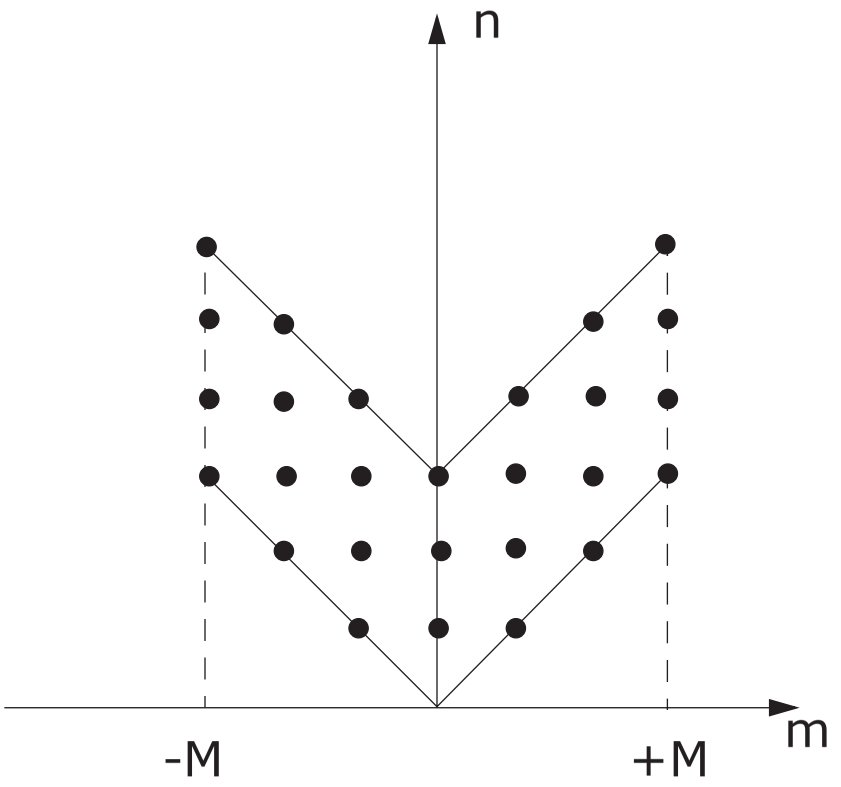
- **triangular**
- **rhomboidal**

TRIANGULAR



n=|m|, N(m)=M

RHOMBOIDAL



N(m)=M+|m|

☞ The arguments in favor of a rhomboidal truncation are mainly historical. Rhomboidal truncation can lead to easier initial programming (the spectral components can be stored in rectangular arrays);

☞ Ellsaesser (1966) showed that for very coarse resolutions, rhomboidal truncations could minimize the variance retained for the rotational kinetic energy at 500mb.

☞ Rhomboidal truncations are only invariant under rotations around the earth's axis and correspond to a uniform resolution only in the east-west direction.

☞ A theoretical advantage of triangular truncations is that they are isotropic, i.e. if $\overline{F_1}$ is an expansion of F triangularly truncated at wavenumber M in one spherical coordinate system (λ_1, θ_1), and $\overline{F_2}$ in another one (λ_2, θ_2) then $\overline{F_1} = \overline{F_2}$. This results from a property of the spherical harmonics

$$Y_n^m(\lambda_1, \theta_1) = \sum_{m' = -n}^{n} C_n^{m, m'} Y_n^{m'}(\lambda_2, \theta_2) \qquad (11)$$

☞ The resolution in a triangularly truncated system is uniform on the sphere and so whatever the position of the pole the truncated field is the same.

The Primitive Equations

The governing primitive equations for U, V, T and q may be written in the form

$$\frac{\partial U}{\partial t} - \zeta V + \dot{\eta}\frac{\partial U}{\partial \eta} + \frac{R_d T_v}{a}\frac{\partial}{\partial \lambda}\ln p + \frac{1}{a}\frac{\partial}{\partial \lambda}(\Phi + E) = P_U \qquad (12)$$

$$\frac{\partial V}{\partial t} + \zeta U + \dot{\eta}\frac{\partial V}{\partial \eta} + \frac{R_d T_v}{a}(1-\mu^2)\frac{\partial}{\partial \mu}\ln p + \frac{(1-\mu^2)}{a}\frac{\partial}{\partial \mu}(\Phi + E) = P_V \qquad (13)$$

$$\frac{\partial T}{\partial t} + \frac{U}{a(1-\mu^2)}\frac{\partial T}{\partial \lambda} + \frac{V}{a}\frac{\partial T}{\partial \eta} + \dot{\eta}\frac{\partial T}{\partial \eta} - \frac{\kappa T_v \omega}{[1+(\delta-1)q]p} = P_T \qquad (14)$$

$$\frac{\partial q}{\partial t} + \frac{U}{a(1-\mu^2)}\frac{\partial q}{\partial \lambda} + \frac{V}{a}\frac{\partial q}{\partial \eta} + \dot{\eta}\frac{\partial q}{\partial \eta} = P_q \qquad (15)$$

From the horizontal momentum equations (12) and (13) we can derive the vorticity and divergence,

$$\frac{\partial \zeta}{\partial t} = \frac{1}{a(1-\mu^2)}\frac{\partial F_v}{\partial \lambda} - \frac{1}{a}\frac{\partial F_u}{\partial \eta}, \tag{16}$$

$$\frac{\partial D}{\partial t} = \frac{1}{a(1-\mu^2)}\frac{\partial F_v}{\partial \lambda} + \frac{1}{a}\frac{\partial F_u}{\partial \eta} - \Delta G, \tag{17}$$

where

$$F_u = \zeta V - \dot{\eta}\frac{\partial U}{\partial \eta} - \frac{R_d T_v}{a}\frac{\partial}{\partial \lambda}\ln p + P_U, \tag{18}$$

$$F_v = -\zeta U - \dot{\eta}\frac{\partial V}{\partial \eta} - \frac{R_d T_v}{a}(1-\mu^2)\frac{\partial}{\partial \mu}\ln p + P_V, \tag{19}$$

and $G = \Phi + E$.

☞ Choosing spherical harmonic expansion functions, we represent ζ and D by a truncated series:

$$\zeta = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \zeta_n^m P_n^m(\mu) e^{im\lambda} \tag{20}$$

$$D = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} D_n^m P_n^m(\mu) e^{im\lambda} \tag{21}$$

☞ The truncated spectral forms of (16) and (17) are,

$$\frac{\partial \zeta_n^m}{\partial t} = \frac{1}{4\pi a} \int_{-1}^{1} \int_{0}^{2\pi} \left(\frac{1}{1-\mu^2} \frac{\partial F_v}{\partial \lambda} - \frac{\partial F_u}{\partial \eta} \right) P_n^m(\mu) e^{-im\lambda} d\lambda d\mu \tag{22}$$

$$\frac{\partial D_n^m}{\partial t} = \frac{1}{4\pi a}\int_{-1}^{1}\int_{0}^{2\pi}\left(\frac{1}{1-\mu^2}\frac{\partial F_u}{\partial \lambda} + \frac{\partial F_v}{\partial \eta}\right)P_n^m(\mu)e^{-im\lambda}d\lambda d\mu - \frac{1}{4\pi a}\int_{-1}^{1}\int_{0}^{2\pi}(\Delta G)P_n^m(\mu)e^{-im\lambda}d\lambda d\mu \tag{23}$$

and integrating by parts we obtain,

$$\frac{\partial \zeta_n^m}{\partial t} = \frac{1}{4\pi a}\int_{-1}^{1}\int_{0}^{2\pi}\left(\frac{im}{1-\mu^2}F_v P_n^m - F_u P_n^m\right)e^{-im\lambda}d\lambda d\mu \tag{24}$$

$$\frac{\partial D_n^m}{\partial t} = \frac{1}{4\pi a}\int_{-1}^{1}\int_{0}^{2\pi}\left(\frac{im}{1-\mu^2}F_u P_n^m + F_v P_n^m\right)e^{-im\lambda}d\lambda d\mu + \frac{n(n+1)}{4\pi a^2}\int_{-1}^{1}\int_{0}^{2\pi}GP_n^m e^{-im\lambda}d\lambda d\mu \tag{25}$$

To compute the wind components we introduce the stream function, ψ and the velocity potential, χ. Then,

$$U = \frac{1}{a}\left\{-(1-\mu^2)\frac{\partial\psi}{\partial\mu} + \frac{\partial\chi}{\partial\lambda}\right\} \tag{26}$$

$$V = \frac{1}{a}\left\{\frac{\partial\psi}{\partial\lambda} + (1-\mu^2)\frac{\partial\chi}{\partial\mu}\right\} \tag{27}$$

with

$$\zeta = \Delta\psi \text{ and } D = \Delta\chi \tag{28}$$

Truncated expansions for ψ and χ are thus given by,

$$\psi = -a^2 \sum_{m=-M}^{M} \sum_{\substack{n=|m| \\ n \neq 0}}^{N(m)} \frac{\zeta_n^m}{n(n+1)} P_n^m(\mu) e^{im\lambda} \tag{29}$$

$$\chi = -a^2 \sum_{m=-M}^{M} \sum_{\substack{n=|m| \\ n \neq 0}}^{N(m)} \frac{D_n^m}{n(n+1)} P_n^m(\mu) e^{im\lambda} \tag{30}$$

☞ Truncated expressions for U and V are,

$$U = a \sum_{m=-M}^{M} \sum_{\substack{n=|m| \\ n \neq 0}}^{N(m)} \frac{1}{n(n+1)} \left\{ \zeta_n^m (1-\mu^2) \frac{dP_n^m}{d\mu} - imD_n^m P_n^m \right\} e^{im\lambda} \tag{31}$$

$$V = -a \sum_{m=-M}^{M} \sum_{\substack{n=|m| \\ n \neq 0}}^{N(m)} \frac{1}{n(n+1)} \left\{ im\zeta_n^m P_n^m + D_n^m (1-\mu^2) \frac{dP_n^m}{d\mu} \right\} e^{im\lambda} \qquad (32)$$

☞ Although the treatment of the temperature and humidity equations is generally similar to that of the other prognostic equations, two different approaches have been adopted for the actual implementation of the technique. For adiabatic models they yield identical results, but the choice of approach has implications for the way the parameterizations of convection and large-scale precipitation are included in a model.

To illustrate the two approaches we write equations (14) and (15) in the forms,

$$\frac{\partial T}{\partial t} = -\frac{U}{a(1-\mu^2)}\frac{\partial T}{\partial \lambda} - \frac{V}{a}\frac{\partial T}{\partial \eta} + R_T \tag{33}$$

$$\frac{\partial q}{\partial t} = -\frac{U}{a(1-\mu^2)}\frac{\partial q}{\partial \lambda} - \frac{V}{a}\frac{\partial q}{\partial \eta} + R_q \tag{34}$$

In the first, and more traditional approach, (33) and (34) are written,

$$\frac{\partial T}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial}{\partial \lambda}(UT) - \frac{1}{a}\frac{\partial}{\partial \eta}(VT) + (DT + R_T), \tag{35}$$

$$\frac{\partial q}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial}{\partial \lambda}(Uq) - \frac{1}{a}\frac{\partial}{\partial \eta}(Vq) + (Dq + R_q), \tag{36}$$

and integration by parts is used to write the spectral equations as,

$$\frac{\partial T_m^n}{\partial t} = -\frac{1}{4\pi a}\int_{-1}^{1}\int_{0}^{2\pi}\left\{\frac{im}{1-\mu^2}(UT)P_n^m - (VT)\frac{dP_n^m}{d\mu}\right\}e^{-im\lambda}\,d\lambda\,d\mu + \frac{1}{4\pi a}\int_{-1}^{1}\int_{0}^{2\pi}(DT + R_T)P_n^m e^{-im\lambda}\,d\lambda\,d\mu \tag{37}$$

$$\frac{\partial q_m^n}{\partial t} = -\frac{1}{4\pi a}\int_{-1}^{1}\int_{0}^{2\pi}\left\{\frac{im}{1-\mu^2}(Uq)P_n^m - (Vq)\frac{dP_n^m}{d\mu}\right\}e^{-im\lambda}\,d\lambda\,d\mu + \frac{1}{4\pi a}\int_{-1}^{1}\int_{0}^{2\pi}(Dq + R_q)P_n^m e^{-im\lambda}\,d\lambda\,d\mu \tag{38}$$

☞ In this approach, tendencies of T and q are not directly available on the Gaussian grid and parameterizations involving condensation have to be performed after transformations to and from spectral space, for example at the beginning of the next timestep.

☞ In the second approach, the additional fields $\partial T/\partial\lambda$, $\partial T/\partial\mu$, $\partial q/\partial\lambda$ and $\partial q/\partial\mu$ are evaluated at grid-points from the spectral expansions of T and q. This allows computation of the total grid-point tendencies as given by the rhs of (35) and (36). This is the approach adopted in the operational ECMW spectral model.

How integrals are evaluated?

☞ The methods used to evaluate the integrals are generally such as to give exact (or non-aliased) results for the contribution of quadratic non-linear terms to the computation of those spectral coefficients retained in the truncation.

- **Longitudinal integral** $C_m(\mu) = \frac{1}{2\pi}\int_0^{2\pi} Ce^{-im\lambda} d\lambda$

☞ The integral is evaluated using a regular grid of points in the east/west, with

$$\lambda_i = \frac{2\pi i}{I} \tag{39}$$

and

$$C_m(\mu_j) = \frac{1}{I}\sum_{i=1}^{I} C(\lambda_i, \mu_j)e^{-im\lambda} \tag{40}$$

☞ The result is exact for $I \geq 3M+1$. In practice, I is chosen to allow efficient FFT techniques to be used to compute the sums, and truncation are usually such that $(3M+1)$ is equal or just less than one such I, for example an integer of the form $2^p 3^q 5^r$ for integers, p, q and r. Thus spectral models have typically been run with zonal truncations given by $M = 15, 21, 31, 42, 63, 84, 95$, etc.

- **Latitudinal integral** $C_n^m = \frac{1}{2}\int_{-1}^{1} C_m(\mu) P_n^m(\mu) d\mu$

☞ the associated Legendre functions $P_n^m(\mu)$ are of the form

$$(1-\mu^2)^{|m|/2} \times (Polynomial\ \ in\ \ (\mu) ofdegree\ \ n-|m|)$$

☞ the longitudinal integral of a product of three spherical harmonics vanishes unless the three zonal wavenumbers, m_1, m_2 and m_3 of the harmonics satisfy $m_1 + m_2 + m_3 = 0$

☞ It follows from these these properties that the integrand $C_m(\mu) P_n^m(\mu)$ is a polynomial in μ, of degree at most $3M$ for triangular truncation and $5M$ for rhomboidal truncation.

☞ The integral is evaluated by Gaussian quadrature, Eliassen(1970)

$$C_n^m = \frac{1}{2}\sum_{j=1}^{J} g_j C_m(\mu_j) P_n^m(\mu_j) \qquad (41)$$

where μ_j are zeros of $P_J^o(\mu)$, that is

$$P_J^o(\mu) = 0, \qquad j = 1, 2, \ldots\ldots, J$$

and $$g_j = \frac{2(1-\mu_j^2)}{\left(JP_{J-1}^0(\mu_j)\right)^2}.$$

☞ This Gaussian quadrature is exact for an integrand which is a polynomial in μ of degree $\leq 2J-1$ and J must satisfy $J \geq (3M+1)/2$ for triangular truncation and $J \geq (5M+1)/2$ for rhomboidal truncation.